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ALGEBRAS OF MULTIPLACE FUNCTIONS

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Preface

Thinking about the future perspectives of general algebra development A. G. Kurosh [84] wrote in his book: "... between the theory of universal algebras and classical parts of general algebra there exists a great uncultivated gap. The explorations began only in some places, isolated even, but between these places there are some, whose exploration is by all means necessary. It should be expected that just in this gap "belonging to no one" the main interests of the general algebra will transfer during the coming decade.

In accordance with general tendencies of modern science, for example, physics, new objects of study, new theories will appear not with the flow of decades, but more often and often. It is impossible to stop this process, and trying to do this would be unreasonable. This process can only be directed. In such an axiomatic science as general algebra a great mind is not needed to create new objects of study. It is more difficult to justify them. It is not even enough to indicate the important ones among the introduced notions. The notion of group is not justified by the existence of symmetrical groups on the arbitrary sets, but by the fact that all the groups are exhausted by symmetrical groups and their subgroups exactly up to an isomorphism."

The exploration of this gap, which has been mentioned above, began with the attempts to transfer the results known for classical algebras (in particular for the groups) to the n-ary case. Therefore, first of all, it is necessary to mark the work by E. L. Post [144], with whose appearance the birth and further development of the theory of n-ary groups are connected. The development of this theory is connected with the publishing of the article of S. A. Tchounikhine [210] and finds its continuation in the works of V. A. Artamonov [4], S. A. Rusakov [154], G. Čupona [16], W. A. Dudek [26, 31] and D. Dimovski [21]. Works simultaneously appeared summarizing the semigroup and quasigroup results in the n-ary case. The development of the polyadic semigroups theory and the theory of n-ary quasigroup began with the work of L. M. Gluskin [56] and V. D. Belousov [6], which are studied nowadays by many algebraists.

Every group is known to be isomorphic to some group of set substitutions and any semigroup is isomorphic to some semigroup of set transformations [14, 84, 105, 107]. Therefore, group theory (respectively semigroup theory) can be considered as an abstract study about groups of substitutions (respectively semigroups substitutions). Such an approach to these theories is justified, first of all, by their multiple applications in geometry, functional analysis, quantum mechanics, etc. Although group theory and semigroup theory deal in particular only with functions of one argument, the wider, but not less important, class of functions remains outside their attention — the functions of many arguments (in other words — the class of multiplace functions). Multiplace functions are known to find various applications not only in mathematics itself, for example, in mathematical analysis, but are also widely used in the theory of multi-valued logics [79, 108], cybernetics [50, 145], and general systems theory [24, 25]. On many multiplace functions, various natural operations are considered, first of all, the superposition operation, i.e., the operation of forming new functions as a result of substitution of one functions into others instead of their arguments. However, the algebraic theory of superpositions of multiplace functions has not been developed during a long period of time. K. Menger paid attention to the abnormality of this state in the mid-1940s of the previous century. He stated, in particular, that the superposition operation of n-place functions (n is a fixed natural number) has a property resembling the associativity, which K. Menger called superassociativity [112–119]. As it has been proved later, this property appeared to be fundamental in the fact that every set with a superassociative $(n+1)$-operation can be isomorphically presented as a set of n-place functions with the operation of superposition. This fact was first proved by R. M. Dicker [20] in 1963, and the particular case of the given theorem was showed by H. Whitlock [258] in 1964.

The theory of algebras of multiplace functions, which was called Menger algebras (in the case of fixed arity functions) and Menger systems (the arity of functions is arbitrary), continued its development in the works by V. A. Artamonov [3], M. I. Burtman [11, 12], F. A. Gadzhiev [42, 44], L. M. Gluskin [51–56], F. A. Ismailov [75–78], F. Kh. Muradov [125–131], L. G. Mustafaev, R. B. Feizullayev [132, 133], E. Redi [147–151], V. S. Trokhimenko [212–244], J. Henno [57–73], Ja. V. Hion [74], B. M. Schein [172, 189, 244], Ja. N. Yaroker [261, 262], W. A. Dudek [29, 30, 32], V. Kafka [80], H. Länger [85–100], H. Lausch, W. Nöbauer, W. Philipp [102, 134, 136], K. L. Skala [195–197]. Simultaneously and independently from algebra theories and Menger systems, beginning with the work of A. I. Mal'cev [106–108], the theory of iterative Post, which found its use in polysemantic logics, was being carried out. We notice in these works [42,44] a close connection between Post algebras and Menger systems.

In solving functional equations, as it was shown by V. D. Belousov [6], in some cases appears the necessity of inserting on the sets of multiplace functions some binary operations of superposition and, as it was found, this can be done in various ways. As a result algebraic constructions, such as $(2, n)$-semigroups [199, 260], positional algebras [6,200], polycategories, and polycycloides [148–151] made their appearance, the exploration of which have just began. And at last we have a number of works in which many vector-valued functions are studied. First of all there are the publications by M. D. Sandik [158, 159], G. Čupona [16], K. Trenčevski and D. Dimovski [211], B. Schweizer, and A. Sklar [191–193], and many others.

Working in the sphere of substantiation of differential geometry, V. V. Vagner in the beginning of the 1960s of the last century came to the conclusion concerning the necessity of an algebraic approach in the study of semigroups of transformations [251–256]. He formulated problems which deal with the abstract characterization of some classes of semigroups of partial transformations on which, besides superposition, some additional operations and relations are defined. He also formulated the task of describing all possible representations of semigroups with the help of partial transformations. The explorations in this direction beginning with the study of V. V. Vagner [246–256] were continued in the works of D. A. Bredihin [8–10], V. S. Garvatckii [45–47], V. A. Molchanov [122, 123], V. N. Salii [155–157], and B. M. Schein [160–187].

The contents of this book is related to the direction that V. V. Vagner followed. Sets of multiplace functions are studied in it, on which some naturally defined operations and relations are introduced. The various classes of algebraic systems of multiplace functions received in this way are described from the abstract point of view. Besides, the description of all representations of Menger algebras with the help of multiplace functions are presented in the book, as well as the formation of each of them.

The book is divided into six chapters. In Chapter 1, the main notions of relations theory, algebraic systems and operations of closing are considered. Special attention is paid to Section 1.4, in which the operations of closing are studied due to their special importance for the next chapters. In Chapter 2, the notion of Menger algebra is introduced and its special classes are studied. Connections with some useful semigroups (Section 2.2) are described first, and then these semigroups are used for the characterization of inverse Menger algebras (Section 2.3). Further on, we investigate Menger algebras for which the diagonal semigroup is a group (Section 2.4) and Menger algebras, which are quasigroups (Section 2.5). Next, Menger algebras of semiclosure operations defined on an ordered set (Section 2.6) are described. The chapter ends with the general theory of representations of Menger algebras by relations and multiplace functions (Section 2.7).

Chapter 3 is devoted to abstract characterization of different classes of Menger algebras of multiplace functions partially ordered by such natural relations as inclusion of functions, inclusion of domains, and the nonempty intersection of domains. In Section 3.6 Menger algebras of n-place functions closed under the restrictive product of functions are studied, and in Section 3.7 – closed under the restrictive product of functions and projectors only.

Studying the concrete class of algebras of functions, it is necessary in many cases to keep some information about fixed properties of functions, which, as it is known, are lost during isomorphism. For example, some of the functions may have stable (fixed) points, others have the same domains or coincide on the common part of their domains, etc. In order to have these properties of functions we must introduce the signature of the corresponding algebras of functions of some relations. Chapter 4 is devoted to the abstract description of such algebras of multiplace functions. First, we describe the subsets of abstract Menger algebras, which correspond to the sets of functions having common stable (fixed) point, next — subsets that correspond to the sets of functions having at least one stable point. Further on, Menger algebras of multiplace functions are studied that coincide on the common part of domains (Section 4.4) or have a nonempty intersection of domains (Section 4.5). The chapter ends with the abstract characterization of some useful equivalence relations.

In Chapter 5, we describe sets of functions closed under Mann's compositions and projection relations on these sets.

In Chapter 6, the sets of functions of various arities are considered. The need to study such sets appears, for example, in many-valued logics, programming, and in the algebraic theory of automaton [50, 106–108, 145, 153]. The theories that study these sets of functions are the theory of Menger systems (Section 6.1), T-Menger systems (Section 6.2), positional algebras (Section 6.3), as well as iterative and pre-iterative algebras of multiplace functions (Section 6.4). In Section 6.5 we describe sets of functions of various arities closed under some composition; in Section 6.6 the same sets of functions but with one distinguished projector. The chapter ends (Section 6.7) with the description of algebras of vector-valued functions, which have many interesting applications in different branches of mathematics.

Our book does not pretend to be complete considering the various questions connected with algebras of multiplace functions. Its contents are completely defined only by the authors' tastes. For readers interested in future study, we present quite a wide bibliography on the given subject at the end of the book.

Wrocław and Vinnytsia, January 2012

Wiesław A. Dudek and Valentin S. Trokhimenko

Contents

Chapter 1

Main concepts

1.1 Elements of theory of relations

Later on, we shall use the following notations of mathematical logic:

$\sim$	—	denotes negation,
$\wedge$	—	conjunction,
$\vee$	—	disjunction,
$\longrightarrow$	—	implication,
$\longleftrightarrow$	—	equivalence,
$\forall$	—	universal quantifier,
$\exists$	—	existential quantifier,

where first operates negation, then conjunction, disjunction, implication, and, at last, equivalence. Conjunction of a finite number of statements $A_n \wedge A_{n+1} \wedge \cdots \wedge A_{n+m}$ will be denoted by $\bigwedge\limits_{i=n}^{n+m} A_i$.

We shall also use ordinary notations from set theory:

$'$	—	denotes a complement,
$\cap$	—	intersection,
$\cup$	—	union,
$\setminus$	—	difference of sets,
$\subset$	—	inclusion,
$\varnothing$	—	the empty set,
$\mathfrak{P}(A)$	—	the family of all subsets of a set A,
$\{a \mid \Pi(a)\}$	—	the set of these elements a which have the property $\Pi(a)$.

The ordered n-tuple of elements $a_1, \ldots, a_n$ is denoted by $(a_1, \ldots, a_n)$, the Cartesian product of sets $A_1, A_2, \ldots, A_n$ — by $A_1 \times A_2 \times \cdots \times A_n$. In the case $A_1 = A_2 = \cdots = A_n = A$, we shall write A^n. The element $(a_1, \ldots, a_n)$ of A^n will be also denoted by $\bar{a}$ or by a^n if all a_i will be equal to a. By $(\bar{a}|_i\, b)$ we denote the element $(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)$.

Every subset ρ of the set $A \times B$ is called a *binary relation* between the elements of the sets A and B. In the case $A = B$, the relation ρ is called *homogeneous*. For $\rho \subset A \times B$, $a \in A$, and $H \subset A$, by $\rho\langle a\rangle$ and $\rho(H)$, we denote the subsets of B defined in the following way:

$$\rho\langle a\rangle = \{b \mid (a,b) \in \rho\}, \qquad \rho(H) = \bigcup\{\, \rho\langle a\rangle \mid a \in H\,\}.$$

If $\rho(H) \subset H$, then H is called a *ρ-saturated subset*. We can easily see that the following conditions are true:

$$\rho_1 \subset \rho_2 \longleftrightarrow (\forall a)\,(\rho_1\langle a\rangle \subset \rho_2\langle a\rangle),$$

$$\rho_1 \subset \rho_2 \longrightarrow \rho_1(H) \subset \rho_2(H),$$

$$\rho'\langle a\rangle = (\rho\langle a\rangle)',$$

$$H_1 \subset H_2 \longrightarrow \rho(H_1) \subset \rho(H_2),$$

$$\rho\Big(\bigcup_{i\in I} H_i\Big) = \bigcup_{i\in I} \rho(H_i), \qquad \rho\Big(\bigcap_{i\in I} H_i\Big) \subset \bigcap_{i\in I} \rho(H_i),$$

$$\Big(\bigcup_{i\in I} \rho_i\Big)(H) = \bigcup_{i\in I} \rho_i(H), \qquad \Big(\bigcap_{i\in I} \rho_i\Big)(H) \subset \bigcap_{i\in I} \rho_i(H).$$

For every relation $\rho \subset A \times B$, there is the so-called *inverse relation* ρ^{-1} defined as

$$\rho^{-1} = \{(b,a) \mid (a,b) \in \rho\} \subset B \times A,$$

which has the following properties:

$$\left(\rho^{-1}\right)^{-1} = \rho,$$

$$\rho_1 \subset \rho_2 \longleftrightarrow \rho_1^{-1} \subset \rho_2^{-1},$$

$$\Big(\bigcup_{i\in I} \rho_i\Big)^{-1} = \bigcup_{i\in I} \rho_i^{-1}, \qquad \Big(\bigcap_{i\in I} \rho_i\Big)^{-1} = \bigcap_{i\in I} \rho_i^{-1}.$$

The *composition (superposition) of binary relations* $\rho \subset A\times B$ and $\sigma \subset B\times C$ is a relation $\sigma \circ \rho \subset A \times C$ defined by the equality

$$\sigma \circ \rho = \{(a,c) \mid (\exists b \in B)\,(a,b) \in \rho \wedge (b,c) \in \sigma\}.$$

It is clear that the composition of relations is an associative operation and

$$\sigma \circ \Big(\bigcup_{i\in I} \rho_i\Big) = \bigcup_{i\in I} (\sigma \circ \rho_i), \qquad \Big(\bigcup_{i\in I} \sigma_i\Big) \circ \rho = \bigcup_{i\in I} (\sigma_i \circ \rho),$$

$$\sigma \circ \Big(\bigcap_{i\in I} \rho_i\Big) \subset \bigcap_{i\in I} (\sigma \circ \rho_i), \qquad \Big(\bigcap_{i\in I} \sigma_i\Big) \circ \rho \subset \bigcap_{i\in I} (\sigma_i \circ \rho),$$

where $\rho_i \subset A \times B$, $\rho \subset A \times B$, $\sigma \subset B \times C$, $\sigma_i \subset B \times C$;

$$\rho_1 \subset \rho_2 \ \wedge\ \sigma_1 \subset \sigma_2 \ \longrightarrow\ \sigma_1 \circ \rho_1 \subset \sigma_2 \circ \rho_2,$$

where $\rho_1, \rho_2 \subset A \times B$, $\sigma_1, \sigma_2 \subset B \times C$;

$$(\sigma \circ \rho)^{-1} = \rho^{-1} \circ \sigma^{-1}, \qquad (\sigma \circ \rho)(H) = \sigma(\rho(H)),$$

where $H \subset A$, $\rho \subset A \times B$, $\sigma \subset B \times C$.

By the *first* and the *second projection* of the relation $\rho \subset A \times B$ we mean the subsets $\mathrm{pr}_1\rho$ and $\mathrm{pr}_2\rho$ of A and B, respectively, defined in the following way:

$$\mathrm{pr}_1\rho = \{a \mid (\exists b \in B)\,(a,b) \in \rho\},$$
$$\mathrm{pr}_2\rho = \{b \mid (\exists a \in A)\,(a,b) \in \rho\}.$$

It is clear that

$$\mathrm{pr}_1\rho = \rho^{-1}(B), \qquad \mathrm{pr}_2\rho = \rho(A),$$
$$\mathrm{pr}_1(\sigma \circ \rho) = \rho^{-1}(\mathrm{pr}_1\sigma) \subset \mathrm{pr}_1\rho,$$
$$\mathrm{pr}_2(\sigma \circ \rho) = \sigma(\mathrm{pr}_2\rho) \subset \mathrm{pr}_2\sigma.$$

Let $\triangle_A$ be the *identical binary relation* on A (in other words: a *diagonal* of the set A), i.e.,

$$\triangle_A = \{(a,a) \mid a \in A\}.$$

A homogeneous binary relation $\rho \subset A \times A$ is called

- *reflexive*, if $\triangle_A \subset \rho$,
- *partially reflexive*, if $\triangle_{\mathrm{pr}_1\rho\, \cup\, \mathrm{pr}_2\rho} \subset \rho$,
- *symmetric*, if $\rho = \rho^{-1}$,
- *transitive*, if $\rho \circ \rho \subset \rho$,
- *antisymmetric*, if $\rho \cap \rho^{-1} \subset \triangle_A$.

A reflexive and transitive binary relation is called a *quasi-order*, an antisymmetric quasi-order is called an *order*. A reflexive and symmetric binary relation is a *quasi-equivalence*, and a symmetric and transitive relation is a *partial equivalence*. A reflexive, symmetric, and transitive binary relation is called an *equivalence relation* or an *equivalence*. Each quasi-order ρ on A induces on A an equivalence $\varepsilon = \rho \cap \rho^{-1}$.

Let $\varepsilon \subset A \times A$ be an equivalence relation, then instead of $(a,b) \in \varepsilon$ we shall often write $a \equiv b\,(\varepsilon)$. For each $a \in A$, the subset $\varepsilon\langle a\rangle$ is called an *ε-class* (an *abstract class* or a *block*) containing a. The set of all ε-classes is denoted by A/ε and is called a *quotient set* or a *factor set* of A with respect to ε. One can show that ε-classes form a *partition* of the set A, i.e.,

(1) $\bigcup\limits_{a \in A} \varepsilon\langle a\rangle = A$,

(2) $\varepsilon\langle a\rangle \cap \varepsilon\langle b\rangle = \varnothing$ for any $a, b \in A$, where $a \not\equiv b$.

Clearly $a \equiv b\,(\varepsilon)$ if and only if $\varepsilon\langle a\rangle = \varepsilon\langle b\rangle$.

An *n-ary relation* (or *n-relation*) between elements of the sets $A_1, A_2, \ldots, A_n$ is a subset ρ of the Cartesian product $A_1 \times A_2 \times \cdots \times A_n$. If $A_1 = A_2 = \cdots = A_n$, then the n-relation ρ is called *homogeneous*. Later on, we shall deal with

$(n+1)$-ary relations, i.e., relations of the form $\rho \subset A_1 \times A_2 \times \cdots \times A_n \times B$. For convenience, we shall consider such relations as binary relations of the form $\rho \subset (A_1 \times A_2 \times \cdots \times A_n) \times B$ (or $\rho \subset \Big(\bigtimes_{i=1}^{n} A_i\Big) \times B$). In the case of homogeneous $(n+1)$-ary relations we shall write $\rho \subset A^n \times A$.

Let $\rho \subset \Big(\bigtimes_{i=1}^{n} A_i\Big) \times B$ be an $(n+1)$-ary relation, $\bar{a} = (a_1, \ldots, a_n)$ an element of $A_1 \times A_2 \times \cdots \times A_n$, $H_i \subset A_i$, $i = 1, \ldots, n$. Then by $\rho\langle a_1, \ldots, a_n\rangle$ and $\rho(H_1, \ldots, H_n)$ we denote the subsets, which are defined in the following way:

$$\rho\langle \bar{a}\rangle = \{b \in B \mid (\bar{a}, b) \in \rho\},$$

$$\rho(H_1, \ldots, H_n) = \bigcup \{\, \rho\langle \bar{a}\rangle \mid \bar{a} \in H_1 \times H_2 \times \cdots \times H_n\}.$$

It is clear that

$$\mathrm{pr}_1 \rho = \{\bar{a} \in \bigtimes_{i=1}^{n} A_i \mid (\exists b \in B)\, (\bar{a}, b) \in \rho\},$$

$$\mathrm{pr}_2 \rho = \{b \in B \mid (\exists \bar{a} \in \bigtimes_{i=1}^{n} A_i)\, (\bar{a}, b) \in \rho\}.$$

With every sequence $\sigma_1, \sigma_2, \ldots, \sigma_n, \rho$ of $(n+1)$-relations such that $\sigma_i \subset A_1 \times \cdots \times A_n \times B_i$, $i = 1, \ldots, n$, and $\rho \subset B_1 \times \cdots \times B_n \times C$, is connected the $(n+1)$-ary relation

$$\rho[\sigma_1 \cdots \sigma_n] \subset A_1 \times \cdots \times A_n \times C,$$

defined as follows:

$$\rho[\sigma_1 \cdots \sigma_n] = \{(\bar{a}, c) \mid (\exists \bar{b}\,)\, (\bar{a}, b_1) \in \sigma_1 \wedge \cdots \wedge (\bar{a}, b_n) \in \sigma_n \wedge (\,\bar{b}, c) \in \rho\},$$

where $\bar{b} = (b_1, \ldots, b_n) \in B_1 \times \cdots \times B_n$. Obviously:

$$\rho[\sigma_1 \cdots \sigma_n](H_1, \ldots, H_n) \subset \rho(\sigma_1(H_1, \ldots, H_n), \ldots, \sigma_n(H_1, \ldots, H_n)\,),$$

$$\rho[\sigma_1 \cdots \sigma_n][\chi_1 \cdots \chi_n] \subset \rho[\sigma_1[\chi_1 \cdots \chi_n] \cdots \sigma_n[\chi_1 \cdots \chi_n]\,],$$

where[1] $\chi_i \subset A_1 \times \cdots \times A_n \times B_i$, $\sigma_i \subset B_1 \times \cdots \times B_n \times C_i$, $i = 1, \ldots, n$ and $\rho \subset C_1 \times \cdots \times C_n \times D$.

The $(n+1)$-operation $O : (\rho, \sigma_1, \ldots, \sigma_n) \mapsto \rho[\sigma_1 \cdots \sigma_n]$ defined as above is called a *Menger superposition* or a *Menger composition* of relations.

Let

$$\overset{n}{\triangle}_A = \{(\underbrace{a, \ldots, a}_{n}) \mid a \in A\}.$$

[1] It is clear that the symbol $\rho[\sigma_1 \cdots \sigma_n][\chi_1 \cdots \chi_n]$ must be read as $\mu[\chi_1 \cdots \chi_n]$, where $\mu = \rho[\sigma_1 \cdots \sigma_n]$.

The homogeneous $(n+1)$-relation $\rho \subset A^{n+1}$ is called

- *reflexive*, if $\overset{n+1}{\triangle}_A \subset \rho$,
- *transitive*, if $\rho[\rho\cdots\rho] \subset \rho$,
- *n-quasi-order*, if it is reflexive and transitive.

Let $\mathfrak{A}_{\mathfrak{H}} = (H^i_b)^{i\in I}_{b\in B}$ be a fixed family of subsets of A uniquely indexed by the elements of the set B. Let ρ be an $(n+1)$-relation on the set A. By $\overset{\wedge}{\rho}_{\mathfrak{H}}$ we denote the relation called $\mathfrak{H}$-*derived from* ρ and defined in the following way:

$$\overset{\wedge}{\rho}_{\mathfrak{H}} \subset \mathfrak{A}^n_{\mathfrak{H}} \times \mathfrak{A}_{\mathfrak{H}},$$

$$\left((H^i_{\bar{b}})^{i\in I}, (H^i_a)^{i\in I}\right) \in \overset{\wedge}{\rho}_{\mathfrak{H}} \longleftrightarrow (\forall i \in I)\ \left(\rho(H^i_{b_1},\ldots,H^i_{b_n}) \subset H^i_a\right),$$

where $(H^i_{\bar{b}})^{i\in I} = ((H^i_{b_1})^{i\in I},\ldots,(H^i_{b_n})^{i\in I})$, $\bar{b} = (b_1,\ldots,b_n)$.

The family of subsets $\mathfrak{H}$ is called *admissible for the* $(n+1)$*-relations* $\sigma, \rho_1, \ldots, \rho_n \subset A^{n+1}$ if

(1) $(\forall i \in I)\,(\sigma[\rho_1\cdots\rho_n](H^i_{\bar{b}}) \subset H^i_a) \longrightarrow \begin{cases} (\exists \bar{c} \in B^n)(\forall i \in I)(\forall k = 1,\ldots,n) \\ \rho_k(H^i_{\bar{b}}) \subset H^i_{c_k} \wedge \sigma(H^i_{\bar{c}}) \subset H^i_a \end{cases}$

(2) $(\forall i \in I)\,(H^i_{b_1} = H^i_{b_2}) \longrightarrow b_1 = b_2$,
where $\bar{c} = (c_1,\ldots,c_n)$.

One can show that for any family $\mathfrak{H}$ which is admissible for the $(n+1)$-relations $\sigma, \rho_1, \ldots, \rho_n$, we have

$$\overset{\wedge}{\overline{\sigma[\rho_1\cdots\rho_n]}}_{\mathfrak{H}} = \overset{\wedge}{\sigma}_{\mathfrak{H}}\,[\overset{\wedge}{\rho}_{1\mathfrak{H}}\cdots\overset{\wedge}{\rho}_{n\mathfrak{H}}],$$

where the relation used on the left side is $\mathfrak{H}$-derived from $\sigma[\rho_1\cdots\rho_n]$.

1.2 Functions and operations

A binary relation $\rho \subset A\times B$ is called *one-valued* (or a *function*) if

$$(\forall a \in A)(\forall b_1, b_2 \in B)\,(\,(a,b_1) \in \rho \wedge (a,b_2) \in \rho \longrightarrow b_1 = b_2).$$

For a function ρ instead of $\rho \subset A\times B$ we write $\rho : A \longrightarrow B$.

The relation ρ is a function if and only if for each $a \in \mathrm{pr}_1\,\rho$ the set $\rho\langle a\rangle$ contains only one element (denoted by $\rho(a)$). A function $f : A \longrightarrow B$, where $\mathrm{pr}_1 f = A$, is called *full* (or a *mapping A into B*).[2] If additionally $\mathrm{pr}_2 = B$,

[2] Otherwise, it is called a *partial mapping*.

then f is called a mapping of A onto B. A binary relation $\rho \subset A \times B$ is called *inversely single-valued* if its inverse relation ρ^{-1} is single-valued. The function $f : A \longrightarrow B$ is called *one-to-one* if f^{-1} is a function, too, i.e.,

$$(\forall a_1, a_2 \in \mathrm{pr}_1 f)\ (\, f(a_1) = f(a_2) \longrightarrow a_1 = a_2\,).$$

For two arbitrary sets A and B by $\mathcal{T}(A, B)$, we denote the family of all mappings from A to B. By $\mathcal{F}(A, B)$ we denote the family of all partial mappings from A to B. The set of all partial mappings A into B, which are one-to-one, is denoted by $\mathcal{R}(A, B)$. It is clear that $\mathcal{R}(A, B) \subset \mathcal{F}(A, B)$ and $\mathcal{T}(A, B) \subset \mathcal{F}(A, B)$.

If $A = B$, then the above sets are denoted by $\mathcal{T}(A)$, $\mathcal{F}(A)$ and $\mathcal{R}(A)$, respectively.[3] It is easy to see that if $f, g \in \mathcal{F}(A)$, then $f \circ g \in \mathcal{F}(A)$. The analogous statement is true for $\mathcal{T}(A)$ and $\mathcal{R}(A)$.

Also it is not difficult to see that for every $f \in \mathcal{F}(A, B)$ and arbitrary sets $Y_1, Y_2 \in \mathfrak{P}(B)$ we have

$$f^{-1}(Y_1 \cup Y_2) = f^{-1}(Y_1) \cup f^{-1}(Y_2),$$

$$f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2),$$

$$f^{-1}(Y_1 \setminus Y_2) = f^{-1}(Y_1) \setminus f^{-1}(Y_2).$$

Every element of the set $\mathcal{F}(\bigtimes_{i=1}^{n} A_i, B)$ is called an *n-ary (partial) function* or a *partial n-operation.*[4] Elements of $\mathcal{T}(\bigtimes_{i=1}^{n} A_i, B)$ are called *full n-ary functions* or *n-operations.*[5] In the case $A = A_1 = \cdots = A_n = B$ these sets are denoted by $\mathcal{F}(A^n, A)$ and $\mathcal{T}(A^n, A)$, respectively. Clearly these sets are closed under Menger composition of n-place functions.

A function $f \in \mathcal{F}(\bigtimes_{i=1}^{n} A_i, B)$ is called *reversive* if for any $i = 1, \ldots, n$ the following condition holds:

$$(\forall\, \bar{a}|_i b_1,\ \bar{a}|_i b_2 \in \mathrm{pr}_1 f)\ (\, f(\bar{a}|_i b_1) = f(\bar{a}|_i b_2) \ \longrightarrow\ b_1 = b_2\,),$$

where $(\bar{a}|_i b)$ denotes $(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)$. The set of all reversive n-place functions is denoted by $\mathcal{R}(\bigtimes_{i=1}^{n} A_i, B)$. A reversive partial n-place operation is

[3] The elements of $\mathcal{T}(A)$ are called *transformations of the set* A, the elements of the other sets are called *partial* and *one-to-one transformations* respectively.

[4] In many papers the term "*n-ary*" is replaced by "*n-place,*" i.e., we have *n-place functions*, *n-place relations*, etc.

[5] In the case $n = 0$, we obtain *nullary operations*, which, in fact, are fixed elements of the set B.

also called a *partial n-ary quasigroup*. The set of all partial n-ary quasigroups is denoted by $\mathcal{R}(A^n, A)$. The Menger composition of the reversive n-ary functions (for $n \geqslant 2$) is not a reversive function in general.

A function $I_i^n \in \mathcal{T}(A^n, A)$ defined by $I_i^n(a_1, \ldots, a_n) = a_i$ for any $a_1, \ldots, a_n \in A$ is called the *n-ary projector* of the set A.

Let $f \in \mathcal{F}(\bigtimes_{i=1}^{n} A_i, B)$ be an arbitrary n-ary function and $H \subset \bigtimes_{i=1}^{n} A_i$. By $f|H$ we denote the *restriction of the function* f to the set H, i.e.,

$$f|H = f \circ \triangle_H.$$

Further, we will consider the restrictions of some functions to the domains of other functions. To study such restrictions, we shall use a binary operation called a *restrictive product of functions* $\rhd$ defined as follows:

$$f \rhd g = g|\operatorname{pr}_1 f = g \circ \triangle_{\operatorname{pr}_1 f}.$$

Now let Φ be an arbitrary subset of n-ary functions defined on the set A. We will consider the following relations on Φ:

- *inclusion*
$$\zeta_\Phi = \{(\varphi, \psi) \in \Phi \times \Phi \mid \varphi \subset \psi\},$$
- *inclusion of domains*
$$\chi_\Phi = \{(\varphi, \psi) \in \Phi \times \Phi \mid \operatorname{pr}_1\varphi \subset \operatorname{pr}_1\psi\},$$
- *equality of domains*
$$\pi_\Phi = \{(\varphi, \psi) \in \Phi \times \Phi \mid \operatorname{pr}_1\varphi = \operatorname{pr}_1\psi\},$$
- *co-definability*
$$\gamma_\Phi = \{(\varphi, \psi) \in \Phi \times \Phi \mid \operatorname{pr}_1\varphi \cap \operatorname{pr}_1\psi \neq \varnothing\},$$
- *connectivity*
$$\kappa_\Phi = \{(\varphi, \psi) \in \Phi \times \Phi \mid \varphi \cap \psi \neq \varnothing\},$$
- *semi-compatibility*
$$\xi_\Phi = \{(\varphi, \psi) \in \Phi \times \Phi \mid \varphi \circ \triangle_{\operatorname{pr}_1\psi} = \psi \circ \triangle_{\operatorname{pr}_1\varphi}\}.$$

An important role in our investigations will play the following subsets of functions

- a *stabilizer of* $a \in A$
$$H_\Phi^a = \{\varphi \in \Phi \mid \varphi(a, \ldots, a) = a\},$$
- a *stationary subset*
$$St(\Phi) = \{\varphi \in \Phi \mid (\exists a \in A)\ \varphi(a, \ldots, a) = a\}.$$

1.3 Algebraic systems

Let $o_\xi \in \mathcal{T}(A^{n_\xi})$ be an arbitrary operation and $\rho_\eta \in \mathfrak{P}(A^{m_\eta})$ an arbitrary relation defined on the set A. Then n_ξ and m_η are called the *arity* of o_ξ and ρ_η, respectively.

By an *algebraic system* we mean a system of the form

$$\mathfrak{A} = (A, o_1, \ldots, o_p, \rho_1, \ldots, \rho_q),$$

where A is a set, $o_1, \ldots, o_p$ are n_i-ary operations on A, $\rho_1, \ldots, \rho_q$ are m_k-ary relation on A. The system $(o_1, \ldots, o_p, \rho_1, \ldots, \rho_q)$ is called the *signature* of $\mathfrak{A}$. The system $(n_1, \ldots, n_p, m_1, \ldots, m_q)$ is called the *type* of $\mathfrak{A}$. Each system of the form $(A, o_1, \ldots, o_p)$ is called an *algebra*. An algebra (G, o), where o is a binary operation, is called a *binary groupoid*. For binary operations will be reserved the symbols: $\cdot$, $+$, $*$, $\wedge$, $\vee$, for relations — ρ, σ, χ, $\leqslant$, $\sqsubset$ and so on. An algebra (G, o), where o is an n-ary operation, is called an *n-ary groupoid*.

If $\mathfrak{A} = (A, o_1, \ldots, o_p, \rho_1, \ldots, \rho_q)$ and $\mathfrak{A}' = (A', o'_1, \ldots, o'_p, \rho'_1, \ldots, \rho'_q)$ are algebraic systems of the same type, then a mapping $P : A \longrightarrow A'$ is called a *homomorphism of* $\mathfrak{A}$ into $\mathfrak{A}'$, if

(1) for all $\xi \in \{1, \ldots, n\}$ and $x_1, \ldots, x_{n_\xi} \in A$ we have

$$P(o_\xi(x_1, \ldots, x_{n_\xi})) = o'_\xi(P(x_1), \ldots, P(x_{n_\xi})),$$

(2) for all $\eta \in \{1, \ldots, m\}$ and $x_1, \ldots, x_{m_\eta} \in A$ we have

$$(x_1, \ldots, x_{m_\eta}) \in \rho_\eta \longleftrightarrow (P(x_1), \ldots, P(x_{m_\eta})) \in \rho'_\eta.$$

If P is an *isomorphism*, i.e., P is a one-to-one homomorphism of A onto A', then $\mathfrak{A}$ and $\mathfrak{A}'$ are *isomorphic*. This fact is denoted by $\mathfrak{A} \cong \mathfrak{A}'$.

By a *semigroup*, we mean a binary groupoid $(G, \cdot)$ with an *associative* operation, i.e., a groupoid $(G, \cdot)$ satisfying the identity

$$(xy)z = x(yz).$$

An element g of a semigroup $(G, \cdot)$ is called *regular* if $gxg = g$ holds for some $x \in G$. A semigroup in which all elements are regular is called *regular*. Two elements $g, \bar{g} \in G$ are *inverse* if

$$g\bar{g}g = g \quad \text{and} \quad \bar{g}g\bar{g} = \bar{g}.$$

An *inverse* semigroup is a regular semigroup in which for every element g there is a uniquely determined element $\bar{g}$ such that g and $\bar{g}$ are inverse. An element $g \in G$ is called *idempotent* if $gg = g$. One can show that a semigroup is inverse if and only if it is regular, and any two its idempotents e and f *commute*, i.e.,

$ef = fe$. By the *identity* (or the *neutral element*) of a groupoid $(G, \cdot)$, we mean an element $e \in G$ such that $eg = ge = g$ for every $g \in G$.

A *group* is a semigroup $(G, \cdot)$ in which the equations $ax = b$ and $ya = b$ are solvable for all $a, b \in G$. These solutions are uniquely determined. A group can be considered as an algebra $(G, \cdot, ^{-1}, e)$ of type $(2, 1, 0)$ satisfying the following three axioms:

$$(xy)z = x(yz),$$

$$xe = ex = x,$$

$$xx^{-1} = x^{-1}x = e.$$

A *semilattice* is a binary groupoid $(G, *)$ in which the following identities hold:

$$(x * y) * z = x * (y * z),$$

$$x * x = x,$$

$$x * y = y * x.$$

A *lattice* is an algebra $(G, \vee, \wedge)$ of type $(2, 2)$, where $\vee$ and $\wedge$ are called, respectively, *join* and *meet* and satisfy for $x, y, z \in G$ the following laws:

- *associativity:*

$$(x \vee y) \vee z = x \vee (y \vee z), \quad (x \wedge y) \wedge z = x \wedge (y \wedge z),$$

- *idempotency:*

$$x \vee x = x, \quad x \wedge x = x,$$

- *commutativity:*

$$x \vee y = y \vee x, \quad x \wedge y = y \wedge x,$$

- *absorption:*

$$x \vee (x \wedge y) = x, \quad x \wedge (x \vee y) = x.$$

Note that a lattice may be also characterized as an algebra $(G, \vee, \wedge)$ of type $(2, 2)$ satisfying identities:

$$x = (y \wedge x) \vee x,$$

$$(((u \wedge v) \wedge w) \vee x) \vee y = (((v \wedge w) \wedge u) \vee y) \vee ((z \vee x) \wedge x).$$

Moreover, in any lattice we have

$$x \vee y = y \longleftrightarrow x \wedge y = x.$$

A lattice $(G, \vee, \wedge)$ in which

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

hold for all $x, y, z \in G$ is called *distributive*.

By a *quasi-ordered set*, we mean the nonempty set A with a quasi-order ω defined on A. If ω is an order, then (A, ω) is called an *ordered set.* An ordered set (A, ω) is *linearly ordered* if $\omega \cup \omega^{-1} = A \times A$. Any two elements of a linearly ordered set are *comparable*, i.e., for any $x, y \in A$ we have $x\omega y$ or $y\omega x$.
The element a of an ordered set $(A, \leqslant)$ is called

- *least*, if $a \leqslant x$ for each $x \in A$,
- *greatest*, if $x \leqslant a$ for each $x \in A$,
- *minimal*, if $x \leqslant a$ for $x \in A$ implies $x = a$,
- *maximal*, if $a \leqslant x$ for $x \in A$ implies $x = a$.

Of course, the least element is minimal, the greatest element is maximal. But an ordered set has at most one least (greatest) element.

If H is a nonempty subset of an ordered set $(A, \leqslant)$, then an element $a \in A$ is called an *upper bound* of H if $h \leqslant a$ for all $h \in H$; and similarly $a \in A$ is called a *lower bound* of H if $a \leqslant h$ for all $h \in H$. The upper bound a of H is called the *least upper bound* of H if for any upper bound b of H we have $a \leqslant b$. Analogously, the lower bound a of H is called the *greatest lower bound* of H if for any lower bound b of H we have $b \leqslant a$. The greatest lower bound of H is denoted by $\bigwedge H$. By $\bigvee H$ we denote the least upper bound of H. An ordered set $(A, \leqslant)$ in which any nonempty subset has the greatest lower bound and the least upper bound is called a *complete lattice.*

1.4 Closure operations

Let $(A, \leqslant)$ be an ordered set. A transformation $f : A \longrightarrow A$ is called

- *extensive*, if $a \leqslant f(a)$ for any $a \in A$,
- *isotone*, if $a_1 \leqslant a_2$ implies $f(a_1) \leqslant f(a_2)$ for any $a_1, a_2 \in A$,
- *idempotent*, if $f \circ f = f$,
- a *semiclosure operation*, if it is extensive and isotone,
- a *closure operation* if it is an idempotent semiclosure operation.

The composition of extensive (isotone) transformations is an extensive (isotone) transformation, too. The analogous statement is true for semiclosure operations. A composition of two closure operations is not, in general, a closure operation.

On the set $\mathcal{T}(A)$ of all transformations of an ordered set $(A, \leqslant)$ we define a binary relation $\prec$ putting

$$f \prec g \longleftrightarrow (\forall a \in A) \, (\, f(a) \leqslant g(a) \,)$$

for all $f, g \in \mathcal{T}(A)$. Obviously, $\prec$ is an order which for any $f, g, h \in \mathcal{T}(A)$ satisfies the implication

$$f \prec g \longrightarrow f \circ h \prec g \circ h.$$

If h is isotone, then also

$$f \prec g \longrightarrow h \circ f \prec h \circ g.$$

Thus, for semiclosure operations f_1, f_2, g_1, g_2 we have

$$f_1 \prec g_1 \wedge f_2 \prec g_2 \longrightarrow f_2 \circ f_1 \prec g_2 \circ g_1.$$

It is easy to show that the following two propositions are true.

Proposition 1.4.1. *Let f be an extensive transformation of an ordered set $(A, \leqslant)$ and $g \in \mathcal{T}(A)$. Then*
(a) *if g is an idempotent transformation, then $f \prec g \longrightarrow g \circ f = g$,*
(b) *if g is a closure operation, then $f \prec g \longleftrightarrow g \circ f = g$,*
(c) *if g is an isotone transformation, then $g \prec g \circ f$ and $g \prec f \circ g$.*

Proposition 1.4.2. *Let f and g be closure operations of an ordered set $(A, \leqslant)$. Then $g \circ f$ is a closure operation if and only if one of the following conditions holds:*
(a) $f \circ g \prec g \circ f$,
(b) $g \circ f = f \circ g \circ f$,
(c) $g \circ f = g \circ f \circ g$.

It follows that for closure operations f and g the compositions $g \circ f$ and $f \circ g$ are simultaneously closure operations if and only if $g \circ f = f \circ g$. If f is an extensive transformation and g is a closure operation, then from $f \prec g$, it follows that $f \circ g$ is a closure operation. In particular, if f, g are closure operations, then

$$f \prec g \longleftrightarrow g \circ f = f \circ g = g.$$

The composition $f_n \circ \cdots \circ f_2 \circ f_1$ of $f_1, f_2, \ldots, f_n \in \mathcal{T}(A)$ will be written in the abbreviated form as $\prod_{i=1}^{n} f_i$.

Theorem 1.4.3. *For closure operations $f_1, f_2, \ldots, f_n$ defined on the same ordered set the following conditions are equivalent:*
(a) $\prod_{i=1}^{n} f_i$ *is a closure operation,*
(b) $\prod_{i=1}^{n} f_i = f_j \circ \Big(\prod_{i=1}^{n} f_i\Big)$ *for any $j = 1, 2, \ldots, n$,*

(c) $\prod_{i=1}^{n} f_i = \Big(\prod_{i=1}^{n-1} f_i\Big) \circ \Big(\prod_{i=1}^{n} f_i\Big)$,

(d) $\Big(\prod_{i=1}^{n-1} f_i\Big) \circ \Big(\prod_{i=2}^{n} f_i\Big) \prec \prod_{i=1}^{n} f_i$,

(e) $\prod_{i=1}^{n} f_i = \Big(\prod_{i=1}^{n} f_i\Big) \circ f_k$ *for any* $k = 1, 2, \ldots, n$.

Proof. Let $\prod_{i=1}^{n} f_i$ be a closure operation, then by Proposition 1.4.1, we have $\prod_{i=1}^{n} f_i \prec f_j \circ \Big(\prod_{i=1}^{n} f_i\Big)$ for all $j = 1, 2, \ldots, n$. On the other hand,

$$\prod_{i=1}^{n} f_i \prec f_1 \circ \Big(\prod_{i=1}^{n} f_i\Big) \prec (f_2 \circ f_1) \circ \Big(\prod_{i=1}^{n} f_i\Big) \prec \cdots \prec \Big(\prod_{i=1}^{j-1} f_i\Big) \circ \Big(\prod_{i=1}^{n} f_i\Big).$$

Hence, $\prod_{i=1}^{n} f_i \prec \Big(\prod_{i=1}^{j-1} f_i\Big) \circ \Big(\prod_{i=1}^{n} f_i\Big)$ and, consequently

$$f_j \circ \Big(\prod_{i=1}^{n} f_i\Big) \prec \Big(\prod_{i=1}^{j} f_i\Big) \circ \Big(\prod_{i=1}^{n} f_i\Big) \prec \Big(\prod_{i=1}^{n} f_i\Big) \circ \Big(\prod_{i=1}^{n} f_i\Big) = \prod_{i=1}^{n} f_i$$

because f_j is isotone. Thus, $\prod_{i=1}^{n} f_i = f_j \circ \Big(\prod_{i=1}^{n} f_i\Big)$. So, (a) implies (b).

The fact that (b) implies (c) is evident.

Now let (c) be true. Then

$$\Big(\prod_{i=1}^{n-1} f_i\Big) \circ \Big(\prod_{i=2}^{n} f_i\Big) \prec \Big(\prod_{i=1}^{n-1} f_i\Big) \circ \Big(\prod_{i=1}^{n} f_i\Big) = \prod_{i=1}^{n} f_i.$$

This proves (d). Since f_1, f_n are isotone, (d) implies

$$\Big(\prod_{i=1}^{n} f_i\Big) \circ \Big(\prod_{i=1}^{n} f_i\Big) = f_n \circ \Big(\prod_{i=1}^{n-1} f_i\Big) \circ \Big(\prod_{i=2}^{n} f_i\Big) \circ f_1 \prec f_n \circ \Big(\prod_{i=1}^{n} f_i\Big) \circ f_1$$
$$= (f_n \circ f_n) \circ \Big(\prod_{i=2}^{n-1} f_i\Big) \circ (f_1 \circ f_1) = \prod_{i=1}^{n} f_i\,.$$

So, $\prod_{i=1}^{n} f_i$ is a closure operation. This means that (d) implies (a).

If (a) is true, then

$$\Big(\prod_{i=1}^{n} f_i\Big) \circ f_k \prec \Big(\prod_{i=1}^{n} f_i\Big) \circ \Big(\prod_{i=1}^{n} f_i\Big) = \prod_{i=1}^{n} f_i,$$

therefore $\left(\prod_{i=1}^{n} f_i\right) \circ f_k \prec \prod_{i=1}^{n} f_i \prec \left(\prod_{i=1}^{n} f_i\right) \circ f_k$, hence, $\prod_{i=1}^{n} f_i = \left(\prod_{i=1}^{n} f_i\right) \circ f_k$ for all $k = 1, 2, \ldots, n$. Thus, (a) implies (e). The converse implication is obvious. □

Let $f_1, f_2, \ldots, f_n$ be closure operations on an ordered set. Theorem 1.4.3 shows that if $\prod_{i=1}^{n} f_i$ and $\prod_{i=1}^{n} f_{\sigma(i)}$, where σ is a permutation of the set $\{1, 2, \ldots, n\}$, are closure operations, then $\prod_{i=1}^{n} f_i = \prod_{i=1}^{n} f_{\sigma(i)}$. Moreover, if

$$\prod_{i=1}^{n} f_i = \Big(\prod_{i=j+1}^{n} f_i\Big) \circ \Big(\prod_{i=1}^{j-1} f_i\Big) \circ f_j,$$

$$f_j \circ \Big(\prod_{i=j+1}^{n} f_i\Big) \circ \Big(\prod_{i=1}^{j-1} f_i\Big) = \Big(\prod_{i=j+1}^{n} f_i\Big) \circ \Big(\prod_{i=1}^{j_1} f_i\Big) \circ f_j$$

is valid for every $j = 1, 2, \ldots, n$, then $\prod_{i=1}^{n} f_i$ is a closure operation. This means that all the products of the form $\prod_{i=1}^{n} f_{\sigma(i)}$ are closure operations if and only if $\prod_{i=1}^{n} f_i = \prod_{i=1}^{n} f_{\sigma(i)}$ for every σ.

A subset H of a complete lattice $(A, \leqslant)$ is called $\bigwedge$-*closed*, if $\bigwedge X \in H$ for every $X \subset H$ (including $X = \varnothing$). By H_f we denote the set of all elements of H which are invariant (fixed) with respect to f, i.e.,

$$H_f = \{a \in H \mid f(a) = a\}.$$

For a closure operation f the set H_f is $\bigwedge$-closed. One can show that if f, g are closure operations, then $f \prec g$ if and only if $H_g \subset H_f$.

Let H be an arbitrary $\bigwedge$-closed subset of a complete lattice $(A, \leqslant)$. By f_H we denote the transformation of the set A defined by the formula

$$f_H(x) = \bigwedge\{h \in H \mid x \leqslant h\}.$$

It is easy to see that f_H is a closure operation and H is the set of all its fixed (invariant) elements. If $f_1, f_2, \ldots, f_n$ are closure operations on a complete lattice, then $H = \bigcap_{i=1}^{n} H_{f_i}$ is a $\bigwedge$-closed subset. In this case we say that the operation f_H is a *closure operation generated by* $f_1, f_2, \ldots, f_n$ and denote it by $\{f_1, f_2, \ldots, f_n\}^c$.

Theorem 1.4.4. *Let $f_1, f_2, \dots, f_n$ be closure operations on a complete lattice $(A, \leqslant)$. If the composition $\prod_{i=1}^{n} f_i$ is a closure operation, then*

$$\{f_1, f_2, \dots, f_n\}^c = \prod_{i=1}^{n} f_i\,.$$

Proof. Let $H = \bigcap_{i=1}^{n} H_{f_i}$. Then $H \subset H_{f_i}$ and $f_i \prec \{f_1, f_2, \dots, f_n\}^c$ for every $1 \leqslant i \leqslant n$. Hence $f \prec \{f_1, f_2, \dots, f_n\}^c$ for $f = \prod_{i=1}^{n} f_i$. But $f_i \prec f$ for every $1 \leqslant i \leqslant n$, thus $H_f \subset H_{f_i}$, and, consequently, $H_f \subset H$. Therefore $\{f_1, f_2, \dots, f_n\}^c \prec f$. Thus, $\{f_1, f_2, \dots, f_n\}^c = f$. □

Corollary 1.4.5. *If $f_1, f_2, \dots, f_n$ are closure operations on a complete lattice such that all compositions $f_i \circ f_j$, where $i, j \in \{1, 2, \dots, n\}$, are closure operations, then*

$$\{f_1, f_2, \dots, f_n\}^c = \prod_{i=1}^{n} f_{\sigma(i)}$$

for every permutation σ of the set $\{1, 2, \dots, n\}$.

Proof. Let σ be an arbitrary permutation of the set $\{1, 2, \dots, n\}$. Since for any i, j the composition $f_{\sigma(i)} \circ f_{\sigma(j)}$ is a closure operation, then according to Proposition 1.4.2, we have

$$f_{\sigma(i)} \circ f_{\sigma(j)} = f_{\sigma(i)} \circ f_{\sigma(j)} \circ f_{\sigma(i)}.$$

Therefore

$$\begin{aligned}
\prod_{i=1}^{n} f_{\sigma(i)} &= \Big(\prod_{i=3}^{n} f_{\sigma(i)}\Big) \circ f_{\sigma(2)} \circ f_{\sigma(1)} \\
&= \Big(\prod_{i=4}^{n} f_{\sigma(i)}\Big) \circ f_{\sigma(3)} \circ f_{\sigma(1)} \circ f_{\sigma(2)} \circ f_{\sigma(1)} \\
&= \Big(\prod_{i=4}^{n} f_{\sigma(i)}\Big) \circ f_{\sigma(1)} \circ f_{\sigma(3)} \circ f_{\sigma(1)} \circ f_{\sigma(2)} \circ f_{\sigma(1)} \\
&= \Big(\prod_{i=4}^{n} f_{\sigma(i)}\Big) \circ f_{\sigma(1)} \circ f_{\sigma(3)} \circ f_{\sigma(2)} \circ f_{\sigma(1)} \\
&= \cdots = f_{\sigma(1)} \circ \Big(\prod_{i=1}^{n} f_{\sigma(i)}\Big) = \cdots = \Big(\prod_{i=1}^{n-1} f_{\sigma(i)}\Big) \circ \Big(\prod_{i=1}^{n} f_{\sigma(i)}\Big).
\end{aligned}$$

By Theorem 1.4.3, $\prod_{i=1}^{n} f_{\sigma(i)}$ is a closure operation. Hence $\prod_{i=1}^{n} f_i = \prod_{i=1}^{n} f_{\sigma(i)}$. Thus, by Theorem 1.4.4, we have $\{f_1, f_2, \dots, f_n\}^c = \prod_{i=1}^{n} f_{\sigma(i)}$. □

Let A be an arbitrary set and $\mathfrak{P}(A)$ be the set of all its subsets ordered by the relation of set-theoretic inclusion.

It is easy to see that $(\mathfrak{P}(A), \subset)$ is a complete lattice. Closure operations on $\mathfrak{P}(A)$ are called *closure operations of subsets of* A. The system (family) $\mathcal{C}$ of subsets of A (i.e., $\mathcal{C} \subset \mathfrak{P}(A)$) is called a *closure system* if it is closed under intersections, i.e.,

$$\bigcap \mathcal{D} \in \mathcal{C} \quad \text{for any subsystem } \mathcal{D} \subset \mathcal{C}.$$

In particular, taking $\mathcal{D} = \varnothing$, we see that A always belongs to $\mathcal{C}$. Since a closure system admits arbitrary intersections, it follows that it is a complete lattice with respect to the inclusion. However, it needs not to be a sublattice of $(\mathfrak{P}(A), \subset)$, since the cup operation in $\mathcal{C}$ is in general different from that of $\mathfrak{P}(A)$.

Each closure system $\mathcal{C}$ of $\mathfrak{P}(A)$ induces on $\mathfrak{P}(A)$ the closure operation $f_{\mathcal{C}} : \mathfrak{P}(A) \longrightarrow \mathfrak{P}(A)$ defined by

$$f_{\mathcal{C}}(X) = \bigcap \{Y \in \mathcal{C} \mid Y \supset X\}.$$

A subset $X \in \mathfrak{P}(A)$ belongs to the closure system $\mathcal{C}$ if and only if $f_{\mathcal{C}}(X) = X$. Such X is called $f_{\mathcal{C}}$*-closed*. The set $f_{\mathcal{C}}(X)$ is called the $f_{\mathcal{C}}$*-closure of* X. It is the least $f_{\mathcal{C}}$-closed subset of A containing X.

One can show that the following theorem is valid.

Theorem 1.4.6. *For every transformation f of $\mathfrak{P}(A)$ the following conditions are equivalent:*
(1) *f is a closure operation of A,*
(2) *$f(f(X)) \subset f(X)$ and $f(X) \cup Y \subset f(X \cup Y)$ for all $X, Y \in \mathfrak{P}(A)$*
(3) *$X \subset f(X)$ and $f(f(X) \cup Y) \subset f(X \cup Y)$ for all $X, Y \in \mathfrak{P}(A)$.*

A transformation $f : \mathfrak{P}(A) \longrightarrow \mathfrak{P}(A)$ is called a $\cup$*-transformation*, if $f(X \cup Y) = f(X) \cup f(Y)$ for any $X, Y \subset A$, and a $\cap$*-transformation*, if $f(X \cap Y) = f(X) \cap f(Y)$ for any $X, Y \subset A$. Closure operations which are ($\cap$-) $\cup$-transformations are called ($\cap$-) $\cup$*-closure operations*.

Proposition 1.4.7. *If f is a transformation of $\mathfrak{P}(A)$ such that $f(\{a\}) = \{a\}$ for every $a \in A$, then the following conditions are equivalent:*
(a) *f is a $\cup$-closure operation,*
(b) *$f(X \cup f(Y)) = f(X) \cup f(Y)$ for any $X, Y \in \mathfrak{P}(A)$.*

Proof. Let f be a $\cup$-closure operation on subsets of A. Since for $X, Y \in \mathfrak{P}(A)$ we have $f(X) \subset f(X \cup f(Y))$ and $f(Y) \subset f(X \cup f(Y))$, then

$$f(X) \cup f(Y) \subset f(X \cup f(Y)) = f(X) \cup f(f(Y)) = f(X) \cup f(Y).$$

Hence $f(X \cup f(Y)) = f(X) \cup f(Y)$. So, (a) implies (b).

Now assume that (b) is true and $X \subset A$. If $a \in X$, then $f(X \cup f(\{a\})) = f(X) \cup f(\{a\})$. But $f(\{a\}) = \{a\}$ for every $a \in A$ implies $f(X \cup \{a\}) = f(X) \cup \{a\}$, i.e., $f(X) = f(X) \cup \{a\}$. Thus $a \in f(X)$ and, consequently, $X \subset f(X)$. Putting $X = Y$ and using the extensiveness of f we obtain $f(f(X)) = f(X)$. Now let $X \subset Y$, then $X \cup f(Y) = f(Y)$, it follows that $f(X \cup f(Y)) = f(Y)$, consequently, $f(X) \subset f(Y)$. So, f is a closure operation on subsets. By Theorem 1.4.6, $X \cup f(Y) \subset f(X \cup Y)$, which gives $f(X \cup f(Y)) \subset f(X \cup Y)$. On the other hand, $f(X \cup Y) \subset f(X \cup f(Y))$, therefore $f(X \cup Y) = f(X \cup f(Y))$. So, $f(X \cup Y) = f(X) \cup f(f(Y)) = f(X) \cup f(Y)$. This proves that f is a $\cup$-closure operation, i.e., (b) implies (a). The proof is complete. □

Proposition 1.4.8. *A transformation $f : \mathfrak{P}(A) \longrightarrow \mathfrak{P}(A)$ is a $\cap$-closure operation if and only if $f(X) = f(\varnothing) \cup X$ for every subset $X \subset A$.*

Proof. Since f is a $\cap$-closure operation, then for every $X \subset A$ we have $f(X) \cap f(X') = f(X \cap X') = f(\varnothing)$. Consequently, $(f(X) \cap f(X')) \cup X = f(\varnothing) \cup X$, which implies $(f(X) \cup X) \cap (f(X') \cup X) = f(\varnothing) \cup X$, i.e., $f(X) \cap A = f(\varnothing) \cup X$. Thus, $f(X) = f(\varnothing) \cup X$ for all $X \in \mathfrak{P}(A)$.

The converse implication is obvious. □

Corollary 1.4.9. *Every $\cap$-closure operation on subsets is a $\cup$-closure operation.*

Let $X, H \in \mathfrak{P}(A)$. By f_H, we denote a $\cap$-closure operation on subsets of A, which is defined by the equality $f_H(X) = H \cup X$. It follows from Proposition 1.4.8 that a composition of two $\cap$-closure operations f_{H_1}, f_{H_2} is a $\cap$-closure operation and $f_{H_1} \circ f_{H_2} = f_{H_1 \cup H_2}$. Hence, the set of all $\cap$-closure operations on subsets forms an idempotent commutative semigroup.

Theorem 1.4.10. *Every commutative idempotent semigroup is isomorphic to a semigroup of the $\cap$-closure operations of subsets of some set.*

Proof. Let $(G, \cdot)$ be a commutative idempotent semigroup. We consider a binary relation $\omega \subset G \times G$ defined by the condition

$$(g_1, g_2) \in \omega \longleftrightarrow g_1 g_2 = g_2.$$

It is easy to see that ω is an order such that

$$\omega\langle g_1 g_2 \rangle = \omega\langle g_1 \rangle \cap \omega\langle g_2 \rangle.$$

To every element $g \in G$ we define a $\cap$-closure operation f_g on the subsets of G by putting

$$f_g(X) = (\omega\langle g \rangle)' \cup X.$$

Then, for every $X \in \mathfrak{P}(G)$ we have:

$$\begin{aligned} f_{g_1 g_2}(X) &= (\omega\langle g_1 g_2\rangle)' \cup X = (\omega\langle g_1\rangle \cap \omega\langle g_2\rangle)' \cup X \\ &= (\omega\langle g_1\rangle)' \cup (\omega\langle g_2\rangle)' \cup X = (\omega\langle g_2\rangle)' \cup ((\omega\langle g_1\rangle)' \cup X) \\ &= (\omega\langle g_2\rangle)' \cup f_{g_1}(X) = f_{g_2}(f_{g_1}(X)) = f_{g_2} \circ f_{g_1}(X). \end{aligned}$$

So, by the commutativity of $(G, \cdot)$, we obtain $f_{g_2 g_1} = f_{g_2} \circ f_{g_1}$.

If $f_{g_1} = f_{g_2}$, then $f_{g_1}(\varnothing) = f_{g_2}(\varnothing)$, therefore $(\omega\langle g_1\rangle)' \cup \varnothing = (\omega\langle g_2\rangle)' \cup \varnothing$, which proves $\omega\langle g_1\rangle = \omega\langle g_2\rangle$. Hence $g_1 = g_2$. Thus, the mapping $g \mapsto f_g$ is an isomorphism. □

Corollary 1.4.11. *Every commutative idempotent semigroup is isomorphic to a semigroup of* $\cup$*-closure operations on the subsets of some set.*

Corollary 1.4.12. *A semigroup* $(G, \cdot)$ *is isomorphic to a semigroup of closure operations on an ordered set if and only if it is a semilattice.*

Now we shall consider some closure operations on the set $\mathfrak{P}(A \times A)$ of all binary relations between elements of A. It is easy to verify that the set of all reflexive binary relations forms a closure system, which by

$$\mathfrak{f}_R(\rho) = \rho \cup \triangle_A,$$

where $\rho \subset A \times A$, defines the operation $\mathfrak{f}_R$ called the *reflexive closure*. The set of all transitive binary relations between the elements of A also forms a closure system. But this system defines the so-called *transitive closure* $\mathfrak{f}_t$ such that

$$\mathfrak{f}_t(\rho) = \bigcup_{n=1}^{\infty} \rho^n,$$

where $\rho^n = \underbrace{\rho \circ \rho \circ \cdots \rho}_{n}$. The set of all quasi-orders between the elements of A defines the closure operation $\mathfrak{f}_Q$, which is called the *quasi-order closure*. Since reflexive closure operations and transitive closure operations commute, i.e., $\mathfrak{f}_t \circ \mathfrak{f}_R = \mathfrak{f}_R \circ \mathfrak{f}_t$, their composition is a closure operation, which coincides with the quasi-order closure. Thus,

$$\mathfrak{f}_Q(\rho) = \Big(\bigcup_{n=1}^{\infty} \rho^n\Big) \cup \triangle_A.$$

The set of all symmetric binary relations defines the *symmetric closure* $\mathfrak{f}_s$, which is determined by the equality

$$\mathfrak{f}_s(\rho) = \rho \cup \rho^{-1}.$$

Since $\mathfrak{f}_s \circ \mathfrak{f}_t \circ \mathfrak{f}_s = \mathfrak{f}_t \circ \mathfrak{f}_s$, the composition $\mathfrak{f}_t \circ \mathfrak{f}_s$ is a closure operation. It coincides with the *partially equivalent closure* $\mathfrak{f}_e$ expressed by the formula

$$\mathfrak{f}_e(\rho) = \bigcup_{n=1}^{\infty} \left(\rho \cup \rho^{-1}\right)^n .$$

Similarly, $\mathfrak{f}_e \circ \mathfrak{f}_R = \mathfrak{f}_R \circ \mathfrak{f}_e$ implies that $\mathfrak{f}_e \circ \mathfrak{f}_R$ is a closure operation. It coincides with the *equivalent closure* $\mathfrak{f}_E$ defined by

$$\mathfrak{f}_E(\rho) = \left(\bigcup_{n=1}^{\infty} \left(\rho \cup \rho^{-1}\right)^n \right) \cup \triangle_A.$$

Let $(A, \leqslant)$ be an ordered set. We say that an n-place transformation $f \in \mathcal{T}(A^n, A)$ is

- *extensive*, if $a_i \leqslant f(\bar{a})$ for all $\bar{a} \in A^n$, $i = 1, \ldots, n$,
- *idempotent*, if $f[f^n] = f$,
- *isotone*, if for all $a_1, a_2 \in A$, $\bar{u} \in A^n$, $i = 1, \ldots, n$

$$a_1 \leqslant a_2 \longrightarrow f(\bar{u}\,|_i\, a_1) \leqslant f(\bar{u}\,|_i\, a_2),$$

- *an n-ary semiclosure operation*, if it is extensive and isotone simultaneously,
- *an n-ary closure operation*, if it is an idempotent semiclosure operation.

On the set $\mathcal{T}(A^n, A)$ we define a binary order $\prec$ putting

$$f \prec g \longleftrightarrow (\forall \bar{a})\,(f(\bar{a}) \leqslant g(\bar{a})).$$

Proposition 1.4.13. *Let $f, g, g_i, u_i \in \mathcal{T}(A^n, A)$, where $i = 1, \ldots, n$, then the following statements are valid:*

(a) *if f is extensive, then $g_i \prec f[g_1 \cdots g_n]$ for all $g_1, \ldots, g_n \in \mathcal{T}(A^n, A)$, $i = 1, \ldots, n$,*

(b) *if f is extensive and g is isotone, then $g \prec g[f^n]$,*

(c) *if f is isotone, then $g_1 \prec g_2 \longrightarrow f[\bar{u}\,|_i\, g_1] \prec f[\bar{u}\,|_i\, g_2]$ for all $g_1, g_2 \in \mathcal{T}(A^n, A)$ and any $\bar{u} \in (\mathcal{T}(A^n, A))^n$, $i = 1, \ldots, n$,*

(d) *if f is an n-ary semiclosure operation, and g is an n-ary closure operation, then $f \prec g \longleftrightarrow g[f^n] = g$.*

Proof. (a) Let f be an extensive n-ary transformation. Then for all $\bar{a} \in A^n$ and $1 \leqslant i \leqslant n$ the condition $g_i(\bar{a}) \leqslant f(g_1(\bar{a}), \ldots, g_n(\bar{a})) = f[g_1 \cdots g_n](\bar{a})$ is satisfied. Hence, $g_i \prec f[g_1 \cdots g_n] = f[\bar{g}]$.

(b) It is obvious.

(c) If $g_1 \prec g_2$, then $g_1(\bar{a}) \leqslant g_2(\bar{a})$ for any $\bar{a} \in A^n$. Therefore by the isotonicity of f we have $f(\bar{u}(\bar{a})|_i g_1(\bar{a})) \leqslant f(\bar{u}(\bar{a})|_i g_2(\bar{a}))$, where $\bar{u}(\bar{a}) = (u_1(\bar{a}), \ldots, \bar{u}_n(\bar{a}))$. Therefore we have $f[\bar{u}|_i g_1](\bar{a}) \leqslant f[\bar{u}|_i g_2](\bar{a})$, hence $f[\bar{u}|_i g_1] \prec f[\bar{u}|_i g_2]$.

(d) Let $f \prec g$. Then $f(\bar{a}) \leqslant g(\bar{a})$ for any $\bar{a} \in A^n$. Hence $g[f^n](\bar{a}) \leqslant g(\bar{a})$. Since f is extensive, $a_i \leqslant f(\bar{a})$ for all $i = 1, \dots, n$. This implies $g(\bar{a}) \leqslant g[f^n](\bar{a})$. So, $g(\bar{a}) = g[f^n](\bar{a})$ for all $\bar{a} \in A^n$, i.e., $g = g[f^n]$. Thus, we have shown that $f \prec g$ implies $g[f^n] = g$.

The converse implication is true by (a). □

Proposition 1.4.14. *If $f, g_1, \dots, g_n$ are extensive (isotone) n-ary transformations of an ordered set, then the n-ary transformation $f[g_1 \cdots g_n]$ is also extensive (isotone).*

Proof. Indeed, if $f, g_1, \cdots, g_n$ are extensive n-ary transformations of an ordered set $(A, \leqslant)$, then $a_i \leqslant g_k(\bar{a}) \leqslant f(g_1(\bar{a}), \cdots, g_n(\bar{a})) = f[g_1 \cdots g_n](\bar{a})$ for all $i, k = 1, \dots, n$, $\bar{a} \in A^n$. So, $f[g_1 \cdots g_n]$ is an extensive transformation.

Now, let $f, g_1, \dots, g_n$ be isotone n-ary transformations. First, we show that $a_1 \leqslant b_1$, …, $a_n \leqslant b_n$ imply $f(a_1, \dots, a_n) \leqslant f(b_1, \dots, b_n)$. From the equality $a_1 \leqslant b_1$ by the isotonicity of f, we obtain $f(a_1, \dots, a_n) \leqslant f(b_1, a_2, \dots, a_n)$. In a similar way from $a_2 \leqslant b_2$, we obtain $f(b_1, \dots, a_n) \leqslant f(b_1, b_2, a_3, \dots, a_n)$ and so on. Finally $a_n \leqslant b_n$ implies $f(b_1, \dots, b_{n-1}, a_n) \leqslant f(b_1, \dots, b_{n-1}, b_n)$. So we have $f(a_1, \dots, a_n) \leqslant f(b_1, \dots, b_n)$.

Now if $c_1 \leqslant c_2$, then $g_k(\bar{u}|_i c_1) \leqslant g_k(\bar{u}|_i c_2)$ for all $i, k = 1, \dots, n$ and $\bar{u} \in A^n$. Hence $f(g_1(\bar{u}|_i c_1), \dots, g_n(\bar{u}|_i c_1)) \leqslant f(g_1(\bar{u}|_i c_2), \dots, g_n(\bar{u}|_i c_2))$, i.e., $f[g_1 ... g_n](\bar{u}|_i c_1) \leqslant f[g_1 ... g_n](\bar{u}|_i c_2)$. So, $f[g_1 ... g_n]$ is an isotone n-ary transformation. □

Proposition 1.4.15. *If $f, g_1, \dots, g_n$ are n-ary closure operations on an ordered set, then $f[\bar{g}]$ is an n-ary closure operation if and only if*

$$g_i[f^n][\bar{g}] = f[\bar{g}]$$

for every $i = 1, \dots, n$.

Proof. The extensivity and isotonicity of $f[\bar{g}]$ can be deduced from Proposition 1.4.14, hence we have to show that $f[\bar{g}]$ is idempotent. Suppose that $f[\bar{g}]$ is idempotent, then

$$f[\bar{g}] \prec g_i[f[\bar{g}] \cdots f[\bar{g}]] = g_i[f^n][\bar{g}],$$

by Proposition 1.4.13.

On the other hand,

$$g_i[f^n][\bar{g}] \prec f[\, g_1[f^n][\bar{g}] \cdots g_n[f^n][\bar{g}]\,] = f[\bar{g}][\, f[\bar{g}] \cdots f[\bar{g}]\,] = f[\bar{g}].$$

So, the equality given in Proposition 1.4.15 holds.

Conversely, if this equality is true, then

$$\begin{aligned} f[\bar{g}][f[\bar{g}] \cdots f[\bar{g}]] &= f[\, g_1[f^n][\bar{g}] \cdots g_n[f^n][\bar{g}]\,] \\ &= f[\, f[\bar{g}] \cdots f[\bar{g}]\,] = f[f^n][\bar{g}] = f[\bar{g}], \end{aligned}$$

which completes the proof. □

1.5 Notes on Chapter 1

Except for the properties described in this chapter, one can find many useful theorems on closure operations in the books [15] and [107]. A more detailed information about closure operations and their applications to the theory of binary relations and partial transformations can be found in the series of papers written by V. V. Vagner, in particular in [253].

Basic properties of semigroups can be found in [14] and [105]. Regular elements of the semigroup of all binary relations are studied in [184].

Corollary 1.4.5 was first proved by T. Tamura in [209], Corollary 1.4.12 by V. T. Kulik [83]. An alternative proof of the last three propositions of this chapter can be found in [237] where Menger algebras of multiplace transformations of ordered sets are investigated.

Chapter 2

Menger algebras of functions

2.1 Definitions and fundamental notions

An $(n+1)$-ary groupoid (G, o), i.e., a nonempty set G with one $(n+1)$-ary operation $o : G^{n+1} \to G$, is called a *Menger algebra of rank* n if it satisfies the following identity (called the *superassociativity*):

$$o(o(x, y_1, \ldots, y_n), z_1, \ldots, z_n) = o(x, o(y_1, z_1, \ldots, z_n), \ldots, o(y_n, z_1, \ldots, z_n)).$$

A Menger algebra of rank 1 is a semigroup.

Since a Menger algebra (as we see later) can be interpreted as an algebra of n-place functions with a Menger composition of such functions, we replace the symbol $o(x, y_1, \ldots, y_n)$ by $x[y_1 \cdots y_n]$ or by $x[\bar{y}]$, i.e., these two symbols will be interpreted as the result of the operation o applied to the elements $x, y_1, \ldots, y_n \in G$.

In this convention, the above superassociativity has the form

$$x[y_1 \cdots y_n][z_1 \cdots z_n] = x[y_1[z_1 \cdots z_n] \cdots y_n[z_1 \cdots z_n]]$$

or shortly

$$x[\bar{y}][\bar{z}] = x[y_1[\bar{z}] \cdots y_n[\bar{z}]],$$

where the left side is read as $\big(x[y_1 \cdots y_n]\big)\,[z_1 \cdots z_n]$.

If we define on a Menger algebra (G, o) the new binary operation

$$x \cdot y = x[y \cdots y],$$

we obtain the so-called *diagonal semigroup* $(G, \cdot)$. An element $g \in G$ is called *idempotent*, if it is idempotent in the diagonal semigroup of (G, o), i.e., if $g[g^n] = g$. An element $e \in G$ is called a *left* (*right*) *diagonal unit* of a Menger algebra (G, o), if it is a left (right) unit of the diagonal semigroup of (G, o), i.e., if the identity $e[g^n] = g$ (respectively, $g[e^n] = g$) holds for all $g \in G$. If e is both a left and a right unit, then it is called a *diagonal unit*. It is clear that a Menger algebra has at most one diagonal unit. Moreover, if a Menger algebra has an element that is a left diagonal unit and an element which is a right diagonal unit, then these elements are equal and no other elements which are left or right diagonal units.

An $(n+1)$-ary groupoid (G, o) with the operation $o(x_0, \ldots, x_n) = x_0$ is a simple example of a Menger algebra of rank n in which all elements are idempotent and right diagonal units. Of course, this algebra has no left units. In the Menger algebra (G, o_n), where $o_n(x_0, \ldots, x_n) = x_n$, all elements are left diagonal units, but this algebra has no right diagonal units. The set $\mathbb{N}_+$ of all nonzero natural numbers with the operation $o(x_0, \ldots, x_n) = x_0 \cdot gcd(x_1, \ldots, x_n)$, where $gcd(\cdots)$ means the greatest common divisor, is an example of a Menger algebra that the diagonal unit. Its diagonal semigroup is commutative, but it is not a group. On the other hand, as it is easy to see, the set $\mathbb{R}_+$ of all positive real numbers with the operation $o(x_0, \ldots, x_n) = x_0 \cdot \sqrt[n]{x_1 x_2 \cdots x_n}$ is a Menger algebra for which its diagonal semigroup is a commutative group.

Proposition 2.1.1. *If a Menger algebra (G, o) has a right diagonal unit e, then every element $c \in G$ satisfying the identity $e = e[c^n]$ is also a right diagonal unit.*

Proof. Indeed, for every $b \in G$ we have

$$b = b[e^n] = b[e[c^n] \cdots e[c^n]] = b[e^n][c^n] = b[c^n],$$

which means that also c is a right diagonal unit. □

It is not difficult to see that the following two lemmas are true.

Lemma 2.1.2. *If a Menger algebra (G, o) has a left diagonal unit e and for every $b \in G$ there exists an element $b' \in G$ such that $e = b'[b^n]$, then the diagonal semigroup of (G, o) is a left cancellative, i.e.,*

$$b[x^n] = b[y^n] \longrightarrow x = y$$

for all $b, x, y \in G$.

Lemma 2.1.3. *If $(G, \cdot)$ is the diagonal semigroup of a Menger algebra (G, o) of rank n, then*

$$(x \cdot y)[\bar{z}] = x \cdot y[\bar{z}]$$

and

$$x[z_1 \cdots z_n] \cdot y = x[(z_1 \cdot y) \cdots (z_n \cdot y)]$$

for all $x, y, z_1, \ldots, z_n \in G$.

Proposition 2.1.4. *If on a semigroup $(G, \cdot)$ one can define an n-ary operation f such that*

(a) $f(g, \ldots, g) = g$ *for all $g \in G$,*

(b) $f(g_1, g_2, \ldots, g_n) \cdot g = f(g_1 \cdot g, g_2 \cdot g, \ldots, g_n \cdot g)$ *for all $g, g_1, \ldots, g_n \in G$, then $(G, \cdot)$ is a diagonal semigroup of some Menger algebra of rank n.*

Proof. If all of the above conditions are satisfied, then on G we can define the $(n+1)$-ary operation

$$o : (x, y_1, \ldots, y_n) \mapsto x[y_1 \cdots y_n],$$

putting $x[y_1 \cdots y_n] = x \cdot f(y_1, \ldots, y_n)$ for $x, y_1, \ldots, y_n \in G$. The operation defined in such a way is superassociative. Indeed, for $x, y_1, \ldots, y_n, z_1, \ldots, z_n \in G$ we have

$$\begin{aligned} x[y_1 \cdots y_n][z_1 \cdots z_n] &= (x \cdot f(y_1, \ldots, y_n)) \cdot f(z_1, \ldots, z_n) \\ &= x \cdot (f(y_1, \ldots, y_n) \cdot f(z_1, \ldots, z_n)) \\ &= x \cdot f(y_1 \cdot f(z_1, \ldots, z_n), y_2 \cdot f(z_1, \ldots, z_n), \ldots, y_n \cdot f(z_1, \ldots, z_n)) \\ &= x \cdot f(y_1[z_1 \cdots z_n], y_2[z_1 \cdots z_n], \ldots, y_n[z_1 \cdots z_n]) \\ &= x[y_1[z_1 \cdots z_n]\, y_2[z_1 \cdots z_n] \cdots y_n[z_1 \cdots z_n]\,], \end{aligned}$$

which means that (G, o) is a Menger algebra of rank n.

If $(G, *)$ is its diagonal semigroup, then for any $x, y \in G$ we have $x * y = x[y \cdots y] = x \cdot f(y, \ldots, y) = x \cdot y$. Hence $(G, *) = (G, \cdot)$, i.e., $(G, \cdot)$ is a diagonal semigroup of (G, o). This completes the proof. □

Corollary 2.1.5. *A semigroup $(G, \cdot)$ with a left unit is a diagonal semigroup of some Menger algebra with a left diagonal unit if and only if we can define on G an n-ary operation satisfying the conditions* (a) *and* (b) *of the above proposition.*

Proof. If $(G, \cdot)$ is a diagonal semigroup of a Menger algebra (G, o) of rank n with a left diagonal unit e, then an n-ary operation

$$f : (x_1, \ldots, x_n) \to e[x_1 \cdots x_n]$$

satisfies the conditions (a) and (b). Indeed, for all $x, x_1, \ldots, x_n, y \in G$, we have $f(x, \ldots, x) = e[x^n] = x$ and

$$\begin{aligned} f(x_1, \ldots, x_n) \cdot y = e[x_1 \cdots x_n] \cdot y &= e[x_1 \cdots x_n][y^n] \\ &= e[x_1[y^n] \cdots x_n[y^n]] = e[(x_1 \cdot y) \cdots (x_n \cdot y)] \\ &= f(x_1 \cdot y, \ldots, x_n \cdot y). \end{aligned}$$

On the other hand, if on a semigroup $(G, \cdot)$ with a left unit there is an n-ary operation f satisfying the conditions (a) and (b), then, according to the Proposition 2.1.4, this semigroup is a diagonal semigroup of a Menger algebra with the operation $x[y_1 \cdots y_n] = x \cdot f(y_1, \ldots, y_n)$. A left unit e of this semigroup is a left diagonal unit of the Menger algebra. Indeed, for any $x \in G$ we have

$$e[x^n] = e \cdot f(x, \ldots, x) = f(x, \ldots, x) = x,$$

which completes the proof. □

Note that non-isomorphic Menger algebras may have the same diagonal semigroup. For this, consider two $(n+1)$-ary operations $o_1(x_0, x_1, \ldots, x_n) = x_0 + x_1$ and $o_2(x_0, x_1, \ldots, x_n) = x_0 + x_2$ defined on a nontrivial commutative semigroup $(G, +)$. Then, as it is easy to see, (G, o_1) and (G, o_2) are an example of non-isomorphic Menger algebras of rank n with the same diagonal semigroup $(G, +)$.

Definition 2.1.6. A Menger algebra (G, o) of the rank n is called *unitary*, if there exist elements $e_1, \ldots, e_n \in G$, called *selectors*, such that

$$x[e_1 \cdots e_n] = x \qquad \text{and} \qquad e_i[x_1 \cdots x_n] = x_i$$

for all $x, x_1, \ldots, x_n \in G$, $i = 1, \ldots, n$.

The set $\mathcal{F}(A^n, A)$ of all n-place functions (also partial) defined on A is closed with respect to the Menger composition of n-place functions. So, it forms a Menger algebra $(\mathcal{F}(A^n, A), O)$ of rank n, which will be called a *Menger algebra of all n-place functions*. Each its subalgebra will be called a *Menger algebra of n-place functions*. The symbol $(\mathcal{T}(A^n, A), O)$ will be reserved for a Menger algebra of *all full n-place functions*, i.e., all n-place mappings of A^n into A. Each subalgebra of this algebra will be called a *Menger algebra of full n-place functions*.

Theorem 2.1.7. *Every unitary Menger algebra (G, o) of rank n is isomorphic to some Menger algebra of full n-place functions on some set in this way that its selectors correspond to the n-place projectors of this set.*

Proof. Let (G, o) be a fixed unitary Menger algebra of rank n with selectors $e_1, \ldots, e_n$. For every element $g \in G$, we define the full n-place function $\lambda_g \in \mathcal{T}(G^n, G)$ putting

$$\lambda_g(x_1, \ldots, x_n) = g[x_1 \cdots x_n]$$

for all $x_1, \ldots, x_n \in G$.

We shall show, that algebras (Λ, O) and (G, o), where $\Lambda = \{\lambda_g \mid g \in G\}$, are isomorphic. Indeed, for all $g, g_1, \ldots, g_n, x_1, \ldots, x_n \in G$ we have

$$\begin{aligned}
\lambda_{g[g_1 \cdots g_n]}(x_1, \ldots, x_n) &= g[g_1 \cdots g_n][x_1 \cdots x_n] \\
&= g[g_1[x_1 \cdots x_n] \cdots g_n[x_1 \cdots x_n]] \\
&= g[\lambda_{g_1}(x_1, \ldots, x_n) \cdots \lambda_{g_n}(x_1, \ldots, x_n)] \\
&= \lambda_g(\lambda_{g_1}(x_1, \ldots, x_n), \ldots, \lambda_{g_n}(x_1, \ldots, x_n)) \\
&= \lambda_g[\lambda_{g_1} \cdots \lambda_{g_n}](x_1, \ldots, x_n),
\end{aligned}$$

which proves that

$$\lambda_{g[g_1\cdots g_n]} = \lambda_g[\lambda_{g_1}\cdots\lambda_{g_n}]$$

for all $g, g_1, \ldots, g_n \in G$.

Moreover, if $\lambda_{g_1} = \lambda_{g_2}$ for some $g_1, g_2 \in G$, then also $\lambda_{g_1}(x_1, \ldots, x_n) = \lambda_{g_2}(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in G$ and, consequently, $\lambda_{g_1}(e_1, \ldots, e_n) = \lambda_{g_2}(e_1, \ldots, e_n)$. This implies $g_1[e_1\cdots e_n] = g_2[e_1\cdots e_n]$ and $g_1 = g_2$. So, we showed, that the mapping $P : g \mapsto \lambda_g$ of the algebra (G, o) onto the algebra (Λ, O) is an isomorphism.

At last, let e_i be the ith selector of (G, o), then

$$\lambda_{e_i}(x_1, \ldots, x_n) = e_i[x_1\cdots x_n] = x_i$$

for any $x_1, \ldots, x_n \in G$. Therefore $\lambda_{e_i} = I_i^n$. This means that selectors are transformed into projectors. □

The following theorem was first proved by R. Dicker [20].

Theorem 2.1.8. *Any Menger algebra of rank n is isomorphic to some Menger algebra of full n-place functions.*

Proof. Let (G, o) be a Menger algebra[1] of rank n and e, c be two different elements not belonging to G. Let $G' = G \cup \{e, c\}$. For every element $g \in G$ we shall define the full n-place function λ'_g on G' by putting

$$\lambda'_g(x_1, \ldots, x_n) = \begin{cases} g[x_1\cdots x_n] & \text{if } x_1, \ldots, x_n \in G, \\ g & \text{if } x_1 = \cdots = x_n = e, \\ c & \text{in all other cases.} \end{cases}$$

We shall prove that

$$\lambda'_{g[g_1\cdots g_n]} = \lambda'_g[\lambda'_{g_1}\cdots\lambda'_{g_n}] \tag{2.1.1}$$

for any $g, g_1, \ldots, g_n \in G$.

Let $x_1, \ldots, x_n \in G$. Then, as it was done in the proof of the previous theorem, we can show that

$$\lambda'_{g[g_1\cdots g_n]}(x_1, \ldots, x_n) = \lambda'_g[\lambda'_{g_1}\cdots\lambda'_{g_n}](x_1, \ldots, x_n).$$

Now let $x_1 = \cdots = x_n = e$, then according to the definition, we have

$$\lambda'_{g[g_1\cdots g_n]}(e, \ldots, e) = g[g_1\cdots g_n]$$

and

$$\begin{aligned} \lambda'_g[\lambda'_{g_1}\cdots\lambda'_{g_n}](e, \ldots, e) &= \lambda'_g(\lambda'_{g_1}(e, \ldots, e), \ldots, \lambda'_{g_n}(e, \ldots, e)) \\ &= \lambda'_g(g_1, \ldots, g_n) = g[g_1\cdots g_n], \end{aligned}$$

[1] In the general case, the given algebra may not be unitary.

which implies

$$\lambda'_{g[g_1\cdots g_n]}(e,\ldots,e) = \lambda'_g[\lambda'_{g_1}\cdots\lambda'_{g_n}](e,\ldots,e).$$

And at last, let $(z_1,\ldots,z_n) \in (G')^n \setminus (G^n \cup \{(e,\ldots,e)\})$, then

$$\lambda'_{g[g_1\cdots g_n]}(z_1,\ldots,z_n) = c$$

and

$$\begin{aligned}\lambda'_g[\lambda'_{g_1}\cdots\lambda'_{g_n}](z_1,\ldots,z_n) &= \lambda'_g(\lambda'_{g_1}(z_1,\ldots,z_n),\ldots,\lambda'_{g_n}(z_1,\ldots,z_n))\\ &= \lambda'_g(c,\ldots,c) = c.\end{aligned}$$

Hence

$$\lambda'_{g[g_1\cdots g_n]}(z_1,\ldots,z_n) = \lambda'_g[\lambda'_{g_1}\cdots\lambda'_{g_n}](z_1,\ldots,z_n).$$

This completes the proof of the identity (2.1.1).

If $\lambda'_{g_1} = \lambda'_{g_2}$, then $\lambda'_{g_1}(e,\ldots,e) = \lambda'_{g_2}(e,\ldots,e)$ and, consequently $g_1 = g_2$. So, the mapping $P : g \mapsto \lambda'_g$ is the isomorphism between the algebra (G, o) and the Menger algebra (Λ', O) of full n-place functions, where $\Lambda' = \{\lambda'_g \mid g \in G\}$. □

Corollary 2.1.9. *Any Menger algebra of rank n is isomorphic to some Menger algebra of n-place functions.*

Let (G, o) be a Menger algebra of rank n. Let us consider the set $T_n(G)$ of all expressions, called *polynomials*, in the alphabet $G \cup \{\,[\,],x\}$, where the square brackets and x do not belong to G, defined as follows:

(a) $x \in T_n(G)$,

(b) if $i \in \{1,\ldots,n\}$, $a, b_1,\ldots,b_{i-1},b_{i+1},\ldots,b_n \in G$, $t \in T_n(G)$, then $a[b_1\cdots b_{i-1}t\,b_{i+1},\ldots b_n] \in T_n(G)$,

(c) $T_n(G)$ contains precisely those polynomials which are defined according to (a) and (b).

Every polynomial $t \in T_n(G)$ defines on (G, o) an *elementary translation* $t : x \mapsto t(x)$. The polynomial and the elementary translation defined by it will be noted by the same letter.

A subset H of G is called

- *stable*, if for all $g, g_1,\ldots,g_n \in G$

$$g, g_1,\ldots,g_n \in H \longrightarrow g[g_1\cdots g_n] \in H,$$

- *quasi-stable*, if for every $g \in G$

$$g \in H \longrightarrow g[g\cdots g] \in H,$$

- *l-unitary*, if for all $g_1, g_2 \in G$

$$g_1[g_2 \cdots g_2] \in H \wedge g_2 \in H \longrightarrow g_1 \in H,$$

- *v-unitary*, if for all $g, g_1, \ldots, g_n \in G$

$$g_1, \ldots, g_n \in H \wedge g[g_1 \cdots g_n] \in H \longrightarrow g \in H,$$

- *a normal v-complex*, if for all $g_1, g_2 \in G$, $t \in T_n(G)$

$$g_1, g_2, t(g_1) \in H \longrightarrow t(g_2) \in H,$$

- *strong*, if for all $g_1, g_2 \in G$, $t_1, t_2 \in T_n(G)$

$$t_1(g_1), t_1(g_2), t_2(g_2) \in H \longrightarrow t_2(g_1) \in H,$$

- *an l-ideal*, if for all $x, h_1, \ldots, h_n \in G$

$$(h_1, \ldots, h_n) \in G^n \setminus (G \setminus H)^n \longrightarrow x[h_1 \cdots h_n] \in H,$$

- *an i-ideal* $(1 \leqslant i \leqslant n)$, if for all $h, u \in G$, $\bar{w} \in G^n$

$$h \in H \longrightarrow u[\bar{w}|_i h] \in H,$$

- *an s-ideal*, if for all $h, x_1, \ldots, x_n \in G$

$$h \in H \longrightarrow h[x_1 \cdots x_n] \in H,$$

- *a v-ideal*, if for all $x, h_1, \ldots, h_n \in G$

$$h_1, \ldots, h_n \in H \longrightarrow x[h_1 \cdots h_n] \in H,$$

- *an sl-ideal*, if H is both an s-ideal and an l-ideal.

It is clear that H is an l-ideal if and only if it is an i-ideal for all $i = 1, \ldots, n$.

In the Menger algebra $(\mathbb{N}_+, o)$ of all nonzero natural numbers with the operation $o(x_0, \ldots, x_n) = x_0 \cdot gcd(x_1, \ldots, x_n)$, the set of all add natural numbers is a stable and quasi-stable sl-ideal but it is not l-unitary and v-unitary. The set of all odd numbers is stable, quasi-stable, l-unitary and v-unitary but it is neither an s-ideal nor a v-ideal.

Proposition 2.1.10. *If H is a normal v-complex, then the quasi-stability of H implies its stability. Similarly, the l-unitarity of H implies its v-unitarity.*

Proof. Indeed, let H be a quasi-stable normal v-complex and let $a, b_1, \ldots, b_n$ be elements of H. Since $a, b_1, a[aa \cdots a] \in H$, therefore also $a[b_1 a \cdots a] \in H$. But this, together with $a, b_2 \in H$, implies $a[b_1 b_2 a \cdots a] \in H$ and so on. Hence $a[b_1 b_2 \ldots b_n] \in H$, i.e., H is stable.

The second part of the proposition can be proved similarly. □

A binary relation $\rho \subset G \times G$, where (G, o) is a Menger algebra of rank n, is called

- *stable*, if for all $x, y, x_i, y_i \in G$, $i = 1, \ldots, n$

$$(x, y), (x_1, y_1), \ldots, (x_n, y_n) \in \rho \longrightarrow (x[\bar{x}], y[\bar{y}]) \in \rho,$$

- *l-regular*, if for any $x, y, z_i \in G$, $i = 1, \ldots, n$

$$(x, y) \in \rho \longrightarrow (x[\bar{z}], y[\bar{z}]) \in \rho,$$

- *v-regular*, if for all $x_i, y_i, z \in G$, $i = 1, \ldots, n$

$$(x_1, y_1), \ldots, (x_n, y_n) \in \rho \longrightarrow (z[\bar{x}], z[\bar{y}]) \in \rho,$$

- *s-regular*, if for all $x, y, z \in G$

$$(y, z) \in \rho \longrightarrow (x[y^n], x[z^n]) \in \rho,$$

- *i-regular*, if for any $u, x, y \in G$, $\bar{w} \in G^n$

$$(x, y) \in \rho \longrightarrow (u[\bar{w}|_i x], u[\bar{w}|_i y]) \in \rho,$$

- *v-negative*, if for all $u, x, y \in G$, $\bar{w} \in G^n$, $i = 1, \ldots, n$

$$(x, u[\bar{w}|_i y]) \in \rho \longrightarrow (x, y) \in \rho,$$

- *l-cancellative*, if for all $x, y \in G$, $\bar{z} \in G^n$

$$(x[\bar{z}], y[\bar{z}]) \in \rho \longrightarrow (x, y) \in \rho,$$

- *v-cancellative*, if for all $x, y, u \in G$, $\bar{w} \in G^n$, $i = 1, \ldots, n$

$$(u[\bar{w}|_i x], u[\bar{w}|_i y]) \in \rho \longrightarrow (x, y) \in \rho$$

- *weakly steady*, if for all $x, y, z \in G$, $t_1, t_2 \in T_n(G)$

$$(x, y), (z, t_1(x)), (z, t_2(y)) \in \rho \longrightarrow (z, t_2(x)) \in \rho,$$

- *steady*, if for all $x, y, z \in G$, $t_1, t_2 \in T_n(G)$

$$(z, t_1(x)), (z, t_1(y)), (z, t_2(y)) \in \rho \longrightarrow (z, t_2(x)) \in \rho.$$

Proposition 2.1.11. *If $\omega \subset G \times G$ is a quasi-order on a Menger algebra (G, o) of rank n, then*
(a) ω *is v-regular if and only if it is i-regular for all $i = 1, \ldots, n$,*
(b) ω *is stable if and only if it is both l-regular and v-regular,*

(c) ω *is v-negative if and only if* $(x[y_1 \cdots y_n], y_i) \in \omega$ *for all* $x, y_i \in G$, $i = 1, \ldots, n$,

(d) ω *is v-negative if and only if* $\delta \subset \omega$, *where*

$$\delta = \{(g_1, g_2) \mid g_1 = t(g_2) \text{ for some } t \in T_n(G)\}.$$

Proof. Let ω be v-regular and $(x, y) \in \omega$. As a quasi-order ω is reflexive, then $(z_1, z_1), \ldots, (z_n, z_n) \in \omega$, which, according to the v-regularity of ω, gives $(u[\bar{z}\,|_i x], u[\bar{z}\,|_i y]) \in \omega$. So, ω is i-regular for every $i = 1, \ldots, n$.

On the other hand, if a quasi-order ω is i-regular for every $i = 1, \ldots, n$ and $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \in \omega$, then the 1-regularity of ω implies $(u[x_1 x_2 \cdots x_n], u[y_1 x_2 \cdots x_n]) \in \omega$. Analogously, $(x_2, y_2) \in \omega$ and the 2-regularity gives $(u[y_1 x_2 \cdots x_n], u[y_1 y_2 \cdots x_n]) \in \omega$, and so on. In the nth step, $(x_n, y_n) \in \omega$ and the n-regularity give $(u[y_1 \cdots y_{n-1} x_n], u[y_1 \cdots y_{n-1} y_n]) \in \omega$. Applying the transitivity of ω, we obtain $(u[x_1 \cdots x_n], u[y_1 \cdots y_n]) \in \omega$, which means the v-regularity of ω. In this way we complete the proof of (a).

By a similar argumentation we can prove (b), (c) and (d). □

Let (G, o) be a Menger algebra of rank n, X — some set with the same cardinality as G, $e_1, \ldots, e_n$ — some pairwise different elements, which do not belong to X. Let $X^* = X \cup \{e_1, \ldots, e_n\}$. Consider the set $\mathbf{\Omega}_n$ of all words in the alphabet $X^* \cup \{(\,,\,)\}$, where the round brackets and coma do not belong to X^*, defined as follows:

(a) if $x \in X^*$, then $x \in \mathbf{\Omega}_n$,

(b) if $\omega, \omega_1, \ldots, \omega_n \in \mathbf{\Omega}_n$, then $\omega(\omega_1, \ldots, \omega_n) \in \mathbf{\Omega}_n$,

(c) $\mathbf{\Omega}_n$ contains precisely those words which are defined according to (a) and (b).

On the set $\mathbf{\Omega}_n$ we define an $(n+1)$-operation O, letting

$$O(\omega, \omega_1, \ldots, \omega_n) = \omega(\omega_1, \ldots, \omega_n)$$

for arbitrary $\omega, \omega_1, \ldots, \omega_n \in \mathbf{\Omega}_n$. The algebra $(\mathbf{\Omega}_n, O)$ will be called the *algebra of words of rank n over X*.

Using a bijection φ from G onto X we can introduce on $(\mathbf{\Omega}_n, O)$ a binary relation π such that

$$\omega(\omega_1, \ldots, \omega_n)(\omega'_1, \ldots, \omega'_n) \eqsim \omega(\omega_1(\omega'_1, \ldots, \omega'_n), \ldots, \omega_n(\omega'_1, \ldots, \omega'_n))(\pi),$$
$$\omega(e_1, \ldots, e_n) \eqsim \omega(\pi),$$
$$e_i(\omega_1, \ldots, \omega_n) \eqsim \omega_i(\pi) \quad \text{for all} \quad i = 1, \ldots, n,$$
$$x \eqsim y(z_1, \ldots, z_n)(\pi), \text{ where } \varphi^{-1}(x) = \varphi^{-1}(y)[\varphi^{-1}(z_1) \cdots \varphi^{-1}(z_n)],$$

for all $\omega, \omega_i, \omega'_i \in \mathbf{\Omega}_n$, $x, y, z_i \in X$, $i = 1, \ldots, n$, where $\omega_1 \eqsim \omega_2(\pi)$ means $(\omega_1, \omega_2) \in \pi$. The relation π does not comprise other elements, besides those, which are defined by the given conditions.

Let ε be the minimal congruence on the algebra $(\mathbf{\Omega}_n, O)$ containing the relation π and let $(\mathbf{\Omega}_n/\varepsilon, O/\varepsilon)$ be the corresponding factor-algebra. As $\pi \subset \varepsilon$, this factor-algebra is a unitary Menger algebra of rank n with selectors $e_1, \ldots, e_n$. The mapping $P : g \mapsto \varepsilon\langle\varphi(g)\rangle$ from G to $\mathbf{\Omega}_n/\varepsilon$ is an isomorphism of (G, o) into $(\mathbf{\Omega}_n/\varepsilon, O/\varepsilon)$. Indeed,

$$P(g[g_1 \cdots g_n]) = \varepsilon\langle\varphi(g[g_1 \cdots g_n])\rangle = \varepsilon\langle\varphi(g)(\varphi(g_1), \ldots, \varphi(g_n))\rangle$$
$$= \varepsilon\langle\varphi(g)\rangle(\varepsilon\langle\varphi(g_1)\rangle, \ldots, \varepsilon\langle\varphi(g_n)\rangle) = P(g)(P(g_1), \ldots, P(g_n)).$$

Thus, P is an homomorphism.

To prove that it is one-to-one assume that $P(g_1) = P(g_2)$, i.e., $\varepsilon\langle\varphi(g_1)\rangle = \varepsilon\langle\varphi(g_2)\rangle$ for some $g_1, g_2 \in G$. Let $\mathbf{\Psi}$ be an isomorphism between a Menger algebra (G, o) and some Menger algebra $(\mathbf{\Phi}, O)$ of full n-place functions defined on some set A. Adjoining to $\mathbf{\Phi}$ all n-place projectors $I_1^n, \ldots, I_n^n$ of A, we obtain a unitary Menger algebra $(\mathbf{\Phi}^*, O)$ generated by the functions from the set $\mathbf{\Phi} \cup \{I_1^n, \ldots, I_n^n\}$. It is clear that the mapping $\chi : \mathbf{\Omega}_n \mapsto \mathbf{\Phi}^*$ determined by the conditions

(a) $\chi(x) = (\psi \circ \varphi^{-1})(x)$ for all $x \in X$,

(b) $\chi(e_i) = I_i^n$ for all $i = 1, \ldots, n$,

(c) $\chi(\omega(\omega_1, \ldots, \omega_n)) = \chi(\omega)[\chi(\omega_1) \cdots \chi(\omega_n)]$ for all $\omega, \omega_1, \ldots, \omega_n \in \mathbf{\Omega}_n$,

is a homomorphism of the algebra $(\mathbf{\Omega}_n, O)$ onto a unitary Menger algebra $(\mathbf{\Phi}^*, O)$ of full n-place functions. Let ε_χ be the kernel of this homomorphism, i.e., $\varepsilon_\chi = \{(\omega_1, \omega_2) \mid \chi(\omega_1) = \chi(\omega_2)\}$. Obviously ε_χ is a congruence containing the relation π. As ε is the minimal congruence containing π, $\varepsilon \subset \varepsilon_\chi$. So, from $\varphi(g_1) \equiv \varphi(g_2)(\varepsilon)$ we have $\varphi(g_1) \equiv \varphi(g_2)(\varepsilon_\chi)$, i.e., $\chi(\varphi(g_1)) = \chi(\varphi(g_2))$. Since $\varphi(g_1), \varphi(g_2) \in X$, we obtain

$$\chi(\varphi(g_1)) = (\psi \circ \varphi^{-1})(\varphi(g_1)) = \psi(\varphi^{-1} \circ \varphi(g_1)) = \psi(g_1)$$

and $\chi(\varphi(g_2)) = \psi(g_2)$ by analogy. So, $\psi(g_1) = \psi(g_2)$. But ψ is one-to-one, hence $g_1 = g_2$, which was to be proved.

This means that a Menger algebra (G, o) of rank n is isomorphically embeddable into a unitary Menger algebra $(\mathbf{\Omega}_n/\varepsilon, O/\varepsilon)$ of the same rank.

The algebra $(\mathbf{\Omega}_n/\varepsilon, O/\varepsilon)$ will be denoted by (G^*, o^*) and will be called the *unitary Menger algebra of rank n obtained from* (G, o) *by external adjoining the full set of selectors*. Note that the algebra (G^*, o^*) is uniquely determined and any unitary Menger algebra of the same rank containing (G, o) as its subalgebra with $G \cup \{e_1, \ldots, e_n\}$ as its generating set is a homomorphic image of (G^*, o^*).

In this way, we proved the following theorem.

Theorem 2.1.12. *Every Menger algebra* (G, o) *of rank* n *can be isomorphically embedded into a unitary Menger algebra* (G^*, o^*) *of the same rank with selectors* $e_1, \ldots, e_n$ *and a generating set* $G \cup \{e_1, \ldots, e_n\}$*, where* $e_i \notin G$ *for all* $i = 1, \ldots, n$*. Every unitary Menger algebra of rank* n *generated by* $G \cup \{e_1, \ldots, e_n\}$ *and containing* (G, o) *as its subalgebra is a homomorphic image of* (G^*, o^*)*.*

As a consequence of some general results on Ω-systems Ja. Henno obtained in [63] the following theorem.

Theorem 2.1.13. *Every finite or countable Menger algebra of rank* $n > 1$ *can be isomorphically embedded into a Menger algebra of the same rank generated by a single element.*

Any stable (respectively, v-regular, l-regular) equivalence relation on a Menger algebra (G, o) is called a *congruence* (respectively, v*-congruence,* l*-congruence*) on (G, o). To any subset H of G we shall associate three binary relations $\varepsilon_v(H)$, $\varepsilon_l(H)$, $\varepsilon_c(H)$ on G and subsets $W_v(H)$, $W_l(H)$, $W_c(H)$ of G defined in the following way:

$$\varepsilon_v(H) = \{(g_1, g_2) \,|\, (\forall t)(t(g_1) \in H \longleftrightarrow t(g_2) \in H)\}, \tag{2.1.2}$$

$$\varepsilon_l(H) = \{(g_1, g_2) \,|\, (\forall \bar{x})(g_1[\bar{x}] \in H \longleftrightarrow g_2[\bar{x}] \in H)\}, \tag{2.1.3}$$

$$\varepsilon_c(H) = \{(g_1, g_2) \,|\, (\forall t)(\forall \bar{x})(t(g_1[\bar{x}]) \in H \longleftrightarrow t(g_2[\bar{x}]) \in H)\}, \tag{2.1.4}$$

$$W_v(H) = \{g \,|\, (\forall t) t(g) \notin H\}, \tag{2.1.5}$$

$$W_l(H) = \{g \,|\, (\forall \bar{x}) g[\bar{x}] \notin H\}, \tag{2.1.6}$$

$$W_c(H) = \{g \,|\, (\forall t)(\forall \bar{x}) t(g[\bar{x}]) \notin H\}, \tag{2.1.7}$$

where $t \in T_n(G)$, $\bar{x} \in B = G^n \cup \{(e_1, \ldots, e_n)\}$, and $e_1, \ldots, e_n$ are selectors of (G^*, o^*). Remark that for every subset H of a Menger algebra (G, o) of rank n the relation $\varepsilon_c(H)$ (respectively, $\varepsilon_v(H)$, $\varepsilon_l(H)$) is a congruence (respectively, v-congruence, l-congruence) and the subset $W_c(H)$ (respectively, $W_v(H)$, $W_l(H)$) is an sl-ideal (respectively, l-ideal, s-ideal) of (G^*, o^*), which is a $\varepsilon_c(H)$-class (respectively, $\varepsilon_v(H)$-class, $\varepsilon_l(H)$-class) or is empty. Indeed it is clear that $\varepsilon_c(H)$ is an equivalence relation. If $g_1 \equiv g_2(\varepsilon_c(H))$ and $\bar{x}$, $\bar{y}$ are arbitrary elements of B, $t \in T_n(G)$, then according to the definition we obtain

$$t(g_1[\bar{y} * \bar{x}]) \in H \longleftrightarrow t(g_2[\bar{y} * \bar{x}]) \in H,$$

where $\bar{y} * \bar{x}$ denotes the vector $(y_1[\bar{x}], \ldots, y_n[\bar{x}])$ and $\bar{y} = (y_1, \ldots, y_n)$. Thus,

$$(t \in T_n(G))(\bar{x} \in B)\ (t(g_1[\bar{y}][\bar{x}]) \in H \longleftrightarrow t(g_2[\bar{y}][\bar{x}]) \in H)\,,$$

i.e., $g_1[\bar{y}] \equiv g_2[\bar{y}](\varepsilon_c(H))$. So, $\varepsilon_c(H)$ is l-regular. Similarly we show that $\varepsilon_c(H)$ is i-regular for any $i = 1, \ldots, n$. Thus it is v-regular, which together with its l-regularity implies that $\varepsilon_c(H)$ is stable. Now, with the help of arguments given above, we show that $W_c(H)$ is an sl-ideal of (G, o) and an $\varepsilon_c(H)$-class under the assumption $W_c(H) \neq \varnothing$.

Let $\boldsymbol{\Theta}$ be a congruence (v-congruence or l-congruence) on (G, o). By a simple verification we can see that the following identities hold:

$$\boldsymbol{\Theta} = \bigcap_{g \in G} \varepsilon_c(\boldsymbol{\Theta}\langle g \rangle), \qquad \boldsymbol{\Theta} = \bigcap_{g \in G} \varepsilon_v(\boldsymbol{\Theta}\langle g \rangle), \qquad \boldsymbol{\Theta} = \bigcap_{g \in G} \varepsilon_l(\boldsymbol{\Theta}\langle g \rangle).$$

Now we consider some special types of v-congruences on Menger algebras, which further on will be used in many constructions. So, let us give on a Menger algebra (G, o) of rank n the relation of quasi-equivalence ρ and some subset A of G. On (G, o) we define two new relations $\varepsilon(\rho, A)$ and $\varepsilon\langle \rho, A \rangle$ putting:

$$\varepsilon(\rho, A) = \overline{\rho \cap A \times A} \cup A' \times A', \tag{2.1.8}$$

where $A' = G \setminus A$ and $\overline{\rho \cap A \times A}$ is a transitive closure of $\rho \cap A \times A$,

$$\varepsilon\langle \rho, A \rangle = \bigcup_{m=0}^{\infty} \varepsilon_m\langle \rho, A \rangle \cup A' \times A', \tag{2.1.9}$$

where $\varepsilon_0\langle \rho, A \rangle = \rho \cap A \times A$ and $(g_1, g_2) \in \varepsilon_{m+1}\langle \rho, A \rangle$ if and only if $g_1 = u[\bar{w}\,|_q c] \in A$, $g_2 = u[\bar{w}\,|_q d] \in A$, $(s[\bar{v}\,|_r c], s[\bar{v}\,|_r d]) \in \varepsilon_m\langle \rho, A \rangle \circ \varepsilon_m\langle \rho, A \rangle$ for some $q, r \in \{1, \ldots, n\}$, $u \in G \cup \{e_q\}$, $s \in G \cup \{e_r\}$, $c, d \in G$, $\bar{w}, \bar{v} \in G^n$, where e_q, e_r are selectors of (G^*, o^*).

It is not difficult to see that these relations are equivalences. Moreover, the following proposition is true.

Proposition 2.1.14. *If a binary relation ρ on a Menger algebra (G, o) is i-regular for any $i = 1, \ldots, n$ and A' is an l-ideal of (G, o), then:*

(a) *$\varepsilon(\rho, A)$ is a v-congruence, if the following implication*

$$(a, b) \in \rho \wedge a \in A \wedge u[\bar{w}\,|_i b] \in A \longrightarrow u[\bar{w}\,|_i a] \in A \tag{2.1.10}$$

holds for all $i = 1, \ldots, n$, $a, b \in G$, $u \in G \cup \{e_i\}$, $\bar{w} \in G^n$.

(b) *$\varepsilon\langle \rho, A \rangle$ is a v-congruence such that $\varepsilon\langle \rho, A \rangle \cap A \times A$ is a v-cancellative, if for all $m = 0, 1, 2, \ldots$, $i = 1, \ldots, n$, $a, b \in G$, $x \in G \cup \{e_i\}$, $\bar{y} \in G^n$ the following condition is satisfied:*

$$(a, b) \in \varepsilon_m\langle \rho, A \rangle \wedge x[\bar{y}\,|_i b] \in A \longrightarrow x[\bar{y}\,|_i a] \in A. \tag{2.1.11}$$

Proof. We prove only (b) because the proof of (a) is analogous.

Let $g_1 \equiv g_2(\varepsilon_m\langle\rho, A\rangle)$ for some m. If $g_1 \equiv g_2(\varepsilon\langle\rho, A\rangle)$ and $g_1, g_2 \in A'$, then $(u[\bar{w}\,|_i g_1], u[\bar{w}\,|_i g_2]) \in A' \times A' \subset \varepsilon\langle\rho, A\rangle$. As A' is an l-ideal, then suppose that $(g_1, g_2) \in \varepsilon_m\langle\rho, A\rangle$ for some m. If $u[\bar{w}\,|_i g_1] \in A$, then according to (2.1.11) we have $u[\bar{w}\,|_i g_2] \in A$ and vice versa. Thus, the elements $u[\bar{w}\,|_i g_1]$, $u[\bar{w}\,|_i g_2]$ belong or do not belong to A simultaneously. If these elements belong to A, then $(g_1, g_2) \in \varepsilon_m\langle\rho, A\rangle \circ \varepsilon_m\langle\rho, A\rangle$. Thus $(u[\bar{w}\,|_i g_1], u[\bar{w}\,|_i g_2]) \in \varepsilon_{m+1}\langle\rho, A\rangle \subset \varepsilon\langle\rho, A\rangle$. The case when these elements do not belong to A is obvious. So, in every case $(u[\bar{w}\,|_i g_1], u[\bar{w}\,|_i g_2]) \in \varepsilon\langle\rho, A\rangle$, i.e., $\varepsilon\langle\rho, A\rangle$ is i-regular for all $i = 1, \ldots, n$. So, the given relation is a v-congruence.

Now let $(u[\bar{w}\,|_i g_1], u[\bar{w}\,|_i g_2]) \in \varepsilon\langle\rho, A\rangle \cap A \times A$. Then $(u[\bar{w}\,|_i g_1], u[\bar{w}\,|_i g_2]) \in \varepsilon_m\langle\rho, A\rangle$ for some m. Since A' is an l-ideal, then $u[\bar{w}\,|_i g_1] \in A$ implies $g_1 \in A$. Analogously we obtain $g_2 \in A$. So, $g_1 = g_1 \in A$, $g_2 = g_2 \in A$, $(u[\bar{w}\,|_i g_1], u[\bar{w}\,|_i g_2]) \in \varepsilon_m\langle\rho, A\rangle \circ \varepsilon_m\langle\rho, A\rangle$, i.e., $(g_1, g_2) \in \varepsilon_{m+1}\langle\rho, A\rangle \subset \varepsilon\langle\rho, A\rangle$. This means that $\varepsilon\langle\rho, A\rangle \cap A \times A$ is a v-cancellative. □

In a similar way, it can be proved that *if H is a strong subset of a Menger algebra (G, o), then the restriction of a v-congruence $\varepsilon_v(H)$ to $W_v'(H)$ is a v-cancellative relation.*

Using mathematical induction, it is not difficult to see that for every $m = 1, 2, \ldots$ the condition $(g_1, g_2) \in \varepsilon_{m+1}\langle\rho, A\rangle$ means that there exist elements $u_i \in G \cup \{e_{q_i}\}$, $s_i \in G \cup \{e_{r_i}\}$, $\bar{w}, \bar{v} \in G^n$, $c_i, d_i, f_i, z_i \in G$, $q_i, r_i \in \{1, \ldots, n\}$ for which

$$u_1[\bar{w}_1|_{q_1} c_1] = g_1 \in A,$$

$$u_1[\bar{w}_1|_{q_1} d_1] = g_2 \in A,$$

$$\bigwedge_{i=1}^{2^m-1} \begin{pmatrix} u_{2i}[\bar{w}_{2i}|_{q_{2i}} c_{2i}] = s_i[\bar{v}_i|_{r_i} c_i] \in A, \\ u_{2i+1}[\bar{w}_{2i+1}|_{q_{2i+1}} c_{2i+1}] = u_{2i}[\bar{w}_{2i}|_{q_{2i}} d_{2i}] \in A, \\ u_{2i+1}[\bar{w}_{2i+1}|_{q_{2i+1}} d_{2i+1}] = s_i[\bar{v}_i|_{r_i} d_i] \in A \end{pmatrix}$$

and

$$\bigwedge_{i=2^m}^{2^{m+1}-1} \begin{pmatrix} (s_i[\bar{v}_i|_{r_i} c_i], z_i) \in \rho \cap A \times A, \\ (z_i, s_i[\bar{v}_i|_{r_i} d_i]) \in \rho \cap A \times A \end{pmatrix}.$$

Let (G, o) be a Menger algebra of rank n, ρ — a binary relation on (G, o), σ — a reflexive binary relation on G. A subset $X \subset G$ is called ρ*-closed* or ρ^**-closed*, if it satisfies respectively:

$$(a, b) \in \rho \ \wedge \ a \in X \ \wedge \ u[\bar{w}\,|_i b] \in X \longrightarrow u[\bar{w}\,|_i a] \in X,$$

$$(a, b) \in \rho \ \wedge \ a \in X \ \wedge \ t(b) \in X \longrightarrow t(a) \in X$$

for all $i = 1, \ldots, n$, $a, b, u \in G$, $\bar{w} \in G^n$, $t \in T_n(G)$. It is easy to see that every ρ^*-closed subset is ρ-closed. The converse statement generally is not true, but if ρ is i-regular for all $i = 1, \ldots, n$ and X' is an l-ideal of the given algebra, then these conditions are equivalent.

A subset X of G is called (ρ, σ)*-closed* (respectively, $(\rho, \sigma)^*$*-closed*), if it is σ-saturated and ρ-closed (respectively, ρ^*-closed). The operations of (ρ, σ)-closure and $(\rho, \sigma)^*$-closure will be denoted respectively by $f_{(\rho,\sigma)}$ and $f^*_{(\rho,\sigma)}$. It is not difficult to see that a subset X will be (ρ, σ)-closed (respectively, $(\rho, \sigma)^*$-closed) if and only if it satisfies respectively the conditions

$$(a, b) \in \rho \;\wedge\; (u[\bar{w}\,|_i a], c) \in \sigma \;\wedge\; a,\, u[\bar{w}\,|_i b] \in X \longrightarrow c \in X,$$

$$(a, b) \in \rho \wedge (t(a), c) \in \sigma \wedge a,\, t(b) \in X \longrightarrow c \in X$$

for all $i = 1, \ldots, n$, $a, b, c \in G$, $u \in G \cup \{e_i\}$, $\bar{w} \in G^n$, $t \in T_n(G)$.

With any subset X of a Menger algebra (G, o) of rank n are associated two sets $E(X)$ and $C(X)$ defined in the following way:

$$g \in E(X) \longleftrightarrow (a, b) \in \rho \;\wedge\; (u[\bar{w}\,|_i a], g) \in \sigma \wedge\; a, u[\bar{w}\,|_i b] \in X,$$

$$g \in C(X) \longleftrightarrow (a, b) \in \rho \;\wedge\; (t(a), g) \in \sigma \;\wedge\; a, t(b) \in X,$$

for some $a, b, g \in G$, $u \in G \cup \{e_i\}$, $\bar{w} \in G^n$, $t \in T_n(G)$, $i \in \{1, \ldots, n\}$. It is obvious that

$$f_{(\rho,\sigma)}(X) = \bigcup_{n=0}^{\infty} \overset{n}{E}(X), \qquad f^*_{(\rho,\sigma)}(X) = \bigcup_{n=0}^{\infty} \overset{n}{C}(X), \tag{2.1.12}$$

where $\overset{n}{E}(X) = E(\overset{n-1}{E}(X))$, $\overset{n}{C}(X) = C(\overset{n-1}{C}(X))$ for $n \geqslant 1$, and $\overset{0}{E}(X) = \overset{0}{C}(X) = X$.

The following proposition can be easily proved by induction.

Proposition 2.1.15. *The condition* $g \in \overset{m}{E}(X)$ *(respectively,* $g \in \overset{m}{C}(X)$*) is satisfied if and only if*

$$(a_1, b_1) \in \rho \;\wedge\; (u_1[\bar{w}_1|_{k_1} a_1], g) \in \sigma,$$

$$\bigwedge_{i=1}^{2^{m-1}-1} \begin{pmatrix} (a_{2i}, b_{2i}) \in \rho \;\wedge\; (u_{2i}[\bar{w}_{2i}|_{k_{2i}} a_{2i}], a_i) \in \sigma, \\ (a_{2i+1}, b_{2i+1}) \in \rho \;\wedge\; (u_{2i+1}[\bar{w}_{2i+1}|_{k_{2i+1}} a_{2i+1}], u_i[\bar{w}_i|_{k_i} b_i]) \in \sigma \end{pmatrix},$$

$$\bigwedge_{i=2^{m-1}}^{2^m-1} (a_i \in X \;\wedge\; u_i[\bar{w}_i|_{k_i} b_i] \in X),$$

or, respectively

$$(a_1, b_1) \in \rho \ \wedge \ (t_1(a_1), g) \in \sigma,$$

$$\bigwedge_{i=1}^{2^{m-1}-1} \begin{pmatrix} (a_{2i}, b_{2i}) \in \rho \ \wedge \ (t_{2i}(a_{2i}), a_i) \in \sigma, \\ (a_{2i+1}, b_{2i+1}) \in \rho \ \wedge \ (t_{2i+1}(a_{2i+1}), t_i(b_i)) \in \sigma \end{pmatrix},$$

$$\bigwedge_{i=2^{m-1}}^{2^m-1} (a_i \in X \ \wedge \ t_i(b_i) \in X),$$

where $k_i \in \{1, \dots, n\}$, $u_i \in G \cup \{e_{k_i}\}$, $a_i, b_i \in G$, $\bar{w}_i \in G^n$, $t_i \in T_n(G)$, $i = 1, 2, \dots, 2^m - 1$.

A subset X of a Menger algebra (G, o) of rank n is called $\langle \rho, \sigma \rangle$*-closed*, where ρ is a binary relation on (G, o), σ — a reflexive binary relation on G, if for any $m = 0, 1, 2, \dots$, $i = 1, \dots, n$, $a, b, c \in G$, $x \in G \cup \{e_i\}$, $\bar{y} \in G^n$ the following implication is true

$$(a, b) \in \varepsilon_m \langle \rho, X \rangle \ \wedge \ (x[\bar{y}|_i a], c) \in \sigma \ \wedge \ x[\bar{y}|_i b] \in X \longrightarrow c \in X.$$

Every subset $X \subset G$ determines the family of subsets $(F_m(X))_{m \in \mathbb{N}}$ such that $F_0(X) = X$ and $g \in F_{m+1}(X)$ if and only if $(a, b) \in \varepsilon_m \langle \rho, F_m(X) \rangle$, $(x[\bar{y}|_i a], g) \in \sigma$ and $x[\bar{y}|_i b] \in F_m(X)$ for some $a, b, \in G$, $x \in G \cup \{e_i\}$, $\bar{y} \in G^n$, $i \in \{1, \dots, n\}$. Using the equation

$$\varepsilon_m \langle \rho, \bigcup_{k=0}^{\infty} F_k(X) \rangle = \bigcup_{k=0}^{\infty} \varepsilon_m \langle \rho, F_k(X) \rangle, \tag{2.1.13}$$

it can be showed that

$$f_{\langle \rho, \sigma \rangle}(X) = \bigcup_{k=0}^{\infty} F_k(X), \tag{2.1.14}$$

where $f_{\langle \rho, \sigma \rangle}$ is a $\langle \rho, \sigma \rangle$-closure operation. It can be showed by induction that the condition $g \in F_{m+1}(X)$ is equivalent to the fact that for some $a_i, b_i \in G$, $x_i \in G \cup \{e_{k_i}\}$, $\bar{y} \in G^n$, $k_i \in \{1, \dots, n\}$, where $i = 0, 1, \dots, m$, we have

$$x_0[\bar{y}_0|_{k_0} b_0] \in X,$$

$$\bigwedge_{i=0}^{m-1} \Big((a_i, b_i) \in \varepsilon_i \langle \rho, F_i(X) \rangle \ \wedge \ (x_i[\bar{y}_i|_{k_i} a_i], x_{i+1}[\bar{y}_{i+1}|_{k_{i+1}} b_{i+1}]) \in \sigma \Big),$$

$$(a_m, b_m) \in \varepsilon_m \langle \rho, F_m(X) \rangle \ \wedge \ (x_m[\bar{y}_m|_{k_m} a_m], g) \in \sigma.$$

Thus, with the help of the above formulas by a finite number of steps, we can always represent the expression $(x, y) \in \varepsilon_m \langle \rho, F_m(X) \rangle$ by some elementary formula $\mathfrak{A}^m_{\langle \rho, \sigma \rangle}(x, y)$.

For $X \subset G$ by $\hat{X}$ will be denoted the least strong subset of (G, o) containing X, which will be called the *strong closure of* X. Of course,

$$\hat{X} = \bigcup_{m=0}^{\infty} \overset{m}{D}(X),$$

where $\overset{0}{D}(X) = X$, $\overset{m}{D}(X) = D(\overset{m-1}{D}(X))$. By induction we can prove that $g \in \overset{m}{D}(X)$ if and only if for $u_i, v_i \in G$, $t'_i, t''_i \in T_n(G)$, $i = 1, \ldots, \frac{3^n-1}{2}$ the following conditions hold

$$g_1 = t''_1(u_1),$$

$$\bigwedge_{i=1}^{\frac{3^{m-1}-1}{2}} \Big(t''_{3i-1}(u_{3i-1}) = t'_i(u_i) \wedge t''_{3i}(u_{3i}) = t'_i(v_i) \wedge t''_{3i+1}(u_{3i+1}) = t''_i(v_i)\Big),$$

$$\bigwedge_{i=\frac{3^{m-1}+1}{2}}^{\frac{3^m-1}{2}} \Big(t'_i(u_i) \in X \wedge t'_i(v_i) \in X \wedge t''_i(v_i) \in X\Big).$$

2.2 Menger semigroups

The close connection between semigroups and Menger algebra was stated already in 1966 by B. M. Schein in his work [172]. He found semigroups of special type, which define Menger algebras. Therefore there is in principle the possibility to study semigroups of such type instead of Menger algebras. But the study of these semigroups is quite difficult, which is why in many questions it is more advisable to simply study Menger algebras, rather than to substitute them with the study of similar semigroups.

Let (G, o) be a Menger algebra of rank n. Let us consider on the set G^n the binary operation $*$, which is defined in the following way:

$$(x_1, \ldots, x_n) * (y_1, \ldots, y_n) = (x_1[y_1 \cdots y_n], \ldots, x_n[y_1 \cdots y_n]),$$

for any $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in G^n$. It is evident that the $(n+1)$-ary operation o is superassociative if and only if the operation $*$ is associative. Thus, to every Menger algebra (G, o) corresponds the semigroup $(G^n, *)$ called the *binary comitant of a Menger algebra* (G, o). It is obvious that binary comitants of isomorphic Menger algebras are isomorphic. Though, as it was mentioned in [172], in the general case the isomorphism of Menger algebras cannot be deduced from the isomorphism of the corresponding binary comitants. This fact leads to the consideration of binary comitants with some additional properties such that the isomorphism of these structures implies the isomorphism of the initial Menger algebras.

L. M. Gluskin observed (see [52] and [53]) that the sets

$$M_1[G] = \{\bar{c} \in G^n \,:\, x[\bar{y}][\bar{c}] = x[\bar{y} * \bar{c}\,] \text{ for all } x \in G,\ \bar{y} \in G^n\}$$

and

$$M_2[G] = \{\bar{c} \in G^n \,:\, x[\bar{c}][\bar{y}] = x[\bar{c} * \bar{y}\,] \text{ for all } x \in G,\ \bar{y} \in G^n\}$$

are either empty or subsemigroups of the binary comitant $(G^n, *)$. The set

$$M_3[G] = \{a \in G \,:\, a[\bar{x}][\bar{y}] = a[\bar{x} * \bar{y}\,] \text{ for all } \bar{x}, \bar{y} \in G^n\}$$

is either empty or a Menger subalgebra of (G, o).

Let us define on the binary comitant $(G^n, *)$ the equivalence relations $\pi_1, \ldots, \pi_n$ letting

$$(x_1, \ldots, x_n) \equiv (y_1, \ldots, y_n)(\pi_i) \longleftrightarrow x_i = y_i$$

for all $x_i, y_i \in G$, $i = 1, \ldots, n$. It is easy to check that these relations have the following properties:

(a) for any elements $\bar{x}_1, \ldots, \bar{x}_n \in G^n$ there is an element $\bar{y} \in G^n$ such that $\bar{x}_i \equiv \bar{y}(\pi_i)$ for all $i = 1, \ldots, n$,

(b) if $\bar{x} \equiv \bar{y}(\pi_i)$ for some $i = 1, \ldots, n$, then $\bar{x} = \bar{y}$, where $\bar{x}, \bar{y} \in G^n$,

(c) relations π_i are *right regular*, i.e.,

$$\bar{x}_1 \equiv \bar{x}_2(\pi_i) \longrightarrow \bar{x}_1 * \bar{y} \equiv \bar{x}_2 * \bar{y}(\pi_i)$$

for all $\bar{x}_1, \bar{x}_2, \bar{y} \in G^n$, $i = 1, \ldots, n$,

(d) $\overset{n}{\triangle}_G$ is a *right ideal* of $(G^n; *)$, i.e.,

$$\bar{x} \in \overset{n}{\triangle}_G \wedge \bar{y} \in G^n \longrightarrow \bar{x} * \bar{y} \in \overset{n}{\triangle}_G$$

(e) for any $i = 1, \ldots n$ every π_i-class contains precisely one element from $\overset{n}{\triangle}_G$.

All systems of the form $(G^n, *, \pi_1, \ldots, \pi_n, \overset{n}{\triangle}_G)$ will be called the *rigged binary comitant* of a Menger algebra (G, o).

Theorem 2.2.1. *Two Menger algebras of the same rank are isomorphic if and only if their rigged binary comitants are isomorphic.*

This theorem is a consequence of a more general result.

Theorem 2.2.2. *The system $(G, \cdot, \varepsilon_1, \ldots, \varepsilon_n, H)$, where $(G, \cdot)$ is a semigroup, $\varepsilon_1, \ldots, \varepsilon_n$ are binary relations on G and $H \subset G$, is isomorphic to the rigged binary comitant of some Menger algebra of rank n if and only if*

(1) *for all* $i = 1, \ldots, n$ *the relations* ε_i *are right regular equivalence relations and for any* $(g_1, \ldots, g_n) \in G^n$ *there is exactly one* $g \in G$ *such that*

$$g_i \equiv g(\varepsilon_i) \;\; \textit{for all} \;\; i = 1, \ldots, n, \tag{2.2.1}$$

(2) H *is a right ideal of a semigroup* $(G, \cdot)$ *and for any* $i = 1, \ldots, n$ *every* ε_i*-class contains exactly one element of* H.

Proof. The fact that the rigged binary comitant satisfies all of the above conditions is obvious. So, we prove only the sufficiency.

Let the system $(G, \cdot, \varepsilon_1, \ldots, \varepsilon_n, H)$ satisfy all the conditions of the theorem. On H we define an $(n+1)$-ary operation

$$o : (h, h_1, \ldots, h_n) \mapsto h[h_1 \cdots h_n],$$

considering that $h[h_1 \cdots h_n] = hg$ for any $h, h_1, \ldots, h_n \in H$, where g is this uniquely determined element of a semigroup $(G, \cdot)$ for which the vector $(h_1, \ldots, h_n)$ satisfies the condition (2.2.1). Obviously $hg \in H$ since H is a right ideal, i.e., $HG \subset H$.

The $(n+1)$-operation o defined in such a way is superassociative. Indeed, if $h'_1, \ldots, h'_n \in H$ and $g' \in G$ is the element for which the vector $(h'_1, \ldots, h'_n)$ satisfies the condition (2.2.1), then, according to the definition of the operation o, we have

$$h[h_1 \cdots h_n][h'_1 \cdots h'_n] = (hg)g'.$$

On the other hand,

$$h[h_1[h'_1 \cdots h'_n] \cdots h_n[h'_1 \cdots h'_n]] = h[h_1 g' \cdots h_n g']. \tag{2.2.2}$$

Since $h_1, \ldots, h_n, g$ satisfy the conditions

$$h_i \equiv g(\varepsilon_i) \;\; \text{for all} \;\; i = 1, \ldots, n,$$

using the right regularity of ε_i we get

$$h_i g' \equiv gg'(\varepsilon_i) \;\; \text{for all} \;\; i = 1, \ldots, n, \tag{2.2.3}$$

which, together with (2.2.2) and (2.2.3), gives

$$h[h_1[h'_1 \cdots h'_n] \cdots h_n[h'_1 \cdots h'_n]] = h(gg').$$

As $(hg)g' = h(gg')$, then

$$h[h_1 \cdots h_n][h'_1 \cdots h'_n] = h[h_1[h'_1 \cdots h'_n] \cdots h_n[h'_1 \cdots h'_n]].$$

This proves the superassociativity of o. So, (H, o) is a Menger algebra of rank n.

To prove that its rigged binary comitant $(H^n, *, \pi_1, \ldots, \pi_n, \overset{n}{\triangle}_H)$ is isomorphic to $(G, \cdot, \varepsilon_1, \ldots, \varepsilon_n, H)$, we consider the mapping $P : H^n \longrightarrow G$, which every vector $(h_1, \ldots, h_n) \in H^n$ is mapped to the element $g \in G$ satisfying the condition (2.2.1). Since for every $i = 1, \ldots, n$, the equivalence class $\varepsilon_i\langle g\rangle$ has only one common element with H, P is one-to-one.

Let $(h_1, \ldots, h_n), (h'_1, \ldots, h'_n) \in H^n$ and

$$g = P(h_1, \ldots, h_n), \quad g' = P(h'_1, \ldots, h'_n).$$

Then, according to the definition of P, $h_i \equiv g(\varepsilon_i)$ and $h'_i \equiv g'(\varepsilon_i)$ for every $i = 1, \ldots, n$. Since all ε_i are right regular, $h_i \equiv g(\varepsilon_i)$ implies $h_i g' \equiv gg'(\varepsilon_i)$. Thus

$$\begin{aligned} P((h_1, \ldots, h_n) * (h'_1, \ldots, h'_n)) &= P(h_1[h'_1, \ldots, h'_n], \ldots, h_n[h'_1, \ldots, h'_n]) \\ &= P(h_1 g', \ldots, h_n g') = gg' = P(h_1, \ldots, h_n) \cdot P(h'_1, \ldots, h'_n). \end{aligned}$$

Now let $(h_1, \ldots, h_n) \equiv (h'_1, \ldots, h'_n)(\pi_i)$, i.e., $h_i = h'_i$, for some $i = 1, \ldots, n$. As $h_i \equiv g(\varepsilon_i)$ and $h'_i \equiv g'(\varepsilon_i)$, $g \equiv g'(\varepsilon_i)$. On the other hand, if $g \equiv g'(\varepsilon_i)$, then obviously $h_i, h'_i \in \varepsilon_i\langle g\rangle$. As $\varepsilon_i\langle g\rangle$ contains exactly one element of H, we have $h_i = h'_i$, and consequently

$$(h_1, \ldots, h_n) \equiv (h'_1, \ldots, h'_n)(\pi_i). \tag{2.2.4}$$

So for every $i = 1, \ldots, n$ the condition (2.2.4) is satisfied if and only if $P(h_1, \ldots, h_n) \equiv P(h'_1, \ldots, h'_n)(\varepsilon_i)$.

And at last, let $(h, \ldots, h) \in \overset{n}{\Delta}_H$. As $h \equiv h(\varepsilon_i)$ for all $h \in H$, $i = 1, \ldots, n$, we have $P(h, \ldots, h) = h$. Hence

$$(h, \ldots, h) \in \overset{n}{\Delta}_H \longleftrightarrow h \in H \longleftrightarrow P(h, \ldots, h) \in H.$$

In this way, we have proved that P is an isomorphism of the rigged binary comitant $(H^n, *, \pi_1, \ldots, \pi_n, \overset{n}{\triangle}_H)$ onto $(G, \cdot, \varepsilon_1, \ldots, \varepsilon_n, H)$. □

Further on, a system of the form $(G, \cdot, \varepsilon_1, \ldots, \varepsilon_n, H)$, satisfying all the conditions of the Theorem 2.2.2, will be called a *Menger semigroup of rank n*. Of course, a Menger semigroup of rank 1 coincides with the semigroup $(G, \cdot)$.

So the theory of Menger algebras can be completely restricted to the theory of Menger semigroups. But we do not have to use this information. It is possible that in some cases it would be advisable to consider Menger algebras, and in others Menger semigroups. Nevertheless, in our opinion, the study of Menger semigroups is more complicated than a study of Menger algebras with one $(n+1)$-ary operation because a Menger semigroup, besides one binary operation, contains $n+1$ relations, which naturally lead to the additional difficulties.

Let (G, o) be a Menger algebra of rank n. Let us define on its binary comitant $(G^n, *)$ the unary operations $\rho_1, \ldots, \rho_n$ such that

$$\rho_i(x_1, \ldots, x_n) = (x_i, \ldots, x_i)$$

for any $x_1, \ldots, x_n \in G$, $i = 1, \ldots, n$. A system $(G^n, *, \rho_1, \ldots, \rho_n)$ obtained in such a way will be called the *selective binary comitant* of Menger algebra (G, o).

Theorem 2.2.3. *For the system $(G, \cdot, p_1, \ldots, p_n)$, where $(G, \cdot)$ is a semigroup and $p_1, \ldots, p_n$ are unary operations on it, a necessary and sufficient condition for this system to be isomorphic to the selective binary comitant of some Menger algebra of rank n is that the following conditions hold:*
(1) *$p_i(x)y = p_i(xy)$ for all $i = 1, \ldots, n$ and $x, y \in G$,*
(2) *$p_i \circ p_j = p_j$ for all $i, j = 1, \ldots, n$,*
(3) *for every vector $(x_1, \ldots, x_n) \in G^n$ there is exactly only one $g \in G$ such that $p_i(x_i) = p_i(g)$ for all $i = 1, \ldots, n$.*

Proof. Since the necessity is easily checked, we prove only the sufficiency. So, let all the conditions of this theorem be satisfied. As $p_i \circ p_j = p_j$ for all $i, j = 1, \ldots, n$, so

$$\mathrm{pr}_2\, p_j = \mathrm{pr}_2\, (p_i \circ p_j) \subset \mathrm{pr}_2\, p_i.$$

Hence, as a consequence, we obtain $\mathrm{pr}_2\, p_i = \mathrm{pr}_2\, p_j$ for all $i, j = 1, \ldots, n$. Let H be the intersection of all $\mathrm{pr}_2\, p_i$ and let $o : (h, h_1, \ldots, h_n) \mapsto h[h_1 \cdots h_n]$ be an $(n+1)$-ary operation defined on H by $h[h_1 \cdots h_n] = hg$, where g is the uniquely determined element of G for which $p_i(h_i) = p_i(g)$ hold for every $i = 1, \ldots, n$.

We shall prove that if (H, o) is a Menger algebra of rank n, its selective binary comitant is isomorphic to $(G, \cdot, p_1, \ldots, p_n)$. For this, let $(h_1, \ldots, h_n)$, $(h'_1, \ldots, h'_n) \in H^n$ and g, g' be two elements of G such that $p_i(h_i) = p_i(g)$ and $p_i(h'_i) = p_i(g')$ for all $i = 1, \ldots, n$. For these elements we have $p_i(h_i g') = p_i(gg')$ for all $i = 1, \ldots, n$, and

$$\begin{aligned} h[h_1 \cdots h_n][h'_1 \cdots h'_n] &= (hg)[h'_1 \cdots h'_n] = (hg)g' = h(gg') \\ &= h[h_1 g' \cdots h_n g'] = h[h_1[h'_1 \cdots h'_n] \cdots h_n[h'_1 \cdots h'_n]]. \end{aligned}$$

So, the operation o is superassociative, i.e., (H, o) is a Menger algebra.

Now let us consider the mapping $\psi : H^n \to G$, which maps every vector $(h_1, \ldots, h_n) \in H^n$ to an element $g \in G$ such that $p_i(h_i) = p_i(g)$ for all $i = 1, \ldots, n$. Of course, there exists only one vector $(p_1(g), \ldots, p_n(g)) \in H^n$ such that $\psi(h_1, \ldots, h_n) = g$, because $p_i(p_i(g)) = p_i \circ p_i(g) = p_i(g)$ for every $i = 1, \ldots, n$. This means that ψ is a one-to-one mapping.

It is also a homomorphism because for $(h_1,\dots,h_n),(h'_1,\dots,h'_n)\in H^n$ and $g=\psi(h_1,\dots,h_n)$, $g'=\psi(h'_1,\dots,h'_n)$, we have

$$\begin{aligned}\psi((h_1,\dots,h_n)*(h'_1,\dots,h'_n)) &= \psi(h_1[h'_1,\dots,h'_n],\dots,h_n[h'_1,\dots,h'_n])\\ &= \psi(h_1g',\dots,h_ng') = gg' = \psi(h_1,\dots,h_n)\cdot\psi(h'_1,\dots,h'_n).\end{aligned}$$

As $p_i(h)=p_i(h)$ for all $h\in H$ and $i=1,\dots,n$, so $\psi(h,\dots,h)=h$. This, together with the fact that $p_i\circ p_i=p_i$ implies $p_i(h)=h$, gives

$$\begin{aligned}\psi(\rho_i(h_1,\dots,h_n)) &= \psi(h_i,\dots,h_i) = h_i\\ &= p_i(h_i) = p_i(g) = p_i(\psi(h_1,\dots,h_n)).\end{aligned}$$

Hence ψ is an isomorphism between the selective binary comitant of the Menger algebra (H,o) and the system $(G,\cdot,p_1,\dots,p_n)$. □

As a simple consequence of the above theorem we obtain

Corollary 2.2.4. *The binary comitant of a Menger algebra is a group if and only if the algebra is a group of rank* 1 *or the algebra is a singleton.*

Proof. Indeed, if (G,o) is a group, its binary comitant coincides with it, and if G is a singleton, then its binary comitant is a trivial group.

Conversely, let the binary comitant of (G,o) be a group. If the rank of (G,o) is 1, then (G,o) coincides with its binary comitant and (G,o) is a group. Let the rank of (G,o) be $n>1$. Suppose that $(G^n,*,\rho_1,\dots,\rho_n)$ is the selective semigroup corresponding to (G,o). Since $(G^n,*)$ is a group, all its idempotent translations are merely the identity transformations of G^n. Since for all $\overline{g}_1,\dots,\overline{g}_n\in G^n$ there exists a unique $\overline{g}\in G^n$ such that $\rho_i(\overline{g}_i)=\rho(\overline{g})$, it follows that $(\overline{g})_i=\overline{g}$ for all $i=1,\dots,n$. Therefore the order of G^n is 1. Hence G is a singleton. □

Further all systems $(G,\cdot,p_1,\dots,p_n)$ satisfying the conditions of our Theorem 2.2.3 will be called *selective semigroups of rank n.*

The proved theorem gives the possibility to reduce the theory of Menger algebras to the theory of selective semigroups. In this way, we received three independent methods for the study of superposition of multiplace functions: Menger algebras, Menger semigroups and selective semigroups. A great number of papers dedicated to the study of Menger algebras have been released lately, but unfortunately the same cannot be said about Menger semigroups and selective semigroups.

2.3 v-regular Menger algebras

A vector $\bar{g} = (g_1, \ldots, g_n) \in G^n$ is called

- *idempotent*, if it satisfies the equality $g_i[\bar{g}] = g_i$ for every $i = 1, \ldots, n$,
- *v-regular*, if the system of equalities

$$\begin{cases} g_1[x^n][\bar{g}] = g_1 \\ g_2[x^n][\bar{g}] = g_2 \\ \cdots\cdots\cdots\cdots \\ g_n[x^n][\bar{g}] = g_n \end{cases}$$

  is true for some $x^n = (x, \ldots, x) \in G^n$.

A Menger algebra (G, o) in which all vectors $(g_1, \ldots, g_n)$ of G^n are v-regular is called *v-regular*. Obviously, the diagonal semigroup of such Menger algebra is regular in the sense of [14, 105], and [186].

Two vectors $\bar{x}$, $\bar{y} \in G^n$ are *pseudo-commutative* if

$$g[\bar{x}][\bar{y}] = g[\bar{y}][\bar{x}]$$

holds for any $g \in G$. An element $x \in G$ is called an *inverse for the vector* $\bar{g} \in G^n$, if it satisfies the following equalities:

$$x[\bar{g}][x^n] = x,$$
$$g_i[x^n][\bar{g}] = g_i \quad \text{for every} \quad i = 1, \ldots, n.$$

Proposition 2.3.1. *Every v-regular vector of a Menger algebra of rank n has an inverse element.*

Proof. Let $\bar{g} = (g_1, \ldots, g_n)$ be a v-regular vector of a Menger algebra (G, o) of rank n. There exists $x \in G$ such that $g_i[x^n][\bar{g}] = g_i$, $i = 1, \ldots, n$. Then $y = x[\bar{g}][x^n]$ is inverse for the vector $\bar{g}$. Indeed,

$$\begin{aligned} y[\bar{g}][y^n] &= (x[\bar{g}][x^n])[\bar{g}][(x[\bar{g}][x^n]) \cdots (x[\bar{g}][x^n])] \\ &= x[(g_1[x^n][\bar{g}]) \cdots (g_n[x^n][\bar{g}])]\,[(x[\bar{g}][x^n]) \cdots (x[\bar{g}][x^n])] \\ &= x[\bar{g}][(x[\bar{g}][x^n]) \cdots (x[\bar{g}][x^n])] = x[(g_1[x^n][\bar{g}]) \cdots (g_n[x^n][\bar{g}])][x^n] \\ &= x[\bar{g}][x^n] = y. \end{aligned}$$

So, $y[\bar{g}][y^n] = y$. Finally we obtain

$$\begin{aligned} g_i[y^n][\bar{g}] &= g_i[(x[\bar{g}][x^n]) \cdots (x[\bar{g}][x^n])][\bar{g}] \\ &= (g_i[x^n][\bar{g}])\,[x^n][\bar{g}] = g_i[x^n][\bar{g}] = g_i \end{aligned}$$

for all $i = 1, \ldots, n$. □

Proposition 2.3.2. *If in a Menger algebra (G, o) of rank n an element g is an inverse for the vector $\bar{g}$, then $g[\bar{g}]$ is an idempotent of this algebra and $(g_1[g^n], \ldots, g_n[g^n])$ is its idempotent vector.*

Proof. Indeed,

$$g[\bar{g}][g[\bar{g}] \cdots g[\bar{g}]] = g[\bar{g}][g^n][\bar{g}] = g[\bar{g}],$$

and so the element $g[\bar{g}]$ is idempotent.

Furthermore,

$$g_i[g^n][g_1[g^n] \cdots g_n[g^n]] = g_i[g^n][\bar{g}][g^n] = g_i[g^n]$$

for any $i = 1, \ldots, n$. Thus the vector $(g_1[g^n], \ldots, g_n[g^n])$ is idempotent. □

Theorem 2.3.3. *For every Menger algebra (G, o) of rank n the following three statements are equivalent:*

(a) *every element of (G, o) is an inverse for each vector of G^n,*

(b) *for all elements $x, y, y_1, \ldots, y_n$ of (G, o)*

$$x[y^n] = y[x^n] \longrightarrow x = y, \tag{2.3.1}$$

$$y_i[x^n][\bar{y}] = y_i, \quad i = 1, \ldots, n, \tag{2.3.2}$$

(c) *every element of (G, o) and every vector of G^n are idempotent and the following identity holds:*

$$x[\bar{y}][\bar{z}] = x[\bar{z}]. \tag{2.3.3}$$

Proof. Let (a) be true. Clearly, by the definition of an inverse element, (2.3.2) holds. Now suppose that $x[y^n] = y[x^n]$ for some $x, y \in G$. Then

$$x[y^n][y[x^n] \cdots y[x^n]] = y[x^n][x[y^n] \cdots x[y^n]].$$

Since x is an inverse for the vector $(y[y^n], \ldots, y[y^n])$, we have

$$x[y^n][y[x^n] \cdots y[x^n]] = x[y^n][y^n][x^n] = x[y[y^n] \cdots y[y^n]][x^n] = x.$$

Similarly, we show that

$$y[x^n][x[y^n] \cdots x[y^n]] = y.$$

Thus $x = y$, and hence (2.3.1) is true. So, (a) implies (b).

Now let (b) be satisfied. Then

$$x[x^n][x^n] = x[x[x^n] \cdots x[x^n]]$$

is true for all $x \in G$, and hence, by (2.3.1), we have $x[x^n] = x$. Thus every element of (G, o) is idempotent. Next consider an arbitrary vector $\bar{x} \in G^n$.

By (2.3.2), $x_i[x_i^n][\bar{x}] = x_i$ for each $i = 1, \ldots, n$, hence $x_i[\bar{x}] = x_i$, $i = 1, \ldots, n$. So, $(x_1, \ldots, x_n)$ is an idempotent vector. Now we show that the equality

$$x[y_1[\bar{z}] \cdots y_n[\bar{z}]][x^n] = x \tag{2.3.4}$$

is satisfied for all $x, y_1, \ldots, y_n \in G$ and $\bar{z} = (z_1, \ldots, z_n) \in G^n$. We have

$$\begin{aligned} x[x[y_1[\bar{z}] \cdots y_n[\bar{z}]][x^n] \cdots x[y_1[\bar{z}] \cdots y_n[\bar{z}]][x^n]] &= x[x^n][y_1[\bar{z}] \cdots y_n[\bar{z}]][x^n] \\ &= x[y_1[\bar{z}] \cdots y_n[\bar{z}]][x^n] = x[y_1[\bar{z}] \cdots y_n[\bar{z}]][x[x^n] \cdots x[x^n]] \\ &= x[y_1[\bar{z}] \cdots y_n[\bar{z}]][x^n][x^n]. \end{aligned}$$

From this, applying (2.3.1), we obtain (2.3.4).

According to (2.3.2) and (2.3.4) we have

$$\begin{aligned} x[\bar{y}][\bar{z}] &= x[\bar{y}][z_1[x^n][\bar{z}] \cdots z_1[x^n][\bar{z}]] \\ &= x[\bar{y}][\bar{z}][x^n][\bar{z}] = x[y_1[\bar{z}] \cdots y_n[\bar{z}]][x^n][\bar{z}] = x[\bar{z}]. \end{aligned}$$

Thus we have proved (2.3.3). Therefore, (b) implies (c).

Let (c) be true. By (2.3.3), $x[\bar{y}][x^n] = x[x^n] = x$. Since the vector $\bar{y}$ is idempotent, $y_i[x^n][\bar{y}] = y_i[\bar{y}] = y_i$ for all $i = 1, \ldots, n$. So, every element x is an inverse for every vector $\bar{y}$. Thus (c) implies (a). □

For $n = 1$, Theorem 2.3.3 becomes a known result for semigroups [105].

Theorem 2.3.4. *Every vector of a v-regular Menger algebra (G, o) has exactly one inverse element if and only if*

$$\left.\begin{aligned} g_i[x^n][\bar{g}] &= g_i \\ g_i[y^n][\bar{g}] &= g_i \\ i &= 1, \ldots, n \end{aligned}\right\} \longrightarrow x[\bar{g}][x^n] = y[\bar{g}][y^n]$$

for all $x, y, g_1, \ldots, g_n \in G$.

Proof. Suppose that every vector of (G, o) has only one inverse, and let the hypothesis of the theorem be true for some $x, y, g_1, \ldots, g_n \in G$. As we have shown in the proof of Proposition 2.3.1, the elements $x[\bar{g}][x^n]$ and $y[\bar{g}][y^n]$ are inverse for the vector $\bar{g}$. By our assumption, $x[\bar{g}][x^n] = y[\bar{g}][y^n]$. So the given condition is true.

Conversely, let the given condition be satisfied. If x and y are inverse elements for a vector $(g_1, \ldots, g_n)$, then the equalities $g_i[x^n][\bar{g}] = g_i$, $g_i[y^n][\bar{g}] = g_i$ are true for all $i = 1, \ldots, n$, hence, by the given condition $x[\bar{g}][x^n] = y[\bar{g}][y^n]$. Since $x[\bar{g}][x^n] = x$ and $y[\bar{g}][y^n] = y$, we obtain $x = y$. □

For $n = 1$ Theorem 2.3.4 gives a known result for semigroups [103].

Elements x, y of a Menger algebra (G, o) of rank n are called *diagonal-commutative*, if they are commutative in the diagonal semigroup of (G, o), i.e., if

$$x[y^n] = y[x^n].$$

If all idempotents of a v-regular Menger algebra (G, o) are diagonal-commutative, then the diagonal semigroup of (G, o) is inverse. A Menger algebra with an inverse diagonal semigroup is called *inverse*.

Proposition 2.3.5. *Any vector of a v-regular inverse Menger algebra with pseudo-commutative idempotent vectors has exactly one inverse element.*

Proof. Let x and y be inverse elements for a vector $\bar{g}$ in a v-regular inverse Menger algebra (G, o) of rank n. By Proposition 2.3.2, the elements $x[\bar{g}]$ and $y[\bar{g}]$ are idempotent. Since (G, o) is inverse, its idempotents are diagonal-commutative, and hence we obtain

$$\begin{aligned} x &= x[\bar{g}][x^n] = x[g_1[y^n][\bar{g}] \cdots g_n[y^n][\bar{g}]][x^n] = x[\bar{g}][y[\bar{g}] \cdots y[\bar{g}]][x^n] \\ &= y[\bar{g}][x[\bar{g}] \cdots x[\bar{g}]][x^n] = y[\bar{g}][x[\bar{g}][x^n] \cdots x[\bar{g}][x^n]] = y[\bar{g}][x^n]. \end{aligned}$$

On the other hand, by Proposition 2.3.2, the vectors $(g_1[x^n], \ldots, g_n[x^n])$ and $(g_1[y^n], \ldots, g_n[y^n])$ are idempotent, and hence, by pseudo-commutativity

$$\begin{aligned} y &= y[\bar{g}][y^n] = y[g_1[x^n][\bar{g}] \cdots g_n[x^n][\bar{g}]][y_1 \cdots y_n] \\ &= y[g_1[x^n] \cdots g_n[x^n]][g_1[y^n] \cdots g_n[y^n]] \\ &= y[g_1[y^n] \cdots g_n[y^n]][g_1[x^n] \cdots g_n[x^n]] \\ &= y[\bar{g}][y^n][\bar{g}][x^n] = y[\bar{g}][x^n]. \end{aligned}$$

Thus we have shown that $x = y$. □

Let (G, o) be a Menger algebra of rank n in which every vector $\bar{x} \in G^n$ has exactly one inverse element $\alpha(\bar{x})$. Then

$$\alpha(\bar{x})[\bar{x}][\alpha(\bar{x}) \cdots \alpha(\bar{x})] = \alpha(\bar{x}),$$

$$x_i[\alpha(\bar{x}) \cdots \alpha(\bar{x})][\bar{x}] = x_i, \quad i = 1, \ldots, n.$$

Proposition 2.3.6. *If (G, o) is a v-regular inverse Menger algebra of rank n with pseudo-commutative idempotent vectors, then*

$$\alpha(\bar{x} * \bar{y}) = \alpha(\bar{y})[\alpha(\bar{x}) \cdots \alpha(\bar{x})], \tag{2.3.5}$$

for all $\bar{x}, \bar{y} \in G^n$, where $\bar{x} * \bar{y}$ denotes the vector $(x_1[\bar{y}], \ldots, x_n[\bar{y}])$.

Proof. First, observe that for any idempotent $e \in G$ the vector $e^n \in G^n$ is idempotent. Moreover, for each vector $\bar{x} \in G^n$, by Proposition 2.3.2, the element $\alpha(\bar{x})[\bar{x}]$ is idempotent and the vector $(x_1 \cdot \alpha(\bar{x}), \ldots, x_n \cdot \alpha(\bar{x}))$ is idempotent, where $x_i \cdot \alpha(\bar{x}) = x_i[\alpha(\bar{x}) \cdots \alpha(\bar{x})]$, $i = 1, \ldots, n$. Denote $\alpha(\bar{x})$ by g_1 and $\alpha(\bar{y})$ by g_2. Then

$$\begin{aligned} x_i[\bar{y}][g_2[g_1^n] \cdots g_2[g_1^n]][\bar{x} * \bar{y}] &= x_i[\bar{y}][g_2^n][g_1^n][\bar{x}][\bar{y}] \\ &= x_i[y_1 g_2 \cdots y_n g_2][g_1[\bar{x}] \cdots g_1[\bar{x}]][\bar{y}] \\ &= x_i[g_1[\bar{x}] \cdots g_1[\bar{x}]][y_1 g_2 \cdots y_n g_2][\bar{y}] \\ &= x_i[g_1^n][\bar{x}][y_1 g_2[\bar{y}] \cdots y_n g_2[\bar{y}]] = x_i[\bar{y}] = x_i[\bar{y}] \end{aligned}$$

for all $i = 1, \ldots, n$.

On the other hand,

$$\begin{aligned} g_2[g_1^n][\bar{x} * \bar{y}][g_2[g_1^n] \cdots g_2[g_1^n]] &= g_2[g_1[\bar{x}] \cdots g_1[\bar{x}]][y_1[g_2^n] \cdots y_n[g_2^n]][g_1^n] \\ &= g_2[y_1[g_2^n] \cdots y_n[g_2^n]][g_1[\bar{x}] \cdots g_1[\bar{x}]][g_1^n] \\ &= g_2[\bar{y}][g_2^n][g_1[\bar{x}][g_1^n] \cdots g_1[\bar{x}][g_1^n]] = g_2[g_1^n]. \end{aligned}$$

Thus $g_2[g_1^n]$ is the inverse element for the vector $\bar{x} * \bar{y}$. Hence, by Proposition 2.3.5, we conclude that the equality (2.3.5) holds. □

Theorem 2.3.7. *If (G, o) is a v-regular Menger algebra of rank n with a right unit, then the following conditions are equivalent:*
(a) *every vector from G^n has exactly one inverse element,*
(b) *every two idempotents of (G, o) are diagonal-commutative.*

Proof. Let (a) hold. Since (G, o) is v-regular, obviously, its diagonal semigroup is regular. In accordance with (a), for every $g \in G$ the vectors g^n have only one inverse element, hence we obtain that the diagonal semigroup is an inverse semigroup. Therefore, its idempotents commutate (cf. [14, 105, 186]). So any two idempotents of (G, o) are diagonal-commutative. Thus (a) implies (b).

Now let (b) hold. We shall prove that for every idempotent vector $\bar{a} \in G^n$ the element $e[\bar{a}]$ is idempotent, where e is a right unit. Indeed, we have

$$e[\bar{a}][e[\bar{a}] \cdots e[\bar{a}]] = e[\bar{a}][e^n][\bar{a}] = e[\bar{a}][\bar{a}] = e[a_1[\bar{a}] \cdots a_n[\bar{a}]] = e[\bar{a}].$$

Let $\bar{b} \in G^n$ be another idempotent vector. Obviously, $e[\bar{b}]$ is an idempotent element in (G, o). Since the idempotents $e[\bar{a}]$ and $e[\bar{b}]$ are diagonal-commutative by (b), for each element $g \in G$ we have

$$\begin{aligned} g[\bar{a}][\bar{b}] &= g[e^n][\bar{a}][e^n][\bar{b}] = g[e[\bar{a}] \cdots e[\bar{a}]][e[\bar{b}] \cdots e[\bar{b}]] \\ &= g[e[\bar{a}][e[\bar{b}] \cdots e[\bar{b}]] \cdots e[\bar{a}][e[\bar{b}] \cdots e[\bar{b}]]] \\ &= g[e[\bar{b}][e[\bar{a}] \cdots e[\bar{a}]] \cdots e[\bar{b}][e[\bar{a}] \cdots e[\bar{a}]]] \\ &= g[e^n][\bar{b}][e^n][\bar{a}] = g[\bar{b}][\bar{a}]. \end{aligned}$$

Thus the idempotent vectors $\bar{a}$ and $\bar{b}$ are pseudo-commutative. By Proposition 2.3.5, condition (a) holds. Thus (b) implies (a). □

Theorem 2.3.8. *For any v-regular Menger algebra (G, o) of rank n the following conditions are equivalent:*

(a) *every vector of G^n has only one inverse element and for every idempotent $e \in G$ and every idempotent vector $\bar{a} \in G^n$ the element $e[\bar{a}]$ is idempotent,*

(b) *(G, o) is an inverse Menger algebra in which the equality*

$$e[\bar{a}] = e[\bar{a}][e^n] \tag{2.3.6}$$

holds for every idempotent $e \in G$ and every idempotent vector $\bar{a} \in G^n$,

(c) *the idempotents of the algebra (G, o) are diagonal-commutative, and idempotent vectors of (G, o) are pseudo-commutative.*

Proof. Suppose that (G, o) is a v-regular Menger algebra of rank n and that condition (a) is true. In this case, the diagonal semigroup of (G, o) is inverse, i.e., the Menger algebra (G, o) is inverse. It is known that idempotents are diagonal-commutative in any inverse Menger algebra. Since $e[\bar{a}]$ is an idempotent for each idempotent $e \in G$ and any idempotent vector $\bar{a} \in G^n$, we have

$$e[\bar{a}] = e[e^n][\bar{a}][e^n] = e[e[\bar{a}] \cdots e[\bar{a}]] = e[\bar{a}][e^n],$$

i.e., the equality (2.3.6) is true. This shows that (a) implies (b).

Now let (b) hold. Clearly, idempotents of (G, o) are diagonal-commutative, therefore we have only to show that idempotent vectors are pseudo-commutative. Let $\bar{a}$ and $\bar{b}$ be idempotent vectors. Consider an arbitrary element $g \in G$. Our algebra is v-regular, and hence for the vector $\bar{g}$ there exists an element $x \in G$ such that $g[x^n][g^n] = g$. Hence

$$g[x[g^n] \cdots x[g^n]] = g.$$

By Proposition 2.3.2, the element $e = x[g^n]$ is idempotent. Therefore $g[e^n] = g$. We have to show that $e[\bar{a}]$ (respectively, $e[\bar{b}]$) is an idempotent. Indeed, e and $\bar{a}$ (respectively, e and $\bar{b}$) satisfy (2.3.6), therefore

$$e[\bar{a}][e[\bar{a}] \cdots e[\bar{a}]] = e[\bar{a}][e^n][\bar{a}] = e[\bar{a}][\bar{a}] = e[a_1[\bar{a}] \cdots a_n[\bar{a}]] = e[\bar{a}] = e[\bar{a}].$$

So, $e[\bar{a}]$ is idempotent. Analogously, we prove that $e[\bar{b}]$ is idempotent too. Also,

$$e[\bar{a}][\bar{b}] = e[\bar{a}][e^n]][\bar{b}] = e[\bar{a}][e[\bar{b}] \cdots e[\bar{b}]] = e[\bar{b}][e[\bar{a}] \cdots e[\bar{a}]] = e[\bar{b}][e^n][\bar{a}] = e[\bar{b}][\bar{a}],$$

hence

$$\begin{aligned} g[\bar{a}][\bar{b}] &= g[e^n][\bar{a}][\bar{b}] = g[e[\bar{a}][\bar{b}] \cdots e[\bar{a}][\bar{b}]] \\ &= g[[e[\bar{b}][\bar{a}] \cdots e[\bar{b}][\bar{a}]] = g[e^n][\bar{b}][\bar{a}] = g[\bar{b}][\bar{a}]. \end{aligned}$$

Thus the vectors $\bar{a}$ and $\bar{b}$ are pseudo-commutative. Therefore the condition (c) is true. So, (b) implies (c).

Finally, if the condition (c) is true, then by Proposition 2.3.5, every vector from G^n has only one inverse element, therefore, we must show that $e[\bar{a}]$ is idempotent for each idempotent $e \in G$ and each idempotent vector $\bar{a}$ of G^n. So, let $e \in G$ be an idempotent and let $\bar{a} \in G^n$ be an idempotent vector. Since e^n is an idempotent vector too, we deduce from the pseudo-commutativity that

$$e[\bar{a}][e[\bar{a}]\cdots e[\bar{a}]] = e[\bar{a}][e^n][\bar{a}] = e[e^n][\bar{a}][\bar{a}] = e[a_1[\bar{a}]\cdots a_n[\bar{a}]] = e[\bar{a}].$$

Thus we have proved that $e[\bar{a}]$ is idempotent. So, (c) implies (a). □

Theorem 2.3.9. *The following conditions are equivalent for any v-regular Menger algebra (G, o) of rank n:*
(a) *(G, o) has only one idempotent,*
(b) *the diagonal semigroup of (G, o) is a group,*
(c) *for all elements $g_1, \ldots, g_n$ from (G, o) there exists only one element $x \in G$ such that $g_i[x^n][\bar{g}] = g_i$, $i = 1, \ldots, n$.*

Proof. Let (a) be satisfied. Since (G, o) is v-regular, its diagonal semigroup is regular too. Therefore, in the accordance with [14] and [105], the diagonal semigroup is a group because it has exactly one idempotent. So, (a) implies (b).

Now let (b) be true. The equation $g_i[x^n][\bar{g}] = g_i$ has the form $g_i x \cdot e[\bar{g}] = g_i$ in the group $(G, \cdot)$, where e is the unit of $(G, \cdot)$ and $i = 1, \ldots, n$. The last equation is equivalent to $x \cdot e[\bar{g}] = e$. It is known that there is only one element x of the group that satisfies this equation. Thus (c) holds and (b) implies (c).

Suppose that (c) is valid, and hence the conditions of Theorem 2.3.4 hold. Therefore every vector from G^n has only one inverse element. Then (G, o) is an inverse Menger algebra, so its idempotents are diagonal-commutative. Let b, c be idempotents of (G, o). Let $a = b[c^n]$. Since $b[c^n] = c[b^n]$, we obtain

$$\begin{aligned} a[a^n] &= b[c^n][b[c^n]\cdots b[c^n]] = b[c^n][c[b^n]\cdots c[b^n]] = b[c[b^n]\cdots c[b^n]][b^n] \\ &= b[c^n][b^n] = b[c[b^n]\cdots c[b^n]] = b[b[c^n]\cdots b[c^n]] = b[b^n][c^n] = b[c^n] = a. \end{aligned}$$

Thus a is idempotent. Similarly, $b[a^n] = a$ and $a[c^n] = a$. Hence, the equalities $a[b^n][a^n] = a[a^n] = a$ and $a[c^n][a^n] = a[a^n] = a$ are true and by the condition (c), we have $b = c$. It follows that (a) holds, and hence (c) implies (a). □

Let (G, o) be an inverse Menger algebra of rank n. Consider on G two binary relations $\leqslant$ and $\sqsubset$ defined in the following way:

$$x \leqslant y \longleftrightarrow y^{-1}x = x^{-1}x, \tag{2.3.7}$$

$$x \sqsubset y \longleftrightarrow x^{-1}x \leqslant y^{-1}y \tag{2.3.8}$$

for all $x, y \in G$, where x^{-1} is an inverse element for x in the diagonal semigroup and $xy = x[y^n]$. It is known (see e.g. [14, 105] or [186]) that $\leqslant$ is an order with the property

$$x \leqslant y \longleftrightarrow xy^{-1}x = x \longleftrightarrow x = yx^{-1}x.$$

Proposition 2.3.10. *In an inverse Menger algebra (G, o) of rank n the relation $\leqslant$ is l-regular and $g[b^n] \leqslant g$ holds for each idempotent $b \in G$ and each element $g \in G$.*

Proof. Let $g \in G$ and b be an idempotent. Since $g^{-1}g$ is an idempotent and idempotents are commutative in the diagonal semigroup, then

$$g(g[b^n])^{-1}g[b^n] = g(gb)^{-1}gb = gbg^{-1}gb = gg^{-1}gbb = gb = g[b^n],$$

and therefore $g[b^n] \leqslant g$.

Now let $x \leqslant y$, i.e., $y^{-1}x = x^{-1}x$, whence we obtain $y^{-1}yy^{-1}x = y^{-1}yx^{-1}x$. Hence $y^{-1}x = y^{-1}yx^{-1}x$, and therefore $x^{-1}x = y^{-1}yx^{-1}x$. Thus $xy^{-1}y = xx^{-1}xy^{-1}y = xy^{-1}yx^{-1}x = xx^{-1}x = x$. So $xe = x$, where $e = y^{-1}y$. Since $x \leqslant y$, then $x = yx^{-1}x$, and therefore $x[\bar{z}] = yx^{-1}x[\bar{z}]$, where $\bar{z} \in G^n$. Next we have

$$\begin{aligned} x[\bar{z}] &= yx^{-1}x \cdot e[\bar{z}] = y \cdot x^{-1}x \cdot e[\bar{z}] \cdot (e[\bar{z}])^{-1} \cdot e[\bar{z}] \\ &= y \cdot e[\bar{z}] \cdot (e[\bar{z}])^{-1} \cdot x^{-1}x \cdot e[\bar{z}] = y[\bar{z}] \cdot (x \cdot e[\bar{z}])^{-1} \cdot x \cdot e[\bar{z}] \\ &= y[\bar{z}] \cdot (x[\bar{z}])^{-1} \cdot x[\bar{z}]. \end{aligned}$$

So $x[\bar{z}] \leqslant y[\bar{z}]$ and $\leqslant$ is l-regular. □

Proposition 2.3.11. *In an inverse Menger algebra the relation $\sqsubset$ is an l-regular quasi-order that contains $\leqslant$.*

Proof. Since $g^{-1}g \leqslant g^{-1}g$ for every element $g \in G$, then $g \sqsubset g$. So $\sqsubset$ is reflexive. Let $x \sqsubset y$ and $y \sqsubset z$, i.e., $x^{-1}x \leqslant y^{-1}y$ and $y^{-1}y \leqslant z^{-1}z$, whence $x^{-1}x \leqslant z^{-1}z$, i.e., $x \sqsubset z$. Thus $\sqsubset$ is a quasi-order.

Suppose that $x \sqsubset y$, i.e., $x^{-1}x \leqslant y^{-1}y$, therefore $xx^{-1}x \leqslant xy^{-1}y$, i.e., $x \leqslant xy^{-1}y$. Since $y^{-1}y$ is an idempotent, $xy^{-1}y \leqslant x$. Thus $x = xy^{-1}y$. We have proved that

$$x \sqsubset y \longleftrightarrow x = xy^{-1}y.$$

Let $e = y^{-1}y$, then, according to the equality $x = xy^{-1}y$,

$$\begin{aligned} x[\bar{z}] &= xy^{-1}y[\bar{z}] = xy^{-1}y \cdot e[\bar{z}] \\ &= xy^{-1}y \cdot e[\bar{z}] \cdot (e[\bar{z}])^{-1} \cdot e[\bar{z}] = x \cdot e[\bar{z}] \cdot (e[\bar{z}])^{-1} \cdot y^{-1}y \cdot e[\bar{z}] \\ &= xy^{-1}y[\bar{z}] \cdot (yy^{-1}y[\bar{z}])^{-1} \cdot yy^{-1}y[\bar{z}] = x[\bar{z}] \cdot (y[\bar{z}])^{-1} \cdot y[\bar{z}]. \end{aligned}$$

Thus $x[\bar{z}] \sqsubset y[\bar{z}]$ and $\sqsubset$ is l-regular.

Now let $x \leqslant y$, i.e., $y^{-1}x = x^{-1}x$, whence

$$y^{-1}yy^{-1}x = y^{-1}yx^{-1}x,$$

therefore $y^{-1}x = y^{-1}yx^{-1}x$. Hence $x^{-1}x = y^{-1}yx^{-1}x$, i.e., $x^{-1}x \leqslant y^{-1}y$. The last inequality means that $x \sqsubset y$. So $\leqslant$ is contained in $\sqsubset$. □

2.4 *i*-solvable Menger algebras

Attempts to transfer the classical notion of groups to the case of n-ary algebras were initiated by various authors many years ago. There are two main directions of such transformation. The first is connected with a generalization of the associativity, the second — with a generalization of solvability of some equations. Menger algebras satisfying the n-ary associativity in the sense of Post [144] are considered, e.g., in [28]. Menger algebras in which the equation $a_0[a_1 \cdots a_{i-1}x_ia_{i+1} \cdots a_n] = b$ has a unique solution x_i for all or only for some $i = 0, 1, \ldots, n$ are investigated by many authors in many directions (see for example [6, 7, 27, 186, 197, 227, 261]).

In this section we describe Menger algebras in which the above equation has a unique solution at some fixed places, in the next — Menger algebras in which the above equation has a unique solution for any i.

Definition 2.4.1. A Menger algebra (G, o) of rank n is called *i-solvable* if the following two equations

$$x[a_1 \cdots a_n] = b \tag{2.4.1}$$

$$a_0[a_1 \cdots a_{i-1}x_ia_{i+1} \cdots a_n] = b \tag{2.4.2}$$

have unique solutions $x, x_i \in G$ for all $a_0, a_1, \ldots, a_n, b \in G$.

A simple example of i-solvable Menger algebras is a Menger algebra (G, o_i) with an $(n+1)$-ary operation $o_i(x_0, x_1, \ldots, x_n) = x_0 + x_i$ defined on a nontrivial commutative group $(G, +)$. The diagonal semigroup of this algebra coincides with the group $(G, +)$. This algebra is j-solvable only for $j = i$, but the algebra (G, o), where $o(x_0, x_1, \ldots, x_n) = x_0 + x_1 + \cdots + x_{k+1}$ and $(G, +)$ is a commutative group of the exponent $k \leqslant n - 1$, is i-solvable for every $i = 1, \ldots, k + 1$. Its diagonal semigroup also coincides with $(G, +)$. The set $\mathbb{Z}_{2n+1}$ of integers modulo $2n + 1$ with the $(2n + 1)$-ary operation

$$o(x_0, x_1, \ldots, x_{2n}) = x_0 - x_1 + \cdots + x_{2n-2} - x_{2n-1} + 2x_{2n} \pmod{(2n+1)}$$

is an example of a Menger algebra which is i-solvable for any $i = 1, \ldots, 2n$. Similarly the set $\mathbb{Z}_{2n}$ with the operation

$$o(x_0, x_1, \ldots, x_{2n-1}) = x_0 + x_1 + x_2 - x_3 + \cdots + x_{2n-2} - x_{2n-1} \pmod{2n}.$$

Lemma 2.4.2. *Any i-solvable Menger algebra has a diagonal unit.*

Proof. According to the definition, in any i-solvable Menger algebra (G, o) for every $a, x \in G$ there are uniquely determined elements e and $\breve{x}$ such that

$$a = e[a^n] \quad \text{and} \quad x = a[a^{i-1}\breve{x}a^{n-i}].$$

Hence we have

$$x = a[a^{i-1}\breve{x}a^{n-i}] = e[a^n][a^{i-1}\breve{x}a^{n-i}] = e[[a^{i-1}\breve{x}a^{n-i}]\cdots[a^{i-1}\breve{x}a^{n-i}]] = e[x^n],$$

i.e., $x = e[x^n]$ for all $x \in G$. Thus e is a left diagonal unit.

Now, using this fact, we obtain

$$x[x^n] = x[e[x^n]\cdots e[x^n]] = x[e^n][x^n],$$

which, by the uniqueness of the solution at the first place, implies $x = e[x^n]$ for all $x \in G$. So e is also a right diagonal unit of (G, o). □

Proposition 2.4.3. *A diagonal semigroup of an i-solvable Menger algebra is a group.*

Proof. By Lemma 2.4.2, a diagonal semigroup $(G, \cdot)$ of an i-solvable Menger algebra (G, o) has the unit e. According to the definition, for every $a \in G$ there exists only one element $x \in G$ such that $e = x \cdot a = x[a^n]$. Denote this element by a^{-1}. Let y be the solution of the equation

$$a[a^{-1}\cdots a^{-1}\, y\, a^{-1}\cdots a^{-1}] = e,$$

where y appears at the place $i + 1$. Then, by Lemma 2.1.3, we have

$$a = e \cdot a = a[a^{-1}\cdots a^{-1}\, y\, a^{-1}\cdots a^{-1}] = a[e\cdots e\,(y \cdot a)\, e\cdots e],$$

which, by the uniqueness of the solution at the place $i + 1$, (see (2.4.2)) gives $y \cdot a = e$. So $y = a^{-1}$ and $a \cdot a^{-1} = e$. This means that $(G, \cdot)$ is a group. □

Corollary 2.4.4. *A group* $(G, \cdot)$ *is a diagonal group of some* i*-solvable Menger algebra if and only if on* G *can be defined an* n*-ary operation* f *satisfying the conditions* (a) *and* (b) *of the Proposition* 2.1.4 *for which the equation*

$$f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = b$$

has a unique solution for all $a_1, \ldots, a_n, b \in G$.

In [29] the following characterization of i-solvable Menger algebras is given.

Theorem 2.4.5. *A Menger algebra of rank n is i-solvable if and only if there exists an element $b \in G$ such that the equations* (2.4.1) *and* (2.4.2) *have unique solutions for all $a_0, \ldots, a_n \in G$.*

Proof. Since, by definition, in every i-solvable Menger algebra of rank n the equations (2.4.1) and (2.4.2) have unique solutions, we prove the converse. For this assume that these equations have unique solutions for all $a_0, \ldots, a_n \in G$ and some $b \in G$. Then for all $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n \in G$ there exists an element $b_i \in G$ such that $a = b[b_1 \cdots b_n]$. On the other hand, for all $a_j \in G$ and $c_j = a_j[b_1 \cdots b_n]$, $j = 1, \ldots, n$, there exists a uniquely determined element $x_1 \in G$ such that $a = x_1[c_1 \cdots c_n]$. Hence

$$a = x_1[c_1 \cdots c_n] = x_1[a_1[b_1 \cdots b_n] \cdots a_n[b_1 \cdots b_n]] = x_1[a_1 \cdots a_n][b_1 \cdots b_n],$$

which gives

$$a = b[b_1 \cdots b_n] = x_1[a_1 \cdots a_n][b_1 \cdots b_n].$$

This, by the uniqueness of the solution of the equation $a = x[b_1 \cdots b_n]$, implies $b = x_1[a_1 \cdots a_n]$. Hence, the equation (2.4.1) is uniquely solvable for all $b, a_1, \ldots, a_n \in G$.

Also (2.4.2) has a unique solution for all $b, a_1, \ldots, a_n \in G$. Indeed, if a, b, a_j, b_j, c_j, where $j = 1, \ldots, n$, are the same as in the previous part of this proof, then

$$\begin{aligned} a &= a_1[c_2 \cdots c_i\, y\, c_{i+1} \cdots c_n] \\ &= a_1[a_2[b_1 \cdots b_n] \cdots a_i[b_1 \cdots b_n]\, y\, a_{i+1}[b_1 \cdots b_n] \cdots a_n[b_1 \cdots b_n]] \end{aligned}$$

for some $y \in G$. By the above, there exists $x \in G$ such that $y = x[b_1 \cdots b_n]$, whence

$$\begin{aligned} a &= a_1[a_2[b_1 \cdots b_n] \cdots a_i[b_1 \cdots b_n]\; x[b_1 \cdots b_n]\; a_{i+1}[b_1 \cdots b_n] \cdots a_n[b_1 \cdots b_n]] \\ &= a_1[a_2 \cdots a_i\, x\, a_{i+1} \cdots a_n][b_1 \cdots b_n]. \end{aligned}$$

Therefore,

$$b[b_1 \cdots b_n] = a_1[a_2 \cdots a_i\, x\, a_{i+1} \cdots a_n][b_1 \cdots b_n],$$

which gives $b = a_1[a_2 \cdots a_i\, x\, a_{i+1} \cdots a_n]$. Hence (2.4.2) has a solution. This solution is unique, because, if also $a_1[a_2 \cdots a_i\, z\, a_{i+1} \cdots a_n] = b$ for some $z \in G$, then from the above $z[b_1 \cdots b_n] = x[b_1 \cdots b_n]$ and, by the first part of this proof, $x = z$. □

We remark that the uniqueness of solutions of (2.4.1) and (2.4.2) cannot be dropped. A nonempty set G with the operation $o(x_0, \ldots, x_n) = a$, where $a \in G$ is fixed, is a simple example of a Menger algebra of rank n in which the equations (2.4.1) and (2.4.2) are solvable, but (G, o) is not i-solvable. The uniqueness of solutions can be omitted in the case of finite algebras. In this case we can also prove the following theorem.

Theorem 2.4.6. *A finite Menger algebra (G, o) of rank n is i-solvable if and only if it has a left diagonal unit e and for all $a_1, \ldots, a_n \in G$ there exist $x, y \in G$ such that*

$$x[a_1 \cdots a_n] = a_0[a_1 \cdots a_{i-1} y\, a_{i+1} \cdots a_n] = e.$$

Proof. In any i-solvable Menger algebra (G, o) for all $e, a_1, \ldots, a_n \in G$ there exist x and y satisfying these two equations. By Lemma 2.4.2, such an algebra has also a left diagonal unit.

To prove the converse, assume that a Menger algebra (G, o) satisfies the conditions formulated in the second part of the above theorem. Then, according to the assumption, for any $b \in G$ there exists $x \in G$ satisfying the equation $e = x[b^n]$. Moreover, for this x one can find $y \in G$ such that $e = y[x^n]$. Hence

$$b = e[b^n] = y[x^n][b^n] = y[x[b^n] \cdots x[b^n]] = y[e^n],$$

i.e., $b = y[e^n]$. Since for all $a_1, \ldots, a_n \in G$ there exists $z \in G$ satisfying the equation $e = z[a_1 \cdots a_n]$, then

$$b = y[e^n] = y[z[a_1 \cdots a_n] \cdots z[a_1 \cdots a_n]] = y[z^n][a_1 \cdots a_n].$$

This means that for all $a_1, \ldots, a_n, b \in G$ and some $u \in G$ (namely for $u = y[z^n]$) we have $b = u[a_1 \cdots a_n]$. Thus (2.4.1) has a solution.

Also (2.4.2) has a solution for all $a_1, \ldots, a_n, b \in G$. Indeed, by the assumption, for any $b \in G$ there exists $u \in G$ such that $e = u[b^n]$ and for any sequence of elements $a_1, \ldots, a_n \in G$ there exists $z \in G$ such that

$$a = u[a_1 \cdots a_1][a_2 \cdots a_i\, z\, a_{i+1} \cdots a_n].$$

Hence

$$u[b^n] = e = u[a_1[a_2 \cdots a_i\, z\, a_{i+1} \cdots a_n] \cdots a_1[a_2 \cdots a_i\, z\, a_{i+1} \cdots a_n]],$$

which, by Lemma 2.1.2, implies $b = a_1[a_2 \cdots a_i\, z\, a_{i+1} \cdots a_n]$. So, (2.4.2) has a solution for all $a_1, \ldots, a_n, b \in G$.

These solutions are unique because the set G is finite. □

Lemma 2.4.7. *Any i-solvable Menger algebra is v-regular.*

Proof. By Proposition 2.4.3, the diagonal semigroup $(G, \cdot)$ of an i-solvable Menger algebra (G, o) is a group. According to the definition of i-solvability, for all $g_1, \ldots, g_n \in G$ and $j = 1, \ldots, n$ there exists $u_j \in G$ such that $g_j = u_j[g_1 \cdots g_n] = u_j[\bar{g}]$. Since $(G, \cdot)$ is a group, for u_j there exists a uniquely determined $x_j \in G$ for which $g_j \cdot x_j = u_j$. So,

$$g_j = u_j[\bar{g}] = (g_j \cdot x_j)[\bar{g}] = g_j \cdot x_j[\bar{g}],$$

by Lemma 2.1.3. Hence $x_j[\bar{g}]$ is the unit of the group $(G, \cdot)$ (for every $j = 1, \ldots, n$). Therefore $x_1 = \cdots = x_n = x$, by the uniqueness of the solution at the first place. Thus $g_j = g_j \cdot x[\bar{g}] = g_j[x[\bar{g}] \cdots x[\bar{g}]] = g_j[x^n][\bar{g}]$ for every $j = 1, \ldots, n$ and all $g_1, \ldots, g_n \in G$. □

Theorem 2.4.8. *A Menger algebra (G, o) of rank n is i-solvable if and only if it is v-regular, has only one idempotent $e \in G$ and the equation*

$$e[a_1 \cdots a_{i-1}\ x_i\ a_{i+1} \cdots a_n] = e \tag{2.4.3}$$

is uniquely solvable for all $a_1, \ldots, a_n \in G$.

Proof. If a Menger algebra (G, o) of rank n is i-solvable, then, according to Proposition 2.4.3, its diagonal semigroup is a group. Hence (G, o) has only one idempotent and (2.4.3) has a unique solution for all $a_1, \ldots, a_n \in G$.

On the contrary, if a Menger algebra (G, o) satisfies all conditions formulated in the second part of the theorem, then, according to Theorem 2.3.9, its diagonal semigroup is a group. Let $a_0, a_1, \ldots, a_n, b$ be an arbitrary sequence of elements of G. We show that the equations (2.4.1) and (2.4.2) have unique solutions. Indeed, the equation $x[a_1 \cdots a_n] = b$ is equivalent to $x \cdot e[a_1 \cdots a_n] = b$, where e is the unit of the diagonal group $(G, \cdot)$. It is clear that the last equation has a unique solution x in $(G, \cdot)$. So (2.4.1) is uniquely solvable in (G, o).

Now we consider the equation (2.4.2), i.e., the equation

$$a_0[a_1 \cdots a_{i-1}\ x_i\ a_{i+1} \cdots a_n] = b,$$

where i is as in (2.4.3). This equation is equivalent to

$$a_0 \cdot e[a_1 \cdots a_{i-1}\ x_i\ a_{i+1} \cdots a_n] = b,$$

i.e., to

$$e[a_1 \cdots a_{i-1}\ x_i\ a_{i+1} \cdots a_n] = a_0^{-1} \cdot b,$$

where a_0^{-1} is the inverse element of a_0 in $(G, \cdot)$. Multiplying the last equation by $b^{-1} \cdot a_0$ and using Lemma 2.1.3 we get

$$e[(a_1 \cdot b^{-1} \cdot a_0) \cdots (a_{i-1} \cdot b^{-1} \cdot a_0)\ (x_i \cdot b^{-1} \cdot a_0)\ (a_{i+1} \cdot b^{-1} \cdot a_0) \cdots (a_n \cdot b^{-1} \cdot a_0)] = e.$$

But, according to (2.4.3), in G there exists precisely one y such as

$$e[(a_1 \cdot b^{-1} \cdot a_0) \cdots (a_{i-1} \cdot b^{-1} \cdot a_0)\ y\ (a_{i+1} \cdot b^{-1} \cdot a_0) \cdots (a_n \cdot b^{-1} \cdot a_0)] = e.$$

So $y = x_i \cdot b^{-1} \cdot a_0$. In the group $(G, \cdot)$ the equation $y = x_i \cdot b^{-1} \cdot a_0$ has a unique solution x_i. Thus, also (2.4.2) has a unique solution. This proves that (G, o) is i-solvable. □

On the diagonal semigroup $(G, \cdot)$ of a Menger algebra (G, o) of rank n with the diagonal unit e we can define a new $(n-1)$-ary operation f putting

$$f(a_1, \ldots, a_{n-1}) = e[a_1 \cdots a_{n-1} e] \tag{2.4.4}$$

for all $a_1, ..., a_{n-1} \in G$. The diagonal semigroup with such an operation f is called the *rigged diagonal semigroup* of (G, o) and is denoted by $(G, \cdot, f)$. It is not difficult to see that in the case when $(G, \cdot)$ is a group, the operation f satisfies the condition

$$a[a_1 \cdots a_n] = a \cdot f(a_1 \cdot a_n^{-1}, \ldots, a_{n-1} \cdot a_n^{-1}) \cdot a_n, \tag{2.4.5}$$

where a_n^{-1} is the inverse of a_n in the group $(G, \cdot)$.

Theorem 2.4.9. *A Menger algebra (G, o) of rank n is i-solvable for some $1 \leqslant i < n$ if and only if in its rigged diagonal group $(G, \cdot, f)$ the equation*

$$f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_{n-1}) = a_n$$

has a unique solution $x \in G$ for all $a_1, \ldots, a_n \in G$.

Proof. If $(G, \cdot, f)$ is the rigged diagonal group of a Menger algebra (G, o), then

$$b = x[a_1 \cdots a_n] = x \cdot f(a_1 \cdot a_n^{-1}, \ldots, a_{n-1} \cdot a_n^{-1}) \cdot a_n$$

has a unique solution $x \in G$, because $(G, \cdot)$ is a group. Similarly, the solvability of the equation

$$\begin{aligned} b &= a_0[a_1 \cdots a_{i-1}\, x\, a_{i+1} \cdots a_n] \\ &= a_0 \cdot f(a_1 \cdot a_n^{-1}, \ldots, a_{i-1} \cdot a_n^{-1},\ x \cdot a_n^{-1},\ a_{i-1} \cdot a_n^{-1}, \ldots, a_{n-1} \cdot a_n^{-1}) \cdot a_n \end{aligned}$$

is equivalent to the solvability of the equation

$$f(a_1 \cdot a_n^{-1}, \ldots, a_{i-1} \cdot a_n^{-1},\ x \cdot a_n^{-1},\ a_{i-1} \cdot a_n^{-1}, \ldots, a_{n-1}) = a_0^{-1} \cdot b \cdot a_n.$$

But the last equation is uniquely solvable by assumption.

The converse statement is obvious. □

Theorem 2.4.10. *A Menger algebra (G, o) of rank n is n-solvable if and only if in its rigged diagonal group $(G, \cdot, f)$ for all $a_1, \ldots, a_{n-1} \in G$ there exists exactly one element $x \in G$ such that*

$$f(a_1 \cdot x, \ldots, a_{n-1} \cdot x) = a_n \cdot x. \tag{2.4.6}$$

Proof. Let a Menger algebra (G, o) of rank n be n-solvable. Then, as it was mentioned above, its diagonal semigroup is a group and

$$a_n \cdot x = f(a_1 \cdot x, \ldots, a_{n-1} \cdot x) = e[(a_1 \cdot x) \cdots (a_{n-1} \cdot x)\, e],$$

where $(G, \cdot, f)$ is the rigged diagonal group of (G, o). Hence, by Lemma 2.1.3,

$$a_n = e[(a_1 \cdot x) \cdots (a_{n-1} \cdot x)\, e] \cdot x^{-1} = e[a_1 \cdots a_{n-1}\, x^{-1}\,].$$

Therefore, there exists an element $x \in G$ which is uniquely determined.

On the other hand, if $(G, \cdot, f)$ is the rigged diagonal group of (G, o), then, by (2.4.5), the equation $b = x[a_1 \cdots a_n]$ has a unique solution $x \in G$. The equation $a_0[a_1 \cdots a_{n-1} x_n] = b$ is equivalent to

$$a_0 \cdot f(a_1 \cdot x_n^{-1}, \ldots, a_{n-1} \cdot x_n^{-1}) \cdot x_n = b$$

and, consequently, to

$$f(a_{1.} \cdot x_n^{-1}, \ldots, a_{n-1} \cdot x_n^{-1}) = a_0 \cdot b \cdot x_n^{-1},$$

which is uniquely solvable by assumption. This proves the n-solvability of (G, o). □

Corollary 2.4.11. *The group $(G, \cdot)$ is the diagonal group of some n-solvable Menger algebra of rank n if and only if we can define on G an $(n-1)$-ary operation f such that $f(e, \ldots, e) = e$ for the unit of $(G, \cdot)$ and the equation* (2.4.6) *has a unique solution $x \in G$ for all $a_1, \ldots, a_n \in G$.*

Proof. If $(G, \cdot)$ is the diagonal group of an n-solvable Menger algebra of rank n, then, according to the above theorem, the operation f from the corresponding rigged diagonal group satisfies these conditions.

Conversely, let define on the group $(G, \cdot)$ an $(n-1)$-ary operation f satisfying all these conditions. Then G with an $(n+1)$-ary operation $o : (x, y_1, \ldots, y_n) \mapsto x[y_1 \cdots y_n]$ such that

$$x[y_1 \cdots y_n] = x \cdot f(y_1 y_n^{-1}, \ldots, y_{n-1} y_n^{-1}) \cdot y_n,$$

where $y_k y_n^{-1}$ means $y_k \cdot y_n^{-1}$, is a Menger algebra of rank n because for all $x, y_1, \ldots, y_n, z_1, \ldots, z_n \in G$ we have

$$\begin{aligned}
&x[y_1[z_1 \cdots z_n] \cdots y_n[z_1 \cdots z_n]] \\
&\quad = x[\, y_1 f(z_1 z_n^{-1}, \ldots, z_{n-1} z_n^{-1}) z_n \ \cdots \ y_n f(z_1 z_n^{-1}, \ldots, z_{n-1} z_n^{-1}) z_n \,] \\
&\quad = x \cdot f\Big((y_1 f(z_1 z_n^{-1}, \ldots, z_{n-1} z_n^{-1}) z_n) \cdot (y_n f(z_1 z_n^{-1}, \ldots, z_{n-1} z_n^{-1}) z_n)^{-1}, \\
&\qquad \ldots, (y_{n-1} f(z_1 z_n^{-1}, \ldots, z_{n-1} z_n^{-1}) z_n) \cdot (y_n f(z_1 z_n^{-1}, \ldots, z_{n-1} z_n^{-1}) z_n)^{-1} \Big) \\
&\qquad \cdot y_n f(z_1 z_n^{-1}, \ldots, z_{n-1} z_n^{-1}) z_n \\
&\quad = x \cdot f(y_1 y_n^{-1}, \ldots, y_{n-1} y_n^{-1}) \cdot y_n f(z_1 z_n^{-1}, \ldots, z_{n-1} z_n^{-1}) z_n \\
&\quad = x[y_1 \cdots y_n][z_1 \cdots z_n].
\end{aligned}$$

The diagonal group of (G, o) coincides with the group $(G, \cdot)$. Indeed,

$$x[y \cdots y] = x \cdot f(yy^{-1}, \ldots, yy^{-1}) \cdot y = x \cdot f(e, \ldots, e) \cdot y = x \cdot e \cdot y = x \cdot y.$$

Moreover,

$$f(x_1, \ldots, x_{n-1}) = e[x_1 \cdots x_{n-1}\ e]$$

for all $x_1, \ldots, x_{n-1} \in G$. So, $(G, \cdot, f)$ is a rigged diagonal group of (G, o), and, by Theorem 2.4.10, (G, o) is an n-solvable Menger algebra of rank n. □

2.5 Group-like Menger algebras

In this section, we describe Menger algebras, which are i-solvable for any $i = 1, \ldots, n$, i.e., Menger algebras in which the equations (2.4.1) and (2.4.2) from the previous section have unique solutions for all $i = 1, \ldots, n$. Such algebras are called *group-like* and were introduced by H. L. Skala [197].

We start with the following example. Let $\mathbb{R}$ be the set of real numbers, n — a fixed natural number. For every $\alpha \in \mathbb{R}$ we define an n-place function f_α on $\mathbb{R}$ putting

$$f_\alpha(x_1, \ldots, x_n) = \frac{x_1 + \cdots + x_n}{n} + \alpha.$$

The set $F = \{f_\alpha \mid \alpha \in \mathbb{R}\}$ with the Menger composition of n-place functions is a group-like Menger algebra. Indeed, for all $f_\alpha, f_{\beta_1}, \ldots, f_{\beta_n} \in F$ and $x_1, \ldots, x_n \in \mathbb{R}$ we have

$$\begin{aligned} f_\alpha[f_{\beta_1} \cdots f_{\beta_n}](x_1, \ldots, x_n) &= f_\alpha(f_{\beta_1}(x_1, \ldots, x_n), \ldots, f_{\beta_n}(x_1, \ldots, x_n)) \\ &= f_\alpha\left(\frac{x_1 + \cdots + x_n}{n} + \beta_1, \ldots, \frac{x_1 + \cdots + x_n}{n} + \beta_n\right) \\ &= \frac{\left(\frac{x_1+\cdots+x_n}{n} + \beta_1\right) + \cdots + \left(\frac{x_1+\cdots+x_n}{n} + \beta_n\right)}{n} + \alpha \\ &= \frac{(x_1 + \cdots + x_n) + (\beta_1 + \cdots + \beta_n)}{n} + \alpha \\ &= \frac{x_1 + \cdots + x_n}{n} + \left(\frac{\beta_1 + \cdots + \beta_n}{n} + \alpha\right) \\ &= f_{\frac{\beta_1+\cdots+\beta_n}{n}+\alpha}(x_1, \ldots, x_n). \end{aligned}$$

Thus

$$f_\alpha[f_{\beta_1} \cdots f_{\beta_n}] = f_{\frac{\beta_1+\cdots+\beta_n}{n}+\alpha}, \tag{2.5.1}$$

which means that the set F is closed with respect to the Menger composition of n-place functions. So, (F, O) is a Menger algebra of rank n.

To prove that it is group-like consider the equation $f_x[f_{\alpha_1} \cdots f_{\alpha_n}] = f_\beta$, where $f_{\alpha_1}, \ldots, f_{\alpha_n}, f_\beta$ are arbitrary functions from F and f_x is the unknown function. According to (2.5.1), this equation is equivalent to

$$f_{\frac{\alpha_1+\cdots+\alpha_n}{n}+x} = f_\beta.$$

Hence

$$\frac{\alpha_1 + \cdots + \alpha_n}{n} + x = \beta$$

and

$$x = \beta - \frac{\alpha_1 + \cdots + \alpha_n}{n},$$

which proves that in F there exists only one function satisfying the above equation. Similarly we can verify that in F there is only one function f_x satisfying the equation

$$f_{\alpha_0}[f_{\alpha_1} \cdots f_{\alpha_{i-1}} f_x f_{\alpha_{i+1}} \cdots f_{\alpha_n}] = f_\beta.$$

Thus (F, O) is a group-like Menger algebra.

Since all functions from F are reversive, this example assures also the existence of group-like Menger algebras of reversive n-place functions.

It is not difficult to see that the algebra (F, O) is isomorphic to the Menger algebra $(\mathbb{R}, o)$, where

$$o(x, y_1, \ldots, y_n) = x[y_1 \cdots y_n] = x + \frac{y_1 + \cdots + y_n}{n}.$$

The basic properties of group-like Menger algebras are a consequence of the corresponding properties of i-solvable Menger algebras. For example, as a consequence of the results from the previous section we obtain

Proposition 2.5.1. *The diagonal semigroup of a group-like Menger algebra is a group.*

Corollary 2.5.2. *A group $(G, \cdot)$ is a diagonal group of some group-like Menger algebra (G, o) of rank n if and only if on G there exists an n-ary quasigroup operation satisfying the conditions* (a) *and* (b) *of the Proposition* 2.1.4.

The following theorem, which is a generalization of the result of H. Skala [197], is a consequence of our Theorem 2.4.5.

Theorem 2.5.3. *A Menger algebra (G, o) of rank n is group-like if and only if there exists an element $b \in G$ such that the equations* (2.4.1) *and* (2.4.2) *have unique solutions for all $a_0, \ldots, a_n \in G$ and $i = 1, \ldots, n$.*

Generally the uniqueness of the solutions of equations (2.4.1) and (2.4.2) cannot be dropped, but in the case of finite Menger algebras we can prove the following theorem (compare with Theorem 2.4.6).

Theorem 2.5.4. *A finite Menger algebra (G, o) of rank n is group-like if and only if it has a left diagonal unit e and for all $a_1, \ldots, a_n \in G$ and $i = 1, \ldots, n$ there exist $x, y_i \in G$ such that*

$$x[a_1 \cdots a_n] = a_0[a_1 \cdots a_{i-1} y\, a_{i+1} \cdots a_n] = e.$$

Similarly, as a consequence of our Theorem 2.4.8, we obtain the following result first proved in [238].

Theorem 2.5.5. *A Menger algebra (G, o) of rank n is group-like if and only if it is v-regular, has only one idempotent $e \in G$ and the equation*

$$e[a_1 \cdots a_{i-1}\ x_i\ a_{i+1} \cdots a_n] = e$$

is uniquely solvable for all $a_1, \ldots, a_n \in G$ and $i = 1, \ldots, n$.

Theorem 2.5.6. *A Menger algebra (G, o) of rank n is group-like if and only if its diagonal semigroup $(G, \cdot)$ is a group with unit e, the operation $f(a_1, \ldots, a_{n-1})$ $= e[a_1 \cdots a_{n-1} e]$ is a quasigroup operation and for all $a_1, \ldots, a_n \in G$ there exists exactly one $x \in G$ satisfying the equation*

$$f(a_1 \cdot x, \ldots, a_{n-1} \cdot x) = a_n \cdot x. \tag{2.5.2}$$

Proof. Let (G, o) be a group-like Menger algebra of rank n. Then its diagonal semigroup is a group and the equation

$$b = f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_{n-1}) = e[a_1 \cdots a_{i-1} x_i a_{i+1}, \ldots, a_{n-1} e]$$

is uniquely solvable for all $b, a_1, \ldots, a_{n-1} \in G$ and $i = 1, \ldots, n-1$. So (G, f) is an $(n-1)$-ary quasigroup. By Theorem 2.4.10, there exists only one $x \in G$ satisfying (2.5.2).

The converse statement is a consequence of Theorems 2.4.9 and 2.4.10. □

Corollary 2.5.7. *The group $(G, \cdot)$ is the diagonal group of some group-like Menger algebra of rank n if and only if on G can be defined an $(n-1)$-ary quasigroup operation f such that $f(e, \ldots, e) = e$ for the unit of the group $(G, \cdot)$ and the equation (2.5.2) has a unique solution $x \in G$ for all $a_1, \ldots, a_n \in G$.*

Proof. The proof is analogous to the proof of the last corollary in the previous section. □

Using condition (2.4.5), we can show that *two group-like Menger algebras of rank n are isomorphic only in the case when their rigged diagonal groups are isomorphic.*

That nonisomorphic group-like Menger algebras may have isomorphic diagonal groups can be seen in the following examples. On the set $\mathbb{Z}_5$ of integers modulo 5 we define three operations

$$o_1(x,y,z) = x - y + 2z \pmod 5,$$
$$o_2(x,y,z) = x - 2y + 3z \pmod 5,$$
$$o_3(x,y,z) = x - 3y + 4z \pmod 5.$$

It is not difficult to verify that $(\mathbb{Z}_5, o_1)$, $(\mathbb{Z}_5, o_2)$ and $(\mathbb{Z}_5, o_3)$ are non-isomorphic group-like Menger algebras with the same diagonal group.

Though groups of every finite order exist, this result is not true for group-like Menger algebras. For example, there exists no 2-place group-like Menger algebra of order $2p$, where p is an odd prime. Before showing this, we prove the following.

Theorem 2.5.8. *A finite group $(G,\diamond)$ is the diagonal group of a group-like Menger algebra of rank $n > 1$ if and only if there exists a reversive mapping φ of G^{n-1} onto G such that $\varphi(e,\ldots,e) = e$, where e is the unit of $(G,\diamond)$, and $\varphi(a_1 \diamond b,\ldots,a_{n-1}\diamond b) \neq \varphi(a_1,\ldots,a_{n-1})\diamond b$ for $b \neq e$.*

Proof. We first prove that the diagonal group of any group-like Menger algebra (G,o) satisfies the condition given in the theorem. It is clear that $\varphi(a_1,\ldots,a_{n-1}){=}e[ea_1\cdots a_{n-1}]$, where e is the unit of the diagonal group $(G,\cdot)$, is a reversive mapping and $\varphi(e,\ldots,e) = e$. Moreover,

$$\varphi(a_1\cdot b,\ldots,a_{n-1}\cdot b) = e[e\,(a_1\cdot b)\,\ldots\,(a_{n-1}\cdot b)] = e[(e\cdot b^{-1})\,a_1\cdots a_{n-1}]\cdot b.$$

Since $e[(e\cdot b^{-1})\,a_1\cdots a_{n-1}] \neq e[e\,a_1\cdots a_{n-1}]$ for $b \neq e$, it follows that $\varphi(a_1\cdot b,\ldots,a_{n-1}\cdot b) \neq \varphi(a_1,\ldots,a_{n-1})\cdot b$ for $b\neq e$.

Conversely, suppose that the finite group $(G,\diamond)$ satisfies the conditions of the theorem. We consider the $(n+1)$-ary operation

$$o(a_0,a_1,\ldots,a_n) = a_0\diamond\varphi(a_2\diamond a_1^{-1},\ldots,a_n\diamond a_1^{-1})\diamond a_1.$$

It can be easily verified by direct calculation that (G,o) is a Menger algebra. The equation

$$b = o(x_0,a_1,\ldots,a_n) = x_0\diamond\varphi(a_2\diamond a_1^{-1},\ldots,a_n\diamond a_1^{-1})\diamond a_1$$

has a unique solution $x_0 = b\diamond a_1^{-1}\diamond\big(\varphi(a_2\diamond a_1^{-1},\ldots,a_n\diamond a_1^{-1})\big)^{-1}$. Because G is finite, in order to prove there exist unique elements $x_1,\ldots,x_n\in G$ such that $o(a_1,\ldots,a_{i-1},x_i,a_{i+1},\ldots,a_n) = b$ it suffices to show that

$$\varphi(a_2\diamond x^{-1},\ldots,a_n\diamond x^{-1})\diamond x \neq \varphi(a_2\diamond y^{-1},\ldots,a_n\diamond y^{-1})\diamond y$$

and

$$\varphi(a_2 \diamond a_1^{-1}, \ldots, x \diamond a_1^{-1}, \ldots, a_n \diamond a_1^{-1}) \neq \varphi(a_2 \diamond a_1^{-1}, \ldots, y \diamond a_1^{-1}, \ldots, a_n \diamond a_1^{-1})$$

for $x \neq y$. Since φ is reversive, the second inequality is obvious. To verify the first suppose

$$\varphi(a_2 \diamond x^{-1}, \ldots, a_n \diamond x^{-1}) \diamond x = \varphi(a_2 \diamond y^{-1}, \ldots, a_n \diamond y^{-1}) \diamond y.$$

This, for $c_i = a_i \diamond x^{-1}$, gives

$$\varphi(c_2, \ldots, c_n) \diamond x \diamond y^{-1} = \varphi(c_2 \diamond x \diamond y^{-1}, \ldots, c_n \diamond x \diamond y^{-1}),$$

which, by the last condition formulated in the theorem, implies $x \diamond y^{-1} = e$, that is $x = y$. Hence (G, o) is group-like. Moreover,

$$a \cdot b = o(a, b, \ldots, b) = a \diamond \varphi(e, \ldots, e) \diamond b = a \diamond b,$$

proves that $(G, \diamond)$ is the diagonal group of the above algebra (G, o). □

Using this theorem, we can solve the problem of existence of finite group-like Menger algebras of rank n. At first observe that for $n = 2k + 1$ the mapping $\varphi(a_1, \ldots, a_{2k}) = a_1 \cdot a_2^{-1} \cdot \ldots \cdot a_{2k-1} \cdot a_{2k}^{-1}$ satisfies the conditions of Theorem 2.5.8. So, for odd n there exist group-like Menger algebras of rank n of an arbitrary finite order. If $n = 2k$ and the order of $(G, \cdot)$ is odd, then the mapping $\varphi(a_1, \ldots, a_{2k-1}) = a_1 \cdot a_2^{-1} \cdot \ldots \cdot a_{2k-3} \cdot a_{2k-2}^{-1} \cdot a_{2k-1}^2$ satisfies the conditions of Theorem 2.5.8. Hence, also in this case there exists a group-like Menger algebra, but only of an arbitrary odd order.

If both n and the order of $(G, \cdot)$ are even there need not necessarily exist a group-like Menger algebra of rank n whose diagonal group is $(G, \cdot)$. For example, the cyclic group of even order and the dihedral groups of order twice an odd integer cannot be the diagonal group of any group-like Menger algebra of rank 2. Indeed, if $(G, \cdot)$ is one of the above groups and a mapping φ satisfies the conditions of the last theorem, then $\varphi(a) \neq \varphi(b)$ for $a \neq b$ and $\varphi(ab) \neq \varphi(a)b$ for $b \neq e$. Putting $c = ab$ in the second condition, we obtain $\varphi(c)c^{-1} \neq \varphi(a)a^{-1}$ for $a \neq c$. Hence the sets $\{\varphi(x) | x \in G\}$ and $\{\varphi(x)x^{-1} | x \in G\}$ are both equal to G itself. Hence if $G = \{a_1, \ldots, a_{2k}\}$, the product $a_1 a_2 \cdots a_{2k} = b$ equals the product of $\varphi(a_1)a_1^{-1}, \ldots, \varphi(a_{2k})a_{2k}^{-1}$ in some order. But since $G = \{\varphi(x) | \, x \in G\}$ $= \{x^{-1} | \, x \in G\}$, therefore b must be equal to the product in some order of all the elements of G taken twice. If G is the cyclic group of order $2k$ generated, for example, by a_1, the product of all the elements of G is equal to a_1^k, while the product of all the elements of G taken twice is $a_1^{2k} = e \neq a_1$. If $(G, \cdot)$ is the dihedral group of order $2k$, where k is odd, then an element of this group can be written in the form $a_1^s a_2^t$, where a_1 and a_2 are generators of $(G, \cdot)$ such that

$a_1^k = a_2^2 = e$ and $0 \leqslant s < k$, $0 \leqslant t < 2$. Since k is odd the product of all the elements of G in any order is of the form $a_1^i a_2$ for some integer i. The product of all elements of G taken twice and in any order is of the form a_1^j for some j. Hence neither of the above groups can be the diagonal group of a group-like Menger algebra.

Since the only groups of order $2p$, where p is a prime, are the cyclic and the dihedral group, it follows that there exist no group-like Menger algebras of order $2p$, where p is an odd prime.

However, group-like Menger algebras of rank 2 of every other order do exist. First, let B be the Boolean group of order 2^k and let $a_1, \ldots, a_k$ be its generators. Consider the sequence $x_1, x_2, \ldots, x_{2^k-1}$ of elements of B defined in the following way: $x_i = a_i$ for $i = 1, \ldots, k$ and $x_i = x_{i-k}x_{i-k+1}$ for $i > k$. The elements x_i are all distinct and different from the identity. Moreover, for $k > 1$, the mapping $\varphi(e) = e$, $\varphi(x_i) = x_{i+1}$, $\varphi(x_{2^k-1}) = x_1$ satisfies the conditions of Theorem 2.5.8. Thus for every $k > 1$, there exists a group-like Menger algebra of rank 2 and order 2^k.

Since the direct product of two group-like Menger algebras is also a group-like Menger algebra of the same rank, the existence of group-like Menger algebras of rank 2 and order 2^k implies the existence of group-like Menger algebras of rank 2 and order $2^k m$, where m is odd and $k > 1$.

Finally, let (G, o) be a group-like Menger algebra of rank n and order m. Let φ be the mapping from Theorem 2.5.8 associated with (G, o). The mapping $\psi(a_1, a_2, \ldots, a_{n+1}) = a_1 a_2^{-1} \varphi(a_3, \ldots, a_{n+1})$ also satisfies the condition of Theorem 2.5.8, which means that there exists a group-like Menger algebra of rank $n+2$ and order m. From the above, it follows by induction that there exists a group-like Menger algebra of rank n and every order $2^k m$, where $k > 1$. Summarizing these results we obtain the following.

Theorem 2.5.9. *There exists a group-like Menger algebra of rank n of every finite order if n is odd; if n is even there exist group-like Menger algebras of any finite order except those of the form $2p$, where p is an odd prime. There are no group-like Menger algebras of rank 2 and finite order $2p$.*

The existence of group-like Menger algebras of order $2p$ and even rank n greater than 2 is as yet undecided.

For group-like Menger algebras, the following implications are obvious:

$$u[\bar{w}|_i x] = u[\bar{w}|_i y] \longrightarrow x = y, \tag{2.5.3}$$

$$x[z_1 \cdots z_n] = y[z_1 \cdots z_n] \longrightarrow x = y. \tag{2.5.4}$$

This means that every elementary translation of a group-like Menger algebra (G, o) is a permutation of the set G. At the beginning of this section we considered an example of a group-like Menger algebra, which elements were n-place reversive functions. It appears that *every group-like Menger algebra of rank n is isomorphic to some group-like Menger algebra of reversive n-place functions.* Indeed, if (G, o) is a group-like Menger algebra of rank n, then the mapping $P : g \mapsto \lambda_g$, as it follows from the proof of Theorem 2.1.7, is an isomorphism[2] of (G, o) onto a group-like Manger algebra (Λ, O), where $\Lambda = \{\lambda_g \mid g \in G\}$. The fact that every λ_g is a reversive n-place function is a consequence of the above implications.

Let (G, o) be a group-like Menger algebra of rank n. According to [227] the subset H of G is called a *T-subgroup* of the diagonal group $(G, \cdot)$, if

$$x, y \in H \longrightarrow t(x)^{-1} \cdot t(y) \in H, \tag{2.5.5}$$

and a *LT-subgroup*, if

$$x, y \in H \longrightarrow t(x[\bar{z}])^{-1} \cdot t(y[\bar{z}]) \in H \tag{2.5.6}$$

for all $x, y \in G$, $\bar{z} \in G^n$, $t \in T_n(G)$, where x^{-1} is the inverse element of x in the diagonal group. Putting $t = \triangle_G$ and $\bar{z} = (e, \ldots, e)$ in (2.5.5) and (2.5.6), where e is the unit of the diagonal group, we see that *every T- and LT-subgroup of $(G, \cdot)$ is its usual subgroup.* Moreover, *every T-subgroup of the diagonal group of (G, o) is a subalgebra of (G, o).* And similarly for every LT-subgroup. Indeed, let H be a T-subgroup of $(G, \cdot)$ and $a_0, a_1, \ldots, a_n \in H$. Choose the polynomial $t_1(x) = a_0[xa_0 \cdots a_0] \in T_n(G)$. As $a_0, a_1 \in H$, then, according to (2.5.5), we get $a_0[a_0a_0 \cdots a_0]^{-1} \cdot a_0[a_1a_0 \cdots a_0] \in H$. Hence $a_0[a_1a_0 \cdots a_0] \in a_0[a_0 \cdots a_0] \cdot H = a_0a_0H \subset H$. Next we choose the polynomial $t_2(x) = a_0[a_1xa_0 \cdots a_0] \in T_n(G)$ and, as in the previous case, from $a_0, a_2 \in H$ we conclude $a_0[a_1a_2a_0 \cdots a_0] \in H$. Continuing with this procedure, after n steps we obtain $a_0[a_1a_2 \cdots a_n] \in H$, which proves that H is a subalgebra of (G, o).

Theorem 2.5.10. *For a group-like Menger algebra (G, o) of rank n the following statements are equivalent:*
(a) *is an LT-subgroup of the diagonal group $(G, \cdot)$,*
(b) *H is a subgroup of $(G, \cdot)$ such that*

$$x \cdot y^{-1} \in H \longrightarrow t(x)^{-1} \cdot t(y) \in H \tag{2.5.7}$$

for all $x, y \in G$, $t \in T_n(G)$,
(c) *H is a subgroup of $(G, \cdot)$ such that*

$$\bigcap_{t \in T_n(G)} t^{-1}(t(x) \cdot H) = Hx \tag{2.5.8}$$

for all $x \in G$, where t^{-1} is an inverse translation of $t \in T_n(G)$,

[2] In the proof that P is one-to-one instead of selectors we must use the diagonal unit of (G, o).

(d) *H is a normal subgroup of* $(G,\cdot)$ *such that*

$$\bigcap_{t\in T_n(G)} t^{-1}(t(x)\cdot H) = xH \tag{2.5.9}$$

for every $x\in G$.

Proof. (a)⟶ (b). Let H be a LT-subgroup of the diagonal group of (G,o). If the assumption of (2.5.7) is satisfied, then, according to (2.5.6), from $x\cdot y^{-1}\in H$ and $e\in H$ we get $t(x\cdot y^{-1}[\bar z])^{-1}\cdot t(e[\bar z])\in H$ for all $t\in T_n(G)$ and $\bar z\in G^n$. This applied to $\bar z=(y,\dots,y)$ gives $t(x\cdot y^{-1}\cdot y)^{-1}\cdot t(e\cdot y)\in H$, i.e., $t(x)^{-1}\cdot t(y)\in H$, which proves (b).

(b)⟶ (c). Now let (b) be satisfied and $y\in t^{-1}(t(x)\cdot H)$ for all $t\in T_n(G)$. Then $t(y)\in t(x)\cdot H$ for all $t\in T_n(G)$. This gives, when $t=\triangle_G$, $y\in xH$, i.e., $x^{-1}y\in H$. Hence $xy^{-1}\in H$ by (2.5.7). Thus, $yx^{-1}=(xy^{-1})^{-1}\in H$, i.e., $y\in Hx$. On the contrary, if $y\in Hx$, then also $xy^{-1}\in H$ and, according to (2.5.7), $t(x)^{-1}\cdot t(y)\in H$ for every $t\in T_n(G)$. Since t is one-to-one, the last relation means that $y\in t^{-1}(t(x)\cdot H)$ for all $t\in T_n(G)$. So, the equality (2.5.8) is satisfied.

(c)⟶ (d). It is not difficult to see that (2.5.8) implies $xH=Hx$. So, H is a normal subgroup of $(G,\cdot)$ and (2.5.9) is true.

(d)⟶ (a). If (d) holds and $x,y\in H$, then $y\in Hx$, which, by Lemma 2.1.3, gives $y[\bar z]\in(Hx)[\bar z]=H\cdot x[\bar z]=x[\bar z]\cdot H$ for arbitrary $\bar z\in G^n$. So, we have $y[\bar z]\in t^{-1}(t(x[\bar z])\cdot H)$ according to (2.5.9). Thus $t(y[\bar z])\in t(x[\bar z])\cdot H$ for all $t\in T_n(G)$, $\bar z\in G^n$. Therefore $t(x[\bar z])^{-1}\cdot t(y[\bar z])\in H$, which means that H is a LT-subgroup of $(G,\cdot)$. □

We consider now the construction of (v-, l-) congruences on group-like Menger algebras. First note that *if* Θ *is a* (v-, l-) *congruence on a group-like Menger algebra* (G,o) *of rank* n*, then the* Θ*-class containing the unit of the diagonal group, is its* (T*-subgroup, subgroup*) LT*-subgroup.* Indeed, if $x,y\in\Theta\langle e\rangle$, where e is the unit of the diagonal group $(G,\cdot)$, then $x\equiv y(\Theta)$ and $t(x[\bar z])\equiv t(y[\bar z])(\Theta)$ for all $t\in T_n(G)$, $\bar z\in G^n$. Hence

$$t(x[\bar z])^{-1}\cdot t(x[\bar z])\equiv t(x[\bar z])^{-1}\cdot t(y[\bar z])(\Theta),$$

i.e., $t(x[\bar z])^{-1}\cdot t(y[\bar z])\in\Theta\langle e\rangle$. So, $\Theta\langle e\rangle$ is a LT-subgroup of $G(G,\cdot)$.

The other two statements can be proved analogously.

Theorem 2.5.11. *Every a* (v-, l-) *congruence* Θ *on a group-like Menger algebra of rank* n *is in one-to-one correspondence with an* (T*-subgroup, subgroup*) LT*-subgroup* H *of its diagonal group, for which the equality* ($\Theta=\varepsilon_v(H)$, $\Theta=\varepsilon_l(H)$) $\Theta=\varepsilon_c(H)$ *holds.*

Proof. First of all we show that for an arbitrary subgroup H of the diagonal group $(G,\cdot)$ of a group-like Menger algebra (G,o) of rank n the following conditions:

$$g_1 \equiv g_2(\varepsilon_c(H)) \longleftrightarrow (\forall t)(\forall \bar{z})\ \left(t(g_1[\bar{z}])^{-1} \cdot t(g_2[\bar{z}]) \in H\right), \tag{2.5.10}$$

$$g_1 \equiv g_2(\varepsilon_v(H)) \longleftrightarrow (\forall t)\ \left(t(g_1)^{-1} \cdot t(g_2) \in H\right), \tag{2.5.11}$$

$$g_1 \equiv g_2(\varepsilon_l(H)) \longleftrightarrow g_1 \cdot g_2^{-1} \in H, \tag{2.5.12}$$

where $t \in T_n(G)$, $\bar{z} \in G^n$, are satisfied. Indeed, since $\varepsilon_c(H)$ is a congruence, $g_1 \equiv g_2(\varepsilon_c(H))$ implies $t(g_1[\bar{z}]) \equiv t(g_2[\bar{z}])(\varepsilon_c(H))$ for all $t \in T_n(G)$, $\bar{z} \in G^n$, which by the v-regularity of $\varepsilon_c(H)$ gives

$$t(g_1[\bar{z}])^{-1} \cdot t(g_1[\bar{z}]) \equiv t(g_1[\bar{z}])^{-1} \cdot t(g_2[\bar{z}])(\varepsilon_c(H)),$$

i.e., $e \equiv t(g_1[\bar{z}])^{-1} \cdot t(g_2[\bar{z}])(\varepsilon_c(H))$. The last congruence means that

$$p(e[\bar{x}]) \in H \longleftrightarrow p\left(t(g_1[\bar{z}])^{-1} \cdot t(g_2[\bar{z}])\right) \in H$$

for any $p \in T_n(G)$, $\bar{x} \in G^n$. Now, replacing p by $\triangle_G$ and $\bar{x}$ by $(e,\dots,e)$, we get

$$t(g_1[\bar{z}])^{-1} \cdot t(g_2[\bar{z}]) \in H.$$

On the other hand, for a subgroup H the condition $t(g_1[\bar{z}])^{-1} \cdot t(g_2[\bar{z}]) \in H$ is equivalent to $t(g_1[\bar{z}]) \cdot H = t(g_2[\bar{z}]) \cdot H$, i.e., to

$$(\forall g)\ \left(g \in t(g_1[\bar{z}]) \cdot H \longleftrightarrow g \in t(g_2[\bar{z}]) \cdot H\right).$$

Hence

$$(\forall g)\ \left(t(g_1[\bar{z}])^{-1} \cdot g \in H \longleftrightarrow t(g_2[\bar{z}])^{-1} \cdot g \in H\right).$$

For $g = e$, the last condition can be written in the form

$$t(g_1[\bar{z}]) \in H \longleftrightarrow t(g_2[\bar{z}]) \in H,$$

which is true for any $t \in T_n(G)$ and $\bar{z} \in G^n$. Thus $g_1 \equiv g_2(\varepsilon_c(H))$. This completes the proof of (2.5.10). The conditions (2.5.11), (2.5.12) are obvious now.

The conditions proved show that if H is an (T-subgroup, subgroup) LT-subgroup of the diagonal group, then H coincides with the ($\varepsilon_v(H)$-, $\varepsilon_l(H)$-) $\varepsilon_c(H)$-class containing the unit e. Indeed, let H be an LT-subgroup. If $x \in H$, then (2.5.6) implies $t(e[\bar{z}])^{-1} \cdot t(x[\bar{z}]) \in H$ for all $t \in T_n(G)$, $\bar{z} \in G^n$. The last relation means that $e \equiv x(\varepsilon_c(H))$, i.e., $x \in \varepsilon_c(H)\langle e\rangle$. Hence $H \subset \varepsilon_c(H)\langle e\rangle$. Conversely, if $x \in \varepsilon_c(H)\langle e\rangle$, then, by (2.5.10), we have $t(e[\bar{z}]) \cdot t(x[\bar{z}]) \in H$ for all $t \in T_n(G)$, $\bar{z} \in G^n$, which by taking $t = \triangle_G$ and $\bar{z} = (e,\dots,e)$ gives $x \in H$. Thus $\varepsilon_c(H)\langle e\rangle \subset H$. So, $H = \varepsilon_c(H)\langle e\rangle$, which was to be proved.

Now let $\boldsymbol{\Theta}$ be an arbitrary congruence on a group-like Menger algebra, H — its equivalence class containing the unit e. Then, as it was mentioned above, H is an LT-subgroup of the diagonal group. Therefore H is a $\varepsilon_c(H)$-class containing e. So, $\boldsymbol{\Theta}\langle e\rangle = \varepsilon_c(H)\langle e\rangle = H$ and

$$\begin{aligned} g_1 \equiv g_2(\boldsymbol{\Theta}) &\longleftrightarrow (\forall t)(\forall \bar{z})\ \ (t(g_1[\bar{z}]) \equiv t(g_2[\bar{z}])(\boldsymbol{\Theta})) \\ &\longleftrightarrow (\forall t)(\forall \bar{z})\ \ \big(t(g_1[\bar{z}])^{-1} \cdot t(g_2[\bar{z}]) \in \boldsymbol{\Theta}\langle e\rangle\big) \\ &\longleftrightarrow (\forall t)(\forall \bar{z})\ \ \big(t(g_1[\bar{z}])^{-1} \cdot t(g_2[\bar{z}]) \in H\big) \longleftrightarrow g_1 \equiv g_2(\varepsilon_c(H)), \end{aligned}$$

i.e., $\boldsymbol{\Theta} = \varepsilon_c(H)$.

In the case of v- and l-congruences the proof is similar. □

As it is not difficult to see, for every subgroup H of the diagonal group of a group-like Menger algebra (G, o) of rank n and every $g \in G$ we have

$$\varepsilon_c(H)\langle g\rangle = \bigcap_{t\in T_n(G)} t^{-1}(t(g)\cdot H). \tag{2.5.13}$$

If H is an LT-subgroup of $(G,\cdot)$, then, by (2.5.9) from Theorem 2.5.10, we get $\varepsilon_v(H)\langle g\rangle = gH$. But $\varepsilon_c(H) \subset \varepsilon_v(H)$, so

$$\varepsilon_c(H)\langle g\rangle \subset \varepsilon_v(H)\langle g\rangle = gH\ (= Hg).$$

Now let $x \in gH$ $(= Hg)$. Then $x[\bar{z}] \in H \cdot g[\bar{z}]$ $(= g[\bar{z}] \cdot H)$ for any $\bar{z} \in G^n$. This, according to (2.5.8), gives $x[\bar{z}] \in t^{-1}(t(g[\bar{z}]) \cdot H)$ for every $t \in T_n(G)$. Hence, $t(g[\bar{z}])^{-1} \cdot t(x[\bar{z}]) \in H$ and $g \equiv x(\varepsilon_c(H))$, i.e., $x \in \varepsilon_c(H)\langle g\rangle$. So, $\varepsilon_c(H)\langle g\rangle = gH$ $(= Hg)$. Thus, each equivalence class of each congruence on a group-like Menger algebra is an equivalence class of its diagonal group. In particular, for each congruence $\boldsymbol{\Theta}$ on a group-like Menger algebra (G, o) and arbitrary $g, g_1, \ldots, g_n \in G$ we have

$$\boldsymbol{\Theta}\langle g[g_1\cdots g_n]\rangle = \boldsymbol{\Theta}\langle g\rangle[\boldsymbol{\Theta}\langle g_1\rangle\cdots\boldsymbol{\Theta}\langle g_n\rangle]. \tag{2.5.14}$$

Indeed, $\boldsymbol{\Theta}\langle x\rangle = xH$ for any $x \in G$, where H is the LT-subgroup, which corresponds to $\boldsymbol{\Theta}$. Thus

$$g[g_1\cdots g_n] \subset g[g_1H\cdots g_nH],$$

and

$$e\cdot g[g_1\cdots g_n]\cdot H \subset H\cdot g[g_1H\cdots g_nH] = (Hg)[g_1H\cdots g_nH].$$

So,

$$\boldsymbol{\Theta}\langle g[g_1\cdots g_n]\rangle \subset \boldsymbol{\Theta}\langle g\rangle[\boldsymbol{\Theta}\langle g_1\rangle\cdots\boldsymbol{\Theta}\langle g_n\rangle].$$

Since the converse inclusion is true for each congruence on an arbitrary Menger algebra, condition (2.5.14) is proved.

Theorem 2.5.12. *For every LT-subgroup H of the diagonal group of a Menger algebra (G,o) of rank n the following conditions are equivalent:*
(a) *a congruence $\varepsilon_c(H)$ is v-cancellative,*
(b) $x\cdot t_1(y)$, $x\cdot t_1(z)$, $u\cdot t_2(z)\in H$ *imply* $u\cdot t_2(y)\in H$ *for any* $t_1,t_2\in T_n(G)$, $x,y,z,u\in G$,
(c) $a[a_1\cdots a_{i-1}\,(a_iH)\,a_{i+1}\cdots a_n] = a[a_1\cdots a_n]\cdot H$ *for all* $i=1,\ldots,n$ *and* $a,a_1,\ldots,a_n\in G$.

Proof. (a)$\longrightarrow$ (b). It is clear that $H=\varepsilon_c(H)\langle e\rangle$ for any LT-subgroup H. If $x\cdot t_1(y)$, $x\cdot t_1(z)$, $u\cdot t_2(z)\in H$, then obviously $x\cdot t_1(y)\equiv x\cdot t_1(z)(\varepsilon_c(H))$ and $y\equiv z(\varepsilon_c(H))$ by the v-cancellativity of the congruence $\varepsilon_c(H)$. Thus $u\cdot t_2(y)\equiv u\cdot t_2(z)(\varepsilon_c(H))$. But $u\cdot t_2(z)\in H$, hence $u\cdot t_2(y)\in H$. This proves (b).

(b)$\longrightarrow$ (c). Since all the $\varepsilon_c(H)$-classes have the form gH, then

$$a[a_1\cdots a_{i-1}\,(a_iH)\,a_{i+1}\cdots a_n]\subset (aH)[a_1H\cdots a_nH] = a[a_1\cdots a_n]\cdot H.$$

To prove the converse inclusion consider arbitrary $y\in a[a_1\cdots a_n]\cdot H$, i.e., such an $y\in G$ for which $a[a_1\cdots a_n]^{-1}\cdot y\in H$. For this y there exists a z such that

$$a[a_1\cdots a_{i-1}\,z\,a_{i+1}\cdots a_n]=y.$$

Then, according to (b), from the conditions

$$\begin{aligned}&a[a_1\cdots a_n]^{-1}\cdot a[a_1\cdots a_{i-1}\,z\,a_{i+1}\cdots a_n]\in H,\\ &a[a_1\cdots a_n]^{-1}\cdot a[a_1\cdots a_n]\in H,\\ &a_i^{-1}\cdot a_i\in H\end{aligned}$$

we conclude $a_i^{-1}\cdot z\in H$, i.e., $z\in a_iH$. Thus

$$y=a[a_1\cdots a_{i-1}\,z\,a_{i+1}\cdots a_n]\in a[a_1\cdots a_{i-1}\,(a_iH)\,a_{i+1}\cdots a_n].$$

So, $a[a_1\cdots a_{i-1}\,(a_iH)\,a_{i+1}\cdots a_n]\supset a[a_1\cdots a_n]\cdot H$. This proves (c).

(c)$\longrightarrow$ (a). Now let (c) be satisfied. If $u[\bar{w}|_i g_1]\equiv u[\bar{w}|_i g_2](\varepsilon_c(H))$, then $u[\bar{w}|_i g_1]\in u[\bar{w}|_i g_2]\cdot H$ and $u[\bar{w}|_i g_1]\in u[\bar{w}|_i g_2H]$ by (c). Thus, according to (2.5.3), $g_1\in g_2H$, i.e., $g_1\equiv g_2(\varepsilon_c(H))$. This completes the proof. □

Proposition 2.5.13. *If H is a T-subgroup of the diagonal group of a group-like Menger algebra (G,o) of rank n, then $\varepsilon_v(H)$ is v-cancellative if and only if*

$$x\cdot t_1(y),\ x\cdot t_1(z),\ t_2(z)\in H\longrightarrow t_2(y)\in H \tag{2.5.15}$$

for all $t_1,t_2\in T_n(g)$ *and* $x,y,z\in G$.

Proof. Indeed, if H is a T-subgroup, then (2.5.15) is a consequence of the v-cancellativity and v-regularity of $\varepsilon_v(H)$ and the fact that $H=\varepsilon_v(H)\langle e\rangle$.

Conversely, let (2.5.15) be satisfied and $u[\bar{w}|_i x] \equiv u[\bar{w}|_i y](\varepsilon_v(H))$ for some $i = 1, \ldots, n$. Suppose that $u[\bar{w}|_i x]$, $u[\bar{w}|_i y] \in \varepsilon_v(H)\langle s\rangle$ for some $s \in G$. Then, according to (2.5.13), we have

$$t(s)^{-1} \cdot t(u[\bar{w}|_i x]),\ t(s)^{-1} \cdot t(u[\bar{w}|_i y]) \in H$$

for every $t \in T_n(G)$. If $t_1(x) \in H$ for some $t_1 \in T_n(G)$, then from the above, according to (2.5.15), we conclude $t_1(y) \in H$. Similarly $t_1(y) \in H$ implies $t_1(x) \in H$. So, $x \equiv y(\varepsilon_v(H))$, which was to be proved. □

It is not difficult to see that for every subgroup H of the diagonal group the relation $\varepsilon_l(H)$ is l-cancellative.

2.6 Antisymmetric Menger algebras

In order to continue the study of n-place transformations of ordered sets, initiated in Section 1.4, we must introduce a number of new notions and definitions. So, let (G, o) be a Menger algebra of rank n and $T_n(G)$ be the set of polynomials defined in Section 2.1. By $t_1 \circ t_2$, where $t_1, t_2 \in T_n(G)$, we will denote the polynomial defined by the equality $t_1 \circ t_2(x) = t_1(t_2(x))$. On (G, o), we introduce two new binary relations $<_\sigma$ and $<_\pi$ putting for any $g_1, g_2 \in G$

$$g_1 <_\sigma g_2 \longleftrightarrow (\exists t \in T_n(G))\ t(g_1) = g_2$$

and

$$g_1 <_\pi g_2 \longleftrightarrow \begin{cases} (\exists m \in \mathbb{N})(\exists x_i \in G)(\exists t_i, t_i' \in T_n(G)) \\ g_1 = t_1(x_1), \\ \bigwedge\limits_{i=1}^{m-1} t_i \circ t_i'(x_i) = t_{i+1}(x_{i+1}), \\ t_m \circ t_m'(x_m) = g_2, \end{cases}$$

where $i = 1, 2, \ldots, n$.[3]

Proposition 2.6.1. *The relations $<_\sigma$ and $<_\pi$ are quasi-orders on (G, o).*

Proof. Let $t = x \in T_n(G)$, then the equality $t(g) = g$ holds for every $g \in G$, i.e., $g <_\sigma g$. Hence $<_\sigma$ is reflexive. If $g_1 <_\sigma g_2$ and $g_2 <_\sigma g_3$ for some $g_1, g_2, g_3 \in G$, then $t_1(g_1) = g_2$ and $t_2(g_2) = g_3$ for some polynomials $t_1, t_2 \in T_n(G)$. Thus $t_2(t_1(g_1)) = g_3$, i.e., $t_2 \circ t_1(g_1) = g_3$, which implies $g_1 <_\sigma g_3$. So $<_\sigma$ is transitive. Therefore $<_\sigma$ is a quasi-order.

Now we consider the relation $<_\pi$. It is clear that for any $g \in G$ the equalities $g = t_1(x_1)$ and $g = t_m \circ t_m'(x_m)$ are satisfied for $m = 1$, $t_1 = t_1' = x \in T_n(G)$

[3] Let us notice that the expression $\bigwedge\limits_{i=p}^{k} A_i$ denotes the empty symbol for $p > k$.

and $x_1 = x_m = g$. Hence $g <_\pi g$, i.e., $<_\pi$ is reflexive. Now let $g_1 <_\pi g_2$ and $g_2 <_\pi g_3$, where $g_1, g_2, g_3 \in G$. Then there are $m, l \in \mathbb{N}$, $t_i, t_i' \in T_n(G)$, $x_i \in G$, $i = 1, 2, \ldots, m+1$ such that

$$g_1 = t_1(x_1), \quad \bigwedge_{i=1}^{m-1} t_i \circ t_i'(x_i) = t_{i+1}(x_{i+1}), \quad t_m \circ t_m'(x_m) = g_2 \tag{2.6.1}$$

and

$$\begin{aligned} & g_2 = t_{m+1}(x_{m+1}), \quad \bigwedge_{i=m+1}^{m+l-1} t_i \circ t_i'(x_i) = t_{i+1}(x_{i+1}), \\ & t_{m+l} \circ t_{m+l}'(x_{m+l}) = g_3. \end{aligned} \tag{2.6.2}$$

Since $t_m \circ t_m'(x_m) = t_{m+1}(x_{m+1})$, from (2.6.1) and (2.6.2) we obtain

$$g_1 = t_1(x_1), \quad \bigwedge_{i=1}^{m+l-1} t_i \circ t_i'(x_i) = t_{i+1}(x_{i+1}), \quad t_{m+l} \circ t_{m+l}'(x_{m+l}) = g_3,$$

which implies $g_1 <_\pi g_3$. So $<_\pi$ is transitive, and, consequently, it is a quasi-order. □

Let (Φ, O) be a Menger algebra of n-place transformations of A, and let x be a variable. Define the set of expressions $\Sigma_n(\Phi)$ in the following way:

(a) $x \in \Sigma_n(\Phi)$,

(b) $f(\bar{a}|_i \sigma(x)) \in \Sigma_n(\Phi)$ for all $i = 1, \ldots, n$, $f \in \Phi$, $\sigma(x) \in \Sigma_n(\Phi)$, $\bar{a} \in A^n$.

It is clear that each expression $\sigma(x)$ from $\Sigma_n(\Phi)$ induces on A a transformation $\sigma : a \mapsto \sigma(a)$.

Definition 2.6.2. A Menger algebra (Φ, O) of n-place transformations of A is called *antisymmetric*, if for all $\sigma_1(x), \sigma_2(x) \in \Sigma_n(\Phi)$ and $a_1, a_2 \in A$ the implication

$$\sigma_1(a_1) = a_2 \ \wedge \ \sigma_2(a_2) = a_1 \longrightarrow a_1 = a_2 \tag{2.6.3}$$

is true.

Definition 2.6.3. A Menger algebra (Φ, O) of n-place transformations of A is called *well-antisymmetric*, if

$$\left.\begin{array}{c} \bigwedge_{i=1}^{m-1} \sigma_i \circ \sigma_i'(a_i) = \sigma_{i+1}(a_{i+1}), \\ \sigma_m \circ \sigma_m'(a_m) = \sigma_1(a_1) \end{array}\right\} \longrightarrow \sigma_1(a_1) = \sigma_2(a_2) \tag{2.6.4}$$

holds for all $m \in \mathbb{N}$, $\sigma_i(x), \sigma_i'(x) \in \Sigma_n(\Phi)$, $a_i \in A$, $i = 1, \ldots, m$.

It is not difficult to see that any well-antisymmetric Menger algebra is antisymmetric.

The theorem presented below is a generalization of Schein's result obtained in [183] for semigroups.

Theorem 2.6.4. *For a Menger algebra (G, o) of rank n, the following conditions are equivalent:*
(1) *(G, o) is isomorphic to a Menger algebra of extensive n-place transformations of some ordered set,*
(2) *(G, o) is isomorphic to an antisymmetric Menger algebra of n-place transformations,*
(3) *(G, o) is ordered by the relation $<_\sigma$,*
(4) *for all $g \in G$, $t_1, t_2 \in T_n(G)$ $\quad t_2 \circ t_1(g) = g$ implies $t_1(g) = g$.*

Proof. (1) $\longrightarrow$ (2). Suppose that (G, o) is isomorphic to a Menger algebra (Φ, O) of extensive n-place transformations of an ordered set $(A, \leqslant)$. Now let $\sigma_1(a_1) = a_2$ and $\sigma_2(a_2) = a_1$ for some $a_1, a_2 \in A$, $\sigma_1(x), \sigma_2(x) \in \Sigma_n(\Phi)$. According to the extensivity of n-place transformations from Φ, it is evident that $a_1 \leqslant \sigma_1(a_1)$ and $a_2 \leqslant \sigma_2(a_2)$. This gives $a_1 \leqslant a_2$ and $a_2 \leqslant a_1$, i.e., $a_1 = a_2$. So (Φ, O) is an antisymmetric Menger algebra of n-place transformations.

(2) $\longrightarrow$ (3). Since Proposition 2.6.1 shows that $<_\sigma$ is a quasi-order, we must show that $<_\sigma$ is antisymmetric. For this, let $g_1 <_\sigma g_2$ and $g_2 <_\sigma g_1$, where $g_1, g_2 \in G$. Then $t_1(g_1) = g_2$ and $t_2(g_2) = g_1$ for some polynomials $t_1, t_2 \in T_n(G)$. Let P be an isomorphism of (G, o) onto an antisymmetric Menger algebra (Φ, O) of n-place transformations of A. Assume that $P(g_1) = f_1$, $P(g_2) = f_2$. If to t_1, t_2 correspond polynomials $p_1, p_2 \in \Phi$, then $p_1(f_1) = f_2$ and $p_2(f_2) = f_1$, i.e., $p_1(f_1)(\bar{a}) = f_2(\bar{a})$ and $p_2(f_2)(\bar{a}) = f_1(\bar{a})$ for any $\bar{a} \in A^n$. Note that to every polynomial p from (Φ, O) corresponds a transformation $\sigma(x) \in \Sigma_n(\Phi)$ such that $\sigma(f(\bar{a})) = p(f)(\bar{a})$ for all $f \in \Phi$ and $\bar{a} \in A^n$. For example, if

$$p(f) = h[g_1 \cdots g_{i-1} g[h_1 \cdots h_{k-1} f\, h_{k+1} \cdots h_n]\, g_{i+1} \cdots g_n],$$

then $\sigma(x) = h(g_1^n(\bar{a})\,|_i g(h_1^n(\bar{a})\,|_k x))$, where the symbol $g_1^n(\bar{a})$ denotes the sequence $g_1(\bar{a}), \ldots, g_n(\bar{a})$. Thus, we have

$$p(f)(\bar{a}) = h(g_1^n(\bar{a})\,|_i g(h_1^n(\bar{a})\,|_k f(\bar{a}))) = \sigma(f(\bar{a})).$$

So, let $\sigma_1(x), \sigma_2(x) \in \Sigma_n(\Phi)$ correspond to p_1, p_2. Then we have

$$\sigma_1(f_1(\bar{a})) = f_2(\bar{a}) \quad \text{and} \quad \sigma_2(f_2(\bar{a})) = f_1(\bar{a}).$$

But (Φ, O) is antisymmetric, so $f_1(\bar{a}) = f_2(\bar{a})$, i.e., $f_1 = f_2$. Therefore $P(g_1) = P(g_2)$, and, consequently $g_1 = g_2$. This proves that $<_\sigma$ is antisymmetric.

(3) $\longrightarrow$ (4). Let $t_2 \circ t_1(g) = g$ for some $g \in G$, $t_1, t_2 \in T_n(G)$. If $t_1(g) = g_1$, then $t_2(g_1) = g$, which gives $g <_\sigma g_1$ and $g_1 <_\sigma g$. Hence $g_1 = g$, i.e., $t_1(g) = g$.

(4) $\longrightarrow$ (1). Let $G' = G \cup \{e, c\}$, where e, c are two different elements not belonging to G. On G' we define an $(n+1)$-ary operation o' letting

$$o'(x, y_1, \ldots, y_n) = \begin{cases} x[\bar{y}] & \text{if } x, y_1, \ldots, y_n \in G, \\ x & \text{if } y_1 = y_2 = \cdots = y_n = e, \\ c & \text{in all other cases.} \end{cases}$$

It is easy to see that (G', o') is a Menger algebra of rank n. By Theorem 2.1.8, the mapping $P_0' : x \mapsto \lambda_x'$, where $x \in G'$, is an isomorphism of (G', o') onto (Λ', O), where $\Lambda' = \{\lambda_x' \mid x \in G'\}$, i.e., an isomorphism onto a Menger algebra of n-place transformations of the set G'.

Consider on (G, o) the quasi-order $<_\sigma$. We must prove that it is antisymmetric. Indeed, if $g_1 <_\sigma g_2$ and $g_2 <_\sigma g_1$, where $g_1, g_2 \in G$, then $t_1(g_1) = g_2$ and $t_2(g_2) = g_1$ for some $t_1, t_2 \in T_n(G)$. Thus $t_1 \circ t_1(g_1) = g_1$, which together with (4) implies $t_1(g_1) = g_1$, i.e., $g_1 = g_2$. So $<_\sigma$ is an order on G. We extend this order to G' putting $e < x$ and $x < c$ for every $x \in G'$, and $g_1 < g_2$ for all $g_1, g_2 \in G$ such that $g_1 <_\sigma g_2$. Any n-place transformation λ_g' of G' is extensive with respect to this new order. Indeed, if $x_1, \ldots, x_n \in G$, then $g[x_1 \cdots x_n] = g[x_1 \cdots x_n]$ implies $x_i <_\sigma g[x_1 \cdots x_n]$, i.e., $x_i < \lambda_g'(x_1, \ldots, x_n)$ for every $i = 1, \ldots, n$. If $x_1 = \cdots = x_n = e$, then $e < g = \lambda_g'(e, \ldots, e)$. In all other cases we have $c = \lambda_g'(x_1, \ldots, x_n)$, i.e., $x_i < c = \lambda_g'(x_1, \ldots, x_n)$, where $i = 1, \ldots, n$. This means that (Λ', O) is a Menger algebra of extensive transformations of the ordered set $(G', <)$.

This completes the proof of Theorem 2.6.4. □

Theorem 2.6.5. *For a Menger algebra (G, o) of rank n the following conditions are equivalent:*

(1) *(G, o) is isomorphic to a Menger algebra of semiclosure operations on some ordered set,*

(2) *(G, o) is isomorphic to a well-antisymmetric Menger algebra of n-place transformations of some ordered set.*

Proof. Let a Menger algebra (G, o) be isomorphic to a Menger algebra (Φ, O) of n-place semiclosure operations on $(A, \leqslant)$. Suppose that all assumptions of condition (2.6.4) are satisfied. Since $a_i \leqslant \sigma_i'(a_i)$, from the isotonicity we obtain $\sigma_i(a_i) \leqslant \sigma \circ \sigma_i'(a_i)$ for every $\sigma_i(x) \in \Sigma_n(\Phi)$. Therefore

$$\sigma_1(a_1) \leqslant \sigma_1 \circ \sigma_1'(a_1) = \sigma_2(a_2) \leqslant \cdots \leqslant \sigma_m \circ \sigma_m'(a_m) = \sigma_1(a_1).$$

Hence $\sigma_1(a_1) = \sigma_2(a_2)$. So (1) implies (2).

To prove the converse, let (G, o) be isomorphic to a well-antisymmetric Menger algebra (Φ, O) of n-place transformations of A. Define on A the binary relation $<_\Phi$ putting

$$a_1 <_\Phi a_2 \longleftrightarrow \begin{cases} (\exists\, \sigma_i(x), \sigma'_i(x) \in \Sigma_n(\Phi))(\exists\, b_j \in A)(\exists\, m \in \mathbb{N}) \\ a_1 = \sigma_1(b_1), \\ \bigwedge\limits_{i=1}^{m-1} \sigma_i \circ \sigma'_i(b_i) = \sigma'_{i+1}(b_{i+1}), \\ \sigma_m \circ \sigma'_m(b_m) = a_2. \end{cases} \tag{2.6.5}$$

It is not difficult to see that $<_\Phi$ is a quasi-order. In fact, $<_\Phi$ is an order because (Φ, O) is well-antisymmetric. Let us show that every element $f \in \Phi$ is an n-place semiclosure of $(A, <_\Phi)$. Indeed, let $f \in \Phi$ and $\bar{a} \in A^n$. Denote by $\sigma_1(x)$ the identity transformation from $\Sigma_n(\Phi)$ and by $\sigma_2(x)$ the transformation $f(\bar{a}\,|_i x)$, where $1 \leqslant i \leqslant n$. Then $a_i = \sigma_1(a_i)$ and $\sigma_2 \circ \sigma_1(a_i) = f(\bar{a})$. So $a_i <_\Phi f(\bar{a})$, which means that f is extensive. Now let $a_1 <_\Phi a_2$, where $a_1, a_2 \in A$. Putting $\sigma(x) = f(\bar{u}\,|_i x)$, where $\bar{u} \in A^n$, and using (2.6.5) we get

$$f(\bar{u}\,|_i a_1) = \sigma \circ \sigma_1(b_1), \qquad \sigma \circ \sigma_m(b_m) = f(\bar{u}\,|_i a_2),$$

$$\bigwedge_{i=1}^{m-1} \sigma \circ \sigma_i \circ \sigma'_i(b_i) = \sigma \circ \sigma_{i+1}(b_{i+1}).$$

Hence $f(\bar{u}\,|_i a_1) <_\Phi f(\bar{u}\,|_i a_2)$. i.e., f is isotone. So f is a semiclosure operation. This completes the proof. □

Definition 2.6.6. A Menger algebra (G, o) of rank n is called *reductive*, if

$$\Big((\forall \bar{x}) g_1[\bar{x}] = g_2[\bar{x}]\Big) \longrightarrow g_1 = g_2$$

where $\bar{x} \in G^n$.

Theorem 2.6.7. *For a reductive Menger algebra (G, o) of rank n the following conditions are equivalent:*

(1) *(G, o) is isomorphic to a Menger algebra of semiclosure operations on some ordered set,*

(2) *(G, o) is isomorphic to a well-antisymmetric Menger algebra of n-place transformations,*

(3) *(G, o) is ordered by the relation $<_\pi$,*

(4) *for all $t_i, t'_i \in T_n(G),\ x_i \in G,\ m \in \mathbb{N},\ i = 1, \ldots, n$*

$$\left.\begin{array}{r} t_1(x_1) = t_m \circ t'_m(x_m), \\ \bigwedge\limits_{i=1}^{m-1} t_i \circ t'_i(x_i) = t_{i+1}(x_{i+1}) \end{array}\right\} \longrightarrow t_1(x_1) = t_2(x_2).$$

Proof. It is evident that the conditions (1) and (2) are equivalent. We prove that (2) implies (3). So let (2) be satisfied. By Proposition 2.6.1, the relation $<_\pi$ is a quasi-order. In fact, it is an order. Indeed, if $g_1 <_\pi g_2$ and $g_2 <_\pi g_1$ for some $g_1, g_2 \in G$, then there exist $m, l \in \mathbb{N}$, $t_i, t'_i \in T_n(G)$, $x_i \in G$, $i = 1, 2, \ldots, m+1$ such that

$$g_1 = t_1(x_1), \quad \bigwedge_{i=1}^{m-1} t_i \circ t'_i(x_i) = t_{i+1}(x_{i+1}), \quad t_m \circ t'_m(x_m) = g_2$$

and

$$g_2 = t_{m+1}(x_{m+1}), \quad \bigwedge_{i=m+1}^{m+l-1} t_i \circ t'_i(x_i) = t_{i+1}(x_{i+1}), \quad t_{m+l} \circ t'_{m+l}(x_{m+l}) = g_1.$$

From the above, we get

$$\bigwedge_{i=m+1}^{m+l-1} t_i \circ t'_i(x_i) = t_{i+1}(x_{i+1}), \qquad t_{m+l} \circ t'_{m+l}(x_{m+l}) = t_1(x_1). \tag{2.6.6}$$

Let P be an isomorphism of (G, o) onto a well-asymmetric Menger algebra (Φ, O) of n-place transformations of A. If the polynomials $p_i, p'_i \in T_n(\Phi)$ correspond to $t_i, t'_i \in T_n(G)$ and $P(x_i) = f_i$ for all $i = 1, \ldots, m+l-1$, then from (2.6.6) we obtain

$$\bigwedge_{i=m+1}^{m+l-1} p_i \circ p'_i(f_i) = p_{i+1}(f_{i+1}), \quad p_{m+l} \circ p'_{m+l}(f_{m+l}) = p_1(f_1),$$

which for $\bar{a} \in A^n$ gives

$$\bigwedge_{i=m+1}^{m+l-1} p_i \circ p'_i(f_i)(\bar{a}) = p_{i+1}(f_{i+1})(\bar{a}), \quad p_{m+l} \circ p'_{m+l}(f_{m+l})(\bar{a}) = p_1(f_1)(\bar{a}).$$

Thus

$$\bigwedge_{i=1}^{m+l-1} \sigma_i \circ \sigma'_i(f(\bar{a})) = \sigma_{i+1}(f_{i+1}(\bar{a})), \quad \sigma_{m+l} \circ \sigma'_{m+l}(f_{m+l}(\bar{a})) = \sigma_1(f_1(\bar{a}))$$

because $p_i(f_i)(\bar{a}) = \sigma_i(f_i(\bar{a}))$ and $p_i \circ p'_i(f_i)(\bar{a}) = \sigma_i \circ \sigma'_i(f_i(\bar{a}))$ for some $\sigma_i(x), \sigma'_i(x) \in \Sigma_n(\Phi)$, $1 \leqslant i \leqslant m+l-1$. Since for an antisymmetric Menger algebra (Φ, O) the above conditions imply $\sigma_i(f_1(\bar{a})) = \sigma_2(f_2(\bar{a}))$, from the above we get

$$\bigwedge_{i=2}^{m+l-1} \sigma_i \circ \sigma'_i(f(\bar{a})) = \sigma_{i+1}(f_{i+1}(\bar{a})), \qquad \sigma_{m+l} \circ \sigma'_{m+l}(f_{m+l}(\bar{a})) = \sigma_2(f_2(\bar{a})),$$

which proves $\sigma_2(f_2(\bar{a})) = \sigma_3(f_3(\bar{a}))$ etc. Consequently we obtain the sequence of equalities

$$\sigma_1(f_1(\bar{a})) = \sigma_2(f_2(\bar{a})) = \cdots = \sigma_{m+1}(f_{m+1}(\bar{a})).$$

So $\sigma_1(f_1(\bar{a})) = \sigma_{m+1}(f_{m+1}(\bar{a}))$, i.e., $p_1(f_1(\bar{a})) = p_{m+1}(f_{m+1}(\bar{a}))$ for every $\bar{a} \in A^n$. This means that $p_1(f_1) = p_{m+1}(f_{m+1})$. Hence $t_1(x_1) = t_{m+1}(x_{m+1})$, i.e., $g_1 = g_2$. Thus $<_\pi$ is an order on (G, o). Therefore (2) implies (3).

Assume now that (3) holds. If for some $t_1, t_i' \in T_n(G)$, $x_i \in G$, $m \in \mathbb{N}$

$$\bigwedge_{i=1}^{m-1} t_i \circ t_i'(x_i) = t_{i+1}(x_{i+1}), \qquad t_m \circ t_m'(x_m) = t_1(x_1),$$

then $t_1(x_1) <_\pi t_2(x_2)$ and $t_2(x_2) <_\pi t_1(x_1)$. This, by (3), gives $t_1(x_1) = t_2(x_2)$. So (3) implies (4).

To prove the last implication observe that for a reductive Menger algebra (G, o) the mapping $P_0 : g \mapsto \lambda_g$, where $\lambda_g(x_1, \ldots, x_n) = g[x_1 \cdots x_n]$ for all $x_1, \ldots, x_n \in G$, is an isomorphism onto a Menger algebra (Λ, O) with $\Lambda = \{\lambda_g \,|\, g \in G\}$. It is not difficult to see that (Λ, O) is, by (4), well-antisymmetric. Therefore, by Theorem 2.6.5, condition (1) is satisfied. Thus (4) implies (1). □

The next theorem gives a characterization of Menger algebras of n-place closure operations on an ordered set.

Theorem 2.6.8. *A Menger algebra (G, o) of rank n is isomorphic to some algebra of n-place closure operations on an ordered set $(G, \leqslant)$ if and only if its diagonal semigroup is a semilattice and*

$$x[y_1 \cdots y_n] = xy_1 \cdots y_n \tag{2.6.7}$$

for any $x, y_1, \ldots, y_n \in G$.

Proof. Necessity. Let (Φ, O) be a Menger algebra of n-place closure operations on the ordered set $(G, \leqslant)$ and let $(\Phi, \cdot)$ be its diagonal semigroup. This semigroup is idempotent because all elements of Φ — as n-place closure operations — satisfy the identity $f[f^n] = f$. Moreover, for every $f, g \in \Phi$, $f[g^n]$ and $g[f^n]$ are also n-place closure operations. Hence, according to Proposition 1.4.15, we have

$$f[g^n] = g[f^n][g^n] \quad \text{and} \quad g[f^n] = f[g^n][f^n].$$

This, by Proposition 1.4.13, gives

$$f[g^n] = g[f^n][g^n] \prec g[f^n][g^n][f^n] = g[f^n][g[f^n] \cdots g[f^n]] = g[f^n],$$

$$g[f^n] = f[g^n][f^n] \prec f[g^n][f^n][g^n] = f[g^n][f[g^n] \cdots f[g^n]] = f[g^n].$$

Hence $f[g^n] = g[f^n]$. Therefore $(\Phi, \cdot)$ is a commutative idempotent semigroup, i.e., it is a semilattice.

To prove that (2.6.7) holds consider $f, g_1, \ldots, g_n \in \Phi$. Then $g_1 \prec f[\bar{g}]$ and

$$fg_1 = f[g_1 \cdots g_1] \prec f[f[\bar{g}] \cdots f[\bar{g}]] = f[f^n][\bar{g}] = f[\bar{g}].$$

Hence $fg_1 \prec f[\bar{g}]$. Next $g_2 \prec f[\bar{g}]$ implies

$$\begin{aligned} fg_1g_2 &= (fg_1)[g_2 \cdots g_2] \prec (fg_1)[f[\bar{g}] \cdots f[\bar{g}]] \\ &= f[g_1[f^n][\bar{g}] \cdots g_1[f^n][\bar{g}]] = f[f[\bar{g}] \cdots f[\bar{g}]] = f[f^n][\bar{g}] = f[\bar{g}]. \end{aligned}$$

Thus, $fg_1g_2 \prec f[\bar{g}]$, etc. After a finite number of steps we get $fg_1 \cdots g_n \prec f[\bar{g}]$. Then it is easy to see that $f \prec fg_1 \cdots g_n$. From the inequalities $a_j \leqslant g_{i+1} \cdots g_n(\bar{a})$, where $i, j = 1, \ldots, n$, $\bar{a} \in A^n$, we get $g_i(\bar{a}) \leqslant g_i \cdots g_n(\bar{a})$. But

$$g_1 \cdots g_n(\bar{a}) \leqslant fg_1 \cdots g_{i-1}(g_i \cdots g_n(\bar{a}), \ldots, g_i \cdots g_n(\bar{a})) = fg_1 \cdots g_n(\bar{a}).$$

Hence $g_i(\bar{a}) \leqslant fg_1 \cdots g_n(\bar{a})$ for all $\bar{a} \in A^n$, i.e., $g_i \prec fg_1 \cdots g_n$, $i = 1, \ldots, n$. In a similar way, from $f \prec fg_1 \cdots g_n$ and $g_i \prec fg_1 \cdots g_n$, $i = 1, \ldots, n$, we conclude

$$f[g_1 \cdots g_n] \prec (fg_1 \cdots g_n)[fg_1 \cdots g_n \cdots fg_1 \cdots g_n] = fg_1 \cdots g_n.$$

So, $f[g_1 \cdots g_n] \prec fg_1 \cdots g_n$, which proves (2.6.7).

Sufficiency. Let all conditions of the theorem be satisfied. Consider on a semilattice $(G, \cdot)$ the order $\leqslant$, where

$$g_1 \leqslant g_2 \longleftrightarrow g_1g_2 = g_2$$

for all $g_1, g_2 \in G$. We will show that a Menger algebra (G, o) is isomorphic to some Menger algebra of n-place closure operations on the ordered set $(G, \leqslant)$. First of all, note that (G, o) is reductive. Indeed, if $g_1[x_1 \cdots g_n] = g_2[x_1 \cdots g_n]$ holds for some $g_1, g_2 \in G$ and all $x_1, \ldots, x_n \in G$, then, according to (2.6.7), we have $g_1x_1 \cdots x_n = g_2x_1 \cdots x_n$. Replacing all x_i by g_1 and then by g_2, we get $g_1 = g_2g_1$ and $g_2 = g_1g_2$, which, by the commutativity of $(G, \cdot)$, gives $g_1 = g_2$. Hence (G, o) is reductive.

Similarly as in the proof of Theorem 2.6.7, we can see that the mapping $P_0 : g \mapsto \lambda_g$, where $\lambda_g(x_1, \ldots, x_n) = g[x_1 \cdots x_n]$ for all $x_1, \ldots, x_n \in G$, is an isomorphism of a reductive Menger algebra (G, o) onto a Menger algebra of n-place transformations (Λ, O), where $\Lambda = \{\lambda_g \,|\, g \in G\}$. Any λ_g is an n-place closure operation on the ordered set $(G, \leqslant)$. Indeed, if $g, x_1, \ldots, x_n \in G$, then $x_igx_1 \cdots x_n = gx_1 \cdots x_n$ holds for every $i = 1, \ldots, n$. Thus $x_i \cdot g[x_1 \cdots x_n] = g[x_1 \cdots x_n]$ and, consequently, $x_i \leqslant g[x_1 \cdots x_n] = \lambda_g(x_1, \ldots, x_n)$. So, λ_g is extensive. Now

$$gx_1 \cdots x_n = g(gx_1 \cdots x_n) \cdots (gx_1 \cdots x_n),$$

which means

$$\lambda_g(x_1,\ldots,x_n) = \lambda_g(\lambda_g(x_1,\ldots,x_n),\ldots,\lambda_g(x_1,\ldots,x_n)).$$

Therefore $\lambda_g(x_1,\ldots,x_n) = \lambda_g[\lambda_g\cdots\lambda_g](x_1,\ldots,x_n)$ for all $x_1,\ldots,x_n \in G$. Hence λ_g is idempotent.

Finally, if $x \leqslant y$, i.e., $xy = y$, where $x, y \in G$, then

$$(gu_1\cdots u_{i-1}xu_{i+1}\cdots u_n)\cdot(gu_1\cdots u_{i-1}yu_{i+1}\cdots u_n) = gu_1\cdots u_{i-1}yu_{i+1}\cdots u_n.$$

Thus $\lambda_g(\bar{u}\,|_i x) \leqslant \lambda_g(\bar{u}\,|_i y)$ for all $\bar{u} \in G^n$, and $i = 1,\ldots,n$. So, λ_g is isotone. So it is an n-place closure operation on $(G,\leqslant)$. □

For $n = 1$, Theorem 2.6.8 gives the characterization of a semigroup of closure operations obtained by V. T. Kulik [83].

2.7 Representations of Menger algebras

Any homomorphism P of a Menger algebra (G,o) of rank n into a Menger algebra $(\mathcal{F}(A^n,A),O)$ of n-place functions (respectively, into a Menger algebra $(\mathfrak{R}(A^{n+1}),O)$ of $(n+1)$-ary relations), where A is an arbitrary set, is called a *representation of* (G,o) *by n-place functions* (respectively, *by* $(n+1)$*-ary relations*). In the case when P is an isomorphism, we say that this representation is *faithful*. Two representations P_1 and P_2 are called *similar*, if there is a bijection f of A_1 onto A_2 such that

$$(\bar{a},b) \in P_1(g) \longleftrightarrow (f(\bar{a}),f(b)) \in P_2(g)$$

for any $g \in G$, $\bar{a} \in A_1^n$, $b \in A_1$, where $f(\bar{a})$ means $(f(a_1),\ldots,f(a_n))$.

With every element g of a Menger algebra (G,o) of rank n we associate an n-place function $\Lambda(g) = \lambda_g^* \cap (B \times G)$, where λ_g^* is an n-place function on (G^*,o^*), such that $\lambda_g^*(\bar{x}) = g[\bar{x}]$ for all $\bar{x} \in B = G^n \cup \{(e_1,\ldots,e_n)\}$, where $e_1,\ldots,e_n$ are selectors in (G^*,o^*). It is easy to see that the mapping $g \longmapsto \Lambda(g)$ is a faithful representation of (G,o) by n-place functions. Further on Λ will be called the *canonical representation of* (G,o) *by translations*.

Let (Φ,O) be a Menger algebra of $(n+1)$-ary relations or n-place functions defined on the set A. We shall say that the *family* $\mathfrak{H}$ *of subsets of* A *is admissible for a Menger algebra* (Φ,O) if it is admissible for all $\sigma,\rho_1,\ldots,\rho_n$ from Φ. A family of subsets of (G^*,o^*), which is admissible for $(\Lambda(G),O)$ is called an *admissible family of subsets of a Menger algebra* (G,o).

Let P be a representation of a Menger algebra (G,o) of rank n by $(n+1)$-ary relations or by n-place functions defined on A, $\mathfrak{H}$ — an admissible family of subsets for $(P(G), O)$, then the representation $\hat{P}_{\mathfrak{H}}$ of (G, o) is called $\mathfrak{H}$-*derived from* P if

$$\hat{P}_{\mathfrak{H}}\,(g) = \overline{\overset{\wedge}{P(g)}}_{\mathfrak{H}}$$

for all $g \in G$.

Consider now the family of subsets $\mathfrak{H}_P = (H_b^{\bar{a}})_{b \in A}^{\bar{a} \in A^n}$ of (G^*, o^*) such that for every $i = 1, \ldots, n$ and $\bar{a} = (a_1, \ldots, a_n) \in A^n$ each $H_{a_i}^{\bar{a}}$ contains a selector e_i of (G^*, o^*), where

$$H_b^{\bar{a}} = \{g \in G \,|\, (\bar{a}, b) \in P(g)\}.$$

Lemma 2.7.1. *If P is a representation of a Menger algebra (G, o) of rank n by an $(n+1)$-ary relations or n-place functions, then the family $\mathfrak{H}_P$ is admissible for (G, o).*

Proof. At first we prove that for every $g[\bar{h}] \in H_c^{\bar{a}}$, where $g \in G$, $\bar{h} \in G^n$, $\bar{a} \in A^n$ and $c \in A$ one can find $\bar{b} \in A^n$ such that $h_i \in H_{b_i}^{\bar{a}}$, for $1 \leqslant i \leqslant n$, and $g[H_{b_1}^{\bar{a}} \cdots H_{b_n}^{\bar{a}}] \subset H_c^{\bar{a}}$. Indeed, if $g[\bar{h}] \in H_c^{\bar{a}}$ then $(\bar{a}, c) \in P(g[\bar{h}])$ and, consequently, $(\bar{a}, c) \in P(g)[P(h_1) \cdots P(h_n)]$. Thus there exists $\bar{b} \in A^n$ such that $(\bar{a}, b_i) \in P(h_i)$, $i = 1, \ldots, n$ and $(\bar{b}, c) \in P(g)$. Hence $h_i \in H_{b_i}^{\bar{a}}$, $i = 1, \ldots, n$. If $z_i \in H_{b_i}^{\bar{a}}$, $i = 1, \ldots, n$, then $(\bar{a}, b_i) \in P(z_i)$, $i = 1, \ldots, n$. But $(\bar{b}, c) \in P(g)$ implies $(\bar{a}, c) \in P(g)[P(z_1) \cdots P(z_n)]$, i.e., $(\bar{a}, c) \in P(g[\bar{z}])$, which gives $g[\bar{z}] \in H_c^{\bar{a}}$. So $g[H_{b_1}^{\bar{a}} \cdots H_{b_n}^{\bar{a}}] \subset H_c^{\bar{a}}$ because z_i was arbitrary.

Now let Λ be the canonical representation of (G, o) by translations. We show that $\mathfrak{H}_P$ is admissible for $\Lambda(g), \Lambda(g_1), \ldots, \Lambda(g_n)$, where $g, g_1, \ldots, g_n \in G$. Suppose that for some $\bar{a} \in A^n$ we have

$$\Lambda(g)[\Lambda(g_1) \cdots \Lambda(g_n)](H_{b_1}^{\bar{a}}, \ldots, H_{b_n}^{\bar{a}}) \subset H_d^{\bar{a}}.$$

Then $\Lambda(g)[\Lambda(g_1) \cdots \Lambda(g_n)](h_1, \ldots, h_n) \in H_d^{\bar{a}}$ for all $h_i \in H_{b_i}^{\bar{a}}$, $i = 1, \ldots, n$. Hence

$$g[g_1[h_1 \cdots h_n] \cdots g_n[h_1 \cdots h_n]] \in H_d^{\bar{a}},$$

i.e., $g[g_1[\bar{h}] \cdots g_n[\bar{h}]] \in H_d^{\bar{a}}$. Therefore one can find $\bar{c} \in A^n$ such that $g_i[\bar{h}] \in H_{c_i}^{\bar{a}}$ for every i and $g[H_{c_1}^{\bar{a}} \cdots H_{c_n}^{\bar{a}}] \subset H_d^{\bar{a}}$. Thus, $\Lambda(g)(H_{c_1}^{\bar{a}} \cdots H_{c_n}^{\bar{a}}) \subset H_d^{\bar{a}}$. So, the first condition of the admissibility is satisfied.

Now let $H_{b_1}^{\bar{a}} = H_{b_2}^{\bar{a}}$ for every $\bar{a} \in A^n$, i.e., let for every $\bar{a} \in A^n$ and every $g \in G^*$, $g \in H_{b_1}^{\bar{a}} \longleftrightarrow g \in H_{b_2}^{\bar{a}}$ be true. From this equivalence for $g = e_1$ and $\bar{a} = (b_1, \ldots, b_n)$, we obtain $e_1 \in H_{b_2}^{\bar{a}}$, which implies $b_1 = b_2$. Thus, the second condition of the admissibility is also satisfied. □

Theorem 2.7.2. *Any representation P of a Menger algebra (G, o) of rank n by $(n+1)$-ary relations (or n-place functions) is similar to a representation $\mathfrak{H}_P$-derived from the canonical representation of (G, o) by translations.*

Proof. Let P be a representation of a Menger algebra (G, o) of rank n by $(n+1)$-ary operations or n-place functions defined on the set A. According to Lemma 2.7.1 the family of subsets $\mathfrak{H}_P$ is admissible for (G, o), hence a representation $\mathfrak{H}_P$-derived from the canonical representation by translation $\overset{\wedge}{\Lambda}_{\mathfrak{H}_P}$ is also a representation of (G, o). We show that $\overset{\wedge}{\Lambda}_{\mathfrak{H}_P}$ and P are similar. Let $f : A \to \mathfrak{H}_P$ be such that $f(b) = (H_b^{\bar{a}})^{\bar{a}\in A^n}$ for every $b \in A$. Then

$$(f(\bar{b}), f(c)) \in \overset{\wedge}{\Lambda}_{\mathfrak{H}_P}(g) \longleftrightarrow ((H_{\bar{b}}^{\bar{a}})^{\bar{a}\in A^n}, (H_c^{\bar{a}})^{\bar{a}\in A^n}) \in \overline{\overset{\wedge}{\Lambda(g)}}_{\mathfrak{H}_P}$$

$$\longleftrightarrow (\forall \bar{a} \in A^n)\ (\Lambda(g)(H_{b_1}^{\bar{a}}, \ldots, H_{b_n}^{\bar{a}}) \subset H_c^{\bar{a}}) \longleftrightarrow (\bar{b}, c) \in P(g),$$

where $f(\bar{b}) = (f(b_1), \ldots, f(b_n))$, $(H_{\bar{b}}^{\bar{a}})^{\bar{a}\in A^n} = ((H_{b_1}^{\bar{a}})^{\bar{a}\in A^n}, \ldots, (H_{b_n}^{\bar{a}})^{\bar{a}\in A^n})$ and $g \in G$. We show that the last equivalence is true. Indeed, for all $\bar{a} \in A^n$ we have $\Lambda(g)(H_{b_1}^{\bar{a}}, \ldots, H_{b_n}^{\bar{a}}) \subset H_c^{\bar{a}}$, i.e.,

$$\lambda_g^*((H_{b_1}^{\bar{a}} \times \cdots \times H_{b_n}^{\bar{a}}) \cap B) \subset H_c^{\bar{a}},$$

where $B = G^n \cup \{\bar{e}\}$, $\bar{e} = (e_1, \ldots, e_n)$. From this, replacing $\bar{a}$ by $\bar{b}$, we get $\lambda_g^*((H_{b_1}^{\bar{b}} \times \cdots \times H_{b_n}^{\bar{b}}) \cap B) \subset H_c^{\bar{b}}$. But $e_i \in H_{b_i}^{\bar{b}}$ for all $i = 1, \ldots, n$. Then $\lambda^*(e_1, \ldots, e_n) \in H_c^{\bar{b}}$. So, $g[\bar{e}] \in H_c^{\bar{b}}$, i.e., $g \in H_c^{\bar{b}}$. Therefore $(\bar{b}, c) \in P(g)$.

On the other hand, if $(\bar{b}, c) \in P(g)$ and $\bar{g} \in H_{b_1}^{\bar{a}} \times H_{b_n}^{\bar{a}} \cap B$, then $(\bar{a}, b_i) \in P(g_i)$, $i = 1, \ldots, n$. Thus

$$(\bar{a}, c) \in P(g)[P(g_1) \cdots P(g_n)] = P(g[\bar{g}]),$$

i.e., $g[\bar{g}] \in H_c^{\bar{a}}$, and consequently, $\Lambda(g)(H_{b_1}^{\bar{a}}, \ldots, H_{b_n}^{\bar{a}}) \subset H_c^{\bar{a}}$, which completes the proof of this implication because $\bar{a}$ and $\bar{g}$ are arbitrary.

Since $f(b_1) = f(b_2)$ implies $H_{b_1}^{\bar{a}} = H_{b_2}^{\bar{a}}$ for every $\bar{a} \in A^n$, then $b_1 = b_2$, which means that $f : A \to \mathfrak{H}_P$ is a bijection. So $\overset{\wedge}{\Lambda}_{\mathfrak{H}_P}$ and P are similar. □

Let $(P_i)_{i\in I}$ be a family of representations of a Menger algebra (G, o) of rank n by n-place functions (by $(n+1)$-relations) on sets $(A_i)_{i\in I}$ respectively, where the sets A_i are pairwise disjoint. The *sum* of $(P_i)_{i\in I}$ is a mapping P, which will be denoted by $\sum_{i\in I} P_i$, from (G, o) into $\mathcal{F}(A^n, A)$ (respectively, into $\mathfrak{R}(A^{n+1})$), where $A = \bigcup_{i\in I} A_i$, defined as $P(g) = \bigcup_{i\in I} P_i(g)$ for every $g \in G$. It is not difficult to see that P is a representation of (G, o).

With every representation P of (G, o) by n-place functions ($(n+1)$-ary relations) we associate the following binary relations on G:

$$\zeta_P = \{(g_1, g_2) \mid P(g_1) \subset P(g_2)\},$$

$$\chi_P = \{(g_1, g_2) \mid \mathrm{pr}_1 P(g_1) \subset \mathrm{pr}_1 P(g_2)\},$$

$$\pi_P = \{(g_1, g_2) \mid \mathrm{pr}_1 P(g_1) = \mathrm{pr}_1 P(g_2)\},$$

$$\gamma_P = \{(g_1, g_2) \mid \mathrm{pr}_1 P(g_1) \cap \mathrm{pr}_1 P(g_2) \neq \varnothing\},$$

$$\kappa_P = \{(g_1, g_2) \mid P(g_1) \cap P(g_2) \neq \varnothing\},$$

$$\xi_P = \{(g_1, g_2) \mid P(g_1) \circ \triangle_{\mathrm{pr}_1 P(g_2)} = P(g_2) \circ \triangle_{\mathrm{pr}_1 P(g_1)}\}.$$

It is easy to see that if P is the sum of the family of representations $(P_i)_{i\in I}$ then $\sigma_P = \bigcap_{i\in I} \sigma_{P_i}$ for $\sigma \in \{\zeta, \chi, \pi, \xi\}$ and $\sigma_P = \bigcup_{i\in I} \sigma_{P_i}$ for $\sigma \in \{\kappa, \gamma\}$.

Definition 2.7.3. A *determining pair* of a Menger algebra (G, o) of rank n is any pair (ε, W), where ε is a partial equivalence on (G^*, o^*), W is a subset of G^* and the following conditions hold:
(1) $G \cup \{e_1, \ldots, e_n\} \subset \mathrm{pr}_1\varepsilon$, where $e_1, \ldots, e_n$ are selectors of (G^*, o^*),
(2) $e_i \notin W$ for all $i = 1, \ldots, n$,
(3) $g[\varepsilon\langle e_1\rangle \cdots \varepsilon\langle e_n\rangle] \subset \varepsilon\langle g\rangle$ for all $g \in G$,
(4) $g[\varepsilon\langle g_1\rangle \cdots \varepsilon\langle g_n\rangle] \subset \varepsilon\langle g[g_1 \cdots g_n]\rangle$ for all $g, g_1, \ldots, g_n \in G$,
(5) if $W \neq \varnothing$, then W is an ε-class and $W \cap G$ is an l-ideal of (G, o).

An example of determining pairs are pairs $(\varepsilon_v^*(H), W_v(H))$, where H is an arbitrary subset of (G, o), $\varepsilon_v^*(H) = \varepsilon_v(H) \cup \{(e_1, e_1), \ldots, (e_n, e_n)\}$, $e_1, \ldots, e_n$ are selectors of (G^*, o^*) [4].

With every determining pair (ε, W) is associated the so-called *simplest representation* $P_{(\varepsilon,W)}$ of (G, o). Let $(H_a)_{a\in A}$ be the family of all ε-classes distinct from W which is indexed by elements from A. Now let $e_i \in H_{b_i}$ for every $i = 1, \ldots, n$, $A_0 = \{a \in A \mid H_a \cap G \neq \varnothing\}$, $\mathfrak{A} = A_0^n \cup \{(b_1, \ldots, b_n)\}$, $B = G^n \cup \{(e_1, \ldots, e_n)\}$. Every $g \in G$ is associated with an n-place function $P_{(\varepsilon,W)}(g)$ on A, which is defined by

$$(\bar{a}, b) \in P_{(\varepsilon,W)}(g) \longleftrightarrow \bar{a} \in \mathfrak{A} \wedge g[H_{a_1} \cdots H_{a_n}] \subset H_b,$$

where $\bar{a} \in A^n$, $b \in A$.

Proposition 2.7.4. *The map $P_{(\varepsilon,W)}$ is a homomorphism of (G, o) into $(\mathcal{F}(A^n, A), O)$.*

Proof. Indeed, if $a = P_{(\varepsilon,W)}(g[\bar{g}])(\bar{a})$ for some $\bar{a} \in A^n$, $a \in A$, $\bar{g} \in G^n$ and $g \in G$, then $\bar{a} \in \mathfrak{A}$ and $g[\bar{g}][H_{a_1} \cdots H_{a_n}] \subset H_a$. Obviously for $h_i \in H_{a_i}$, where $i = 1, \ldots, n$, we have $g[g_1[\bar{h}] \cdots g_n[\bar{h}]] = g[\bar{g}][\bar{h}] \in H_a$, which implies

$$g[g_1[H_{a_1} \cdots H_{a_n}] \cdots g_n[H_{a_1} \cdots H_{a_n}]] \subset H_a.$$

[4] For the definition of $\varepsilon_v(H)$ and $W_v(H)$ see Section 2.1.

Assume that $g_i[H_{a_1}\cdots H_{a_n}] \subset H_{c_i}$, $i = 1,\ldots,n$. Then clearly $\bar{c} \in \mathfrak{A}$. Thus $g[H_{c_1}\cdots H_{c_n}] \subset H_a$ and $g_i[H_{a_1}\cdots H_{a_n}] \subset H_{c_i}$ for all $i = 1,\ldots,n$, where $\bar{a},\bar{c} \in \mathfrak{A}$. Therefore $(\bar{a}, c_i) \in P_{(\varepsilon,W)}(g_i)$ and $(\bar{c}, a) \in P_{(\varepsilon,W)}(g)$, i.e.,

$$a = P_{(\varepsilon,W)}(g)[P_{(\varepsilon,W)}(g_1)\cdots P_{(\varepsilon,W)}(g_n)](\bar{a}).$$

So we have

$$P_{(\varepsilon,W)}(g[g_1\cdots g_n]) \subset P_{(\varepsilon,W)}(g)[P_{(\varepsilon,W)}(g_1)\cdots P_{(\varepsilon,W)}(g_n)].$$

The reverse inclusion can be proved analogously. □

It is easy to show that a determining pair (ε, W) defines the simplest representation of a Menger algebra (G, o) of rank $n \geqslant 2$ by reversive n-place functions (i.e., by partial n-ary quasi-groups) if and only if the following condition is satisfied:

$$g[\bar{w}\,|_i\, x] \equiv g[\bar{w}\,|_i\, y](\varepsilon) \,\wedge\, g[\bar{w}\,|_i\, x] \notin W \longrightarrow x \equiv y(\varepsilon) \qquad (2.7.1)$$

where $g, x, y \in G$, $\bar{w} \in G^n$, $i = 1,\ldots,n$. Of course, this condition is satisfied by a determining pair $(\varepsilon_v^*(H), W_v(H))$, where H is a strong subset of (G, o).

Moreover, for a determining pair (ε, W) of (G, o) the following formulas are true:

$$(g_1, g_2) \in \zeta_{(\varepsilon,W)} \longleftrightarrow (\forall \bar{x})(g_1[\bar{x}] \notin W \longrightarrow g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon)),$$

$$(g_1, g_2) \in \chi_{(\varepsilon,W)} \longleftrightarrow (\forall \bar{x})(g_1[\bar{x}] \notin W \longrightarrow g_2[\bar{x}] \notin W),$$

$$(g_1, g_2) \in \pi_{(\varepsilon,W)} \longleftrightarrow (\forall \bar{x})(g_1[\bar{x}] \notin W \longleftrightarrow g_2[\bar{x}] \notin W),$$

$$(g_1, g_2) \in \xi_{(\varepsilon,W)} \longleftrightarrow (\forall \bar{x})(g_1[\bar{x}], g_2[\bar{x}] \notin W \longrightarrow g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon)),$$

$$(g_1, g_2) \in \gamma_{(\varepsilon,W)} \longleftrightarrow (\exists \bar{x})(g_1[\bar{x}] \notin W \wedge g_2[\bar{x}] \notin W),$$

$$(g_1, g_2) \in \kappa_{(\varepsilon,W)} \longleftrightarrow (\exists \bar{x})(g_1[\bar{x}] \notin W \wedge g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon)),$$

where $\bar{x} \in B = G^n \cup \{\bar{e}\}$, $\sigma_{(\varepsilon,W)} = \sigma_{P_{(\varepsilon,W)}}$ for $\sigma \in \{\zeta, \chi, \pi, \xi, \gamma, \kappa\}$.

Let P and P_i, $i \in I$, be representations of a Menger algebra (G, o) by n-place functions. A representation P is called the *union* of the family $(P_i)_{i\in I}$, if $P(g) = \bigcup_{i\in I} P_i(g)$ for all $g \in G$.

Theorem 2.7.5. *Every representation of a Menger algebra of rank n by n-place functions is the union of some family of its simplest representations.*

Proof. Let P be a representation of a Menger algebra (G, o) of rank n by n-place functions defined on the set A, c — some fixed element not belonging to A. With every $g \in G$ we associate an n-ary operation $P^*(g)$ on the set $A^* = A \cup \{c\}$ such that

$$P^*(g)(\bar{a}) = \begin{cases} P(g)(\bar{a}) & \text{if } \bar{a} \in \mathrm{pr}_1 P(g), \\ c & \text{if } \bar{a} \notin \mathrm{pr}_1 P(g). \end{cases}$$

It is not difficult to see that P^* is a representation of (G, o) by n-ary operations on A^* and $\varphi : P(g) \longmapsto P^*(g)$ is an isomorphism of $(P(G), O)$ onto $(P^*(G), O)$. (G^*, o^*) is generated by $G \cup \{e_1, \ldots, e_n\}$. So putting $P^*(e_i) = I_i^n$ for $i = 1, \ldots, n$, where I_i^n is an ith n-place projector on A^*, we extend P^* to (G^*, o^*).

With every $\bar{a} \in A^n$ we associate an equivalence relation $\Theta_{\bar{a}}$ on G^* defined by the formula

$$\Theta_{\bar{a}} = \{(x, y) \mid P^*(x)(\bar{a}) = P^*(y)(\bar{a})\}.$$

It is easy to see that $\Theta_{\bar{a}}$ is v-regular, $\overset{*}{H}{}_b^{\bar{a}} = \{x \in G^* \mid (\bar{a}, b) \in P^*(x)\}$ are $\Theta_{\bar{a}}$-classes such that $H_b^{\bar{a}} = \overset{*}{H}{}_b^{\bar{a}} \cap G$ for every $b \in A$. On G^* we introduce also a partial equivalence

$$\varepsilon_{\bar{a}}^* = \Theta_{\bar{a}} \cap \Theta_{\bar{a}}(G \cup \{e_1, \ldots, e_n\}) \times \Theta_{\bar{a}}(G \cup \{e_1, \ldots, e_n\}),$$

where $e_1, \ldots, e_n$ are the selectors in (G^*, o^*). It is obvious that the restriction of $\varepsilon_{\bar{a}}^*$ to G coincides with

$$\varepsilon_{\bar{a}} = \{(g_1, g_2) \mid P(g_1)\langle \bar{a} \rangle = P(g_2)\langle \bar{a} \rangle\} \subset G \times G$$

and the pair $(\varepsilon_{\bar{a}}^*, W_{\bar{a}}^*)$, where

$$W_{\bar{a}}^* = \Theta_{\bar{a}}(W_{\bar{a}}) \quad \text{and} \quad W_{\bar{a}} = \{g \mid P(g)\langle \bar{a} \rangle = \varnothing\} \subset G,$$

is the determining pair.

Now we prove that a representation P is an union of the simplest representations induced by the family $((\varepsilon_{\bar{a}}^*, W_{\bar{a}}^*))_{\bar{a} \in A^n}$ of determining pairs, i.e., that

$$P(g) = \bigcup_{\bar{a} \in A^n} P_{(\varepsilon_{\bar{a}}^*, W_{\bar{a}}^*)}(g)$$

for every $g \in G$. For this let $g \in G$ and $(\bar{b}, c) \in P(g)$, i.e., $g \in H_c^{\bar{b}}$. Then $g[e_1 \cdots e_n] = g$ and $e_i \in \overset{*}{H}{}_{b_i}^{\bar{b}}$ for $i = 1, \ldots, n$. Hence, by the v-regularity of $\Theta_{\bar{b}}$, we have

$$g[\overset{*}{H}{}_{b_1}^{\bar{b}} \cdots \overset{*}{H}{}_{b_n}^{\bar{b}}] \subset \overset{*}{H}{}_c^{\bar{b}} \ .$$

Therefore $(\bar{b}, c) \in P_{(\varepsilon^*_{\bar{b}}, W^*_{\bar{b}})}(g)$, i.e.,

$$P(g) \subset \bigcup_{\bar{a} \in A^n} P_{(\varepsilon^*_{\bar{a}}, W^*_{\bar{a}})}(g).$$

To prove the reverse inclusion, let $(\bar{b}, c) \in P_{(\varepsilon^*_{\bar{a}}, W^*_{\bar{a}})}(g)$ for some $\bar{a} \in A^n$. Then obviously $g[\overset{*}{H}{}^{\bar{a}}_{b_1} \cdots \overset{*}{H}{}^{\bar{a}}_{b_n}] \subset \overset{*}{H}{}^{\bar{a}}_{c}$, i.e., $g[h_1 \cdots h_n] \in H^{\bar{a}}_c$ and $h_i \in H^{\bar{a}}_{b_i}$ for $i = 1, \ldots, n$. But $(\bar{a}, c) \in P(g[h_1 \cdots h_n]) = P(g)[P(h_1) \cdots P(h_n)]$. So, there is $\bar{d} \in A^n$ such that $(\bar{d}, c) \in P(g)$ and $(\bar{a}, d_i) \in P(h_i)$, $i = 1, \ldots, n$. Since $(\bar{a}, b_i) \in P(h_i)$, $i = 1, \ldots, n$ and $P(h_i)$ is a function, we have $d_i = b_i$, $i = 1, \ldots, n$, which gives $\bar{b} = \bar{d}$ and $(\bar{b}, c) \in P(g)$. So,

$$\bigcup_{\bar{a} \in A^n} P_{(\varepsilon^*_{\bar{a}}, W^*_{\bar{a}})}(g) \subset P(g).$$

This completes the proof that a representation P is the union of the family of simplest representations. □

2.8 Notes on Chapter 2

In literature, the diagonal unit is also called the *special element* (see [27,29]) or the neutral element of (G, o) (see [197]) but it is not the neutral element in the sense of n-ary structures, i.e., it does not satisfy the identity $o(\overset{(i-1)}{e}, x, \overset{(n-i)}{e}) = x$.

Any i-solvable Menger algebra (G, o) of rank n which is associative in the sense of E. L. Post [144] is an $(n+1)$-ary group derived from its diagonal group, i.e., it has the form $o(x_0, x_1, \ldots, x_n) = x_0 \cdot x_1 \cdot \ldots \cdot x_n$, where $(G, \cdot)$ is the diagonal group of (G, o). In this case, the diagonal group coincides with the group $(G, *)$, where $x * y = o(x, e, \ldots, e, y)$ and e is the diagonal unit of (G, o) (see [27]). Group-like Menger algebras of order $n > 2$ are derived from a cyclic group of order 2. No two-element group-like Menger algebras of order 2 exist (see [29]). Group-like Menger algebras with operations satisfying some commutative law are described in [27–29].

Menger algebras are special case of the so-called *Dicker algebras*, i.e., $(n+1)$-ary groupoids satisfying the identity

$$o(x_1^i, o(y_0^n), x_{i+1}^n) = o(y_0, o(x_1^i, y_1, x_{i+1}^n), o(x_1^i, y_2, x_{i+1}^n), \ldots, o(x_1^i, y_n, x_{i+1}^n)),$$

where $i = 0, 1, \ldots, n$. Dicker algebras were introduced in [20]. Some results on Dicker $(n+1)$-ary quasigroups and loops one can find in [6, 7]. Properties of i-solvable ith Dicker algebras are described in [30].

H. Länger in [95] and [99] considered abstract algebras with one n-ary operation satisfying some other associative laws.

A Menger algebra is called *s-v-simple* if it posses no proper s-ideals and v-ideals. Of course, every group-like Menger algebra is s-v-simple. A Menger algebra is s-v-simple if and only if its diagonal semigroup is a group [261]. So, any i-solvable Menger algebra is s-v-simple too [27].

A Menger algebra is *completely simple* if it posses minimal s-ideals and v-ideals but has no proper s-ideals which are v-ideals. Ja. N. Yaroker proved in [261] that each such Menger algebra can be decomposed into a disjoint union of completely simple Menger algebras with pairwise isomorphic diagonal groups. Moreover, a mapping $h : G_1 \longrightarrow G_2$ of two s-v-simple Menger algebras is an isomorphism if and only if it is an isomorphism of the corresponding rigged diagonal groups.

The role of constant elements with some additional properties is described in [87,91,98]. Menger algebras in which every subset is a subalgebra are studied in [196]. The results obtained are very similar to the corresponding results for semigroups.

Characterizations of various types of ideals of Menger algebras of multiplace functions can be found in the series of papers written by L. M. Gluskin. For example, in [51] all densely embedded ideals of Menger algebras of all full and all partial n-place functions are described, and it is proved that automorphisms of such Menger algebras are inner. Congruences of Menger algebras of linear mappings are described in [11], finitely generated subalgebras of this algebra — in [12]. Congruences of group-like Menger algebras are completely characterized by congruences of its diagonal group [227]. J. Henno [57] gives a natural generalization of the Green relations in semigroups to the case of Menger algebras and proves that their properties are the same as in the case of semigroups (see also [41,67]). An abstract characterization of Menger algebras of relations in terms of densely embedded ideals can be found in [262].

Strong connections between superassociativity of the composition of multiplace functions and the associativity of the corresponding binary comitant were firstly noted by P. Hall in his lectures 1947 – 1949. But the first abstract characterization of the associativity of (G, o) by the associativity of $(G^n, *)$ was given by R. Dicker [20]. In [167] binary comitants were used by B. M. Schein to characterize the sets of binary relations. Description of some cases of binary comitants (called also *induced groupoids*) can be found in [88].

The first abstract characterization of semigroups of extensive transformations of an ordered set was given by B. M. Schein [183]. The first characterization of ordered Menger algebras was done in [213]. Lattice ordered semigroups of functions were described in [181]. An abstract characterization of the class of all semigroups of semiclosure operations on ordered sets was obtained by V. S. Trokhimenko [233]. He proved that those semigroups cannot be characterized by means of finite system of elementary axioms. In [237] all these results are extended to

Menger algebras of extensive n-place mappings and n-place semiclosure operations.

The method of representation of semigroups by binary relations was proposed by B. M. Schein [162], who proved that every such representation can be expressed in terms of the derivative of the canonical ω-fold representation for some cardinal ω.

Schein's method of construction of representations is based on Vagner's construction (for details see [246]) of some family of right regular equivalences (such that one of its classes is a right ideal) with the property that one class of this equivalence is a right ideal. Schein's construction (see [161]) instead of giving a family of right regular equivalences gives a family of the determining pairs. Using this family he find abstract characterizations of binary semigroups of various transformations of a given set (see for example [160, 161, 164, 171]). In Section 2.7 we extend all these results to Menger algebras.

Menger algebras with selectors can be identified with some systems of all multiplace endomorphisms of a universal algebra (see [64,74,147]). E. Redi [147] proved that a Menger algebra is isomorphic to the system of all multiplace endomorphisms of a universal algebra if and only if it has a full system of selectors.

W. Nöbauer and W. Philipp consider [134] the set of all one-place mappings of a fixed universal algebra into itself with a Menger composition and proved that for $n > 1$ this algebra is simple in the sense that it possesses no congruences other the equality and the universal relation [136].

Regular and idempotent elements in Menger algebras of terms are characterized in [19].

Chapter 3

Ordered Menger algebras

3.1 Menger algebras of relations

Let (Φ, O) be a Menger algebra of $(n+1)$-ary relations (in particular, of n-place functions). Let ζ_Φ and χ_Φ denote the inclusion relation and the relation of inclusion of domains (see Section 2.1) between the elements of Φ. Algebraic systems of the form $(\Phi, O, \zeta_\Phi, \chi_\Phi)$, (Φ, O, ζ_Φ), (Φ, O, χ_Φ) will be called: *fundamentally ordered projection (f.o.p.) Menger algebras, fundamentally ordered (f.o.) Menger algebras and projection quasi-ordered (p.q-o.) Menger algebras of* $(n+1)$*-relations.*

Let (G, o) be a Menger algebra of rank n, $e_1, e_2, \ldots, e_n$ — n distinct elements, which do not belong to G. For these elements we let $g[e_1 \cdots e_n] = g$ for every $g \in G$.

For every $a \in G$ we define the following three sets:

$$G_a = G \times \{a\}, \qquad \widetilde{G}_a = G_a \cup \{e_{1a}, \ldots, e_{na}\}, \qquad B_a = G_a^n \cup \{\bar{e}_a\},$$

where $\bar{e}_a = (e_{1a}, \ldots, e_{na})$, $e_{ia} = (e_i, a)$, $i = 1, \ldots, n$. It is easy to see that $\widetilde{G}_{a_1} \cap \widetilde{G}_{a_2} = \varnothing$ for $a_1 \neq a_2$.

The first characterization of f.o.p. Menger algebras of $(n+1)$-relations is given by

Theorem 3.1.1. *An algebraic system* (G, o, ζ, χ)*, where* (G, o) *is a Menger algebra of rank* n *and* ζ*,* χ *are relations defined on* G*, is isomorphic to some f.o.p. Menger algebra of* $(n+1)$*-relations if and only if* ζ *is a stable order and* χ *is an l-regular and v-negative quasi-order containing* ζ*.*

Proof. We prove only the sufficiency because the necessity of conditions given in the theorem follows from general properties of relations described in § 1.1.

Let $a \in G$ be fixed. For every $g \in G$ we define an $(n+1)$-relation $P_a(g) \subset \widetilde{G}_a \times \widetilde{G}_a$ letting

$$(\bar{x}_a, y_a) \in P_a(g) \longleftrightarrow a \sqsubset y \leqslant g[\bar{x}] \tag{3.1.1}$$

for all $\bar{x}_a = (x_{1a}, \ldots, x_{na}) \in B_a$, $y_a \in G_a$, where y_a means (y, a) for every $y \in G$. The condition $a \sqsubset y \leqslant g[\bar{x}]$ is equivalent to the conjunction

$$a \sqsubset y \ \wedge \ y \leqslant g[\bar{x}],$$

where $x \sqsubset y \longleftrightarrow (x,y) \in \chi$, and $x \leqslant y \longleftrightarrow (x,y) \in \zeta$.

We will prove that $P_a : g \longmapsto P_a(g)$ is the representation of (G,o) by $(n+1)$-relations. Let $(\bar{x}_a, z_a) \in P_a(g[g_1 \cdots g_n])$ for some $g, g_1, \ldots, g_n \in G$. Then

$$a \sqsubset z \leqslant g[g_1 \cdots g_n][\bar{x}] = g[g_1[\bar{x}] \cdots g_n[\bar{x}]].$$

From this, by the v-negativity of χ, we obtain $a \sqsubset g_i[\bar{x}] \leqslant g_i[\bar{x}]$ for every $i = 1, \ldots, n$. Thus, $(\bar{x}_a, g_i[\bar{x}]_a) \in P_a(g_i)$, $i = 1, \ldots, n$, and, consequently, $(g_1[\bar{x}]_a, \ldots, g_n[\bar{x}]_a, z_a) \in P_a(g)$, that is,

$$(\bar{x}_a, z_a) \in P_a(g)[P_a(g_1) \cdots P_a(g_n)].$$

Conversely, the last condition implies the existence of a $\bar{y}_a = (y_{1a}, \ldots, y_{na})$ in G_a^n such that $(\bar{x}_a, y_{ia}) \in P_a(g_i)$, $i = 1, \ldots, n$, and $(\bar{y}_a, z_a) \in P_a(g)$, i.e., $a \sqsubset y_i \leqslant g_i[\bar{x}]$, $i = 1, \ldots, n$ and $a \sqsubset z \leqslant g[\bar{y}]$. Applying the stability of ζ to $y_i \leqslant g_i[\bar{x}]$, $i = 1, \ldots, n$, we get

$$g[\bar{y}] \leqslant g[g_1[\bar{x}] \cdots g_n[\bar{x}]] = g[g_1 \cdots g_n][\bar{x}].$$

This implies $a \sqsubset z \leqslant g[\bar{y}] \leqslant g[g_1 \cdots g_n][\bar{x}]$, hence $(\bar{x}_a, z_a) \in P_a(g[g_1 \cdots g_n])$. Therefore

$$P_a(g[g_1 \cdots g_n]) = P_a(g)[P_a(g_1) \cdots P_a(g_n)],$$

i.e., P_a is the representation of (G,o).

Now let P be the sum of the family of representations $(P_a)_{a \in G}$. We prove that in this case $\zeta = \zeta_P$ and $\chi = \chi_P$. Indeed, for any $g_1, g_2 \in G$ the condition $P(g_1) \subset P(g_2)$ means that for all $a \in G$, $\bar{x}_a \in B_a$, $y_a \in G_a$ from $a \sqsubset y \leqslant g_1[\bar{x}]$ follows $a \sqsubset y \leqslant g_2[\bar{x}]$, which for $a = y = g_1$, $\bar{x} = \bar{e}$ gives $g_1 \leqslant g_2$. Conversely, if $g_1 \leqslant g_2$ and $a \sqsubset y \leqslant g_1[\bar{x}]$, then the stability of ζ implies $a \sqsubset y \leqslant g_2[\bar{x}]$. So, $\zeta = \zeta_P$. The second equality can be proved similarly.

Finally, if $P(g_1) = P(g_2)$, i.e., $P(g_1) \subset P(g_2)$ and $P(g_2) \subset P(g_1)$, then $g_1 \leqslant g_2$ and $g_2 \leqslant g_1$. Therefore $g_1 = g_2$. So, P is an isomorphism of (G, o, ζ, χ) onto a f.o.p. Menger algebra $(\Phi, O, \zeta_\Phi, \chi_\Phi)$ of $(n+1)$-relations, where $\Phi = \{P(g) \,|\, g \in G\}$. □

For $\chi = G \times G$ Theorem 3.1.1 shows that *each stable ordered Menger algebra of rank n is isomorphic to some f.o. Menger algebra of $(n+1)$–relations*. For $n = 1$ it is the result obtained by K.A. Zaretsky [263] for ordered semigroups of binary relations. Moreover, if in Theorem 3.1.1 instead of ζ we take the diagonal of G, then we obtain that *each p.q-o. Menger algebra of $(n+1)$-relations is a Menger algebra of rank n with the l-regular and v-negative quasi-order.*

As in the case of semigroups [263], the further study of Menger algebras of $(n+1)$-relations needs the study of some special types (for example, reflexive, transitive and others). First, we consider the case of reflexive relations.

Theorem 3.1.2. *An algebraic system (G, o, ζ, χ), where (G, o) is a Menger algebra of rank $n \geqslant 2$, while ζ, χ are binary relations on G, is isomorphic to some f.o.p. Menger algebra of reflexive $(n+1)$-relations if and only if the conditions of Theorem* 3.1.1 *are satisfied and* $x \leqslant y[x^n]$ *for all* $x, y \in G$.

Proof. The necessary part is obvious, since for any two reflexive $(n+1)$-relations ρ and σ we have $\rho \subset \sigma[\rho \cdots \rho]$. We prove the sufficiency using the notation of the previous theorem. Let a be an arbitrary element of G and let $H_a = \{x_a \mid a \sqsubset x\}$, $\widetilde{H}_a = H_a \cup \{e_{1a}, \ldots, e_{na}\}$, $H = \bigcup\{\widetilde{H}_a \mid a \in G\}$. For every $g \in G$ we define the $(n+1)$-relation $P_a^*(g)$ on $\widetilde{H}_a$ by letting:

$$P_a^*(g) = P_a(g) \cup \left\{ \overset{n+1}{e_{1a}}, \ldots, \overset{n+1}{e_{na}} \right\}, \tag{3.1.2}$$

where $\overset{n+1}{e_{ia}}$ denotes the $(n+1)$–tuple $(e_{ia}, \ldots, e_{ia})$, and $P_a(g)$ is defined by (3.1.1). Then $P_a(g) \subset (\widetilde{H}_a)^{n+1}$. Indeed, if $(\bar{x}_a, y_a)$ belongs to $P_a(g)$, then $a \sqsubset y \leqslant g[\bar{x}]$, which, by the v-negativity of χ, implies $\bar{x}_a \in (\widetilde{H}_a)^n$ and $y_a \in \widetilde{H}_a$. Hence $(\bar{x}_a, y_a) \in (\widetilde{H}_a)^{n+1}$. Let P^* be the sum of the representations of the family $(P_a^*)_{a \in G}$, then, for every $g \in G$, $P^*(g)$ is an $(n+1)$–relation on H. In the same way as in the proof of Theorem 3.1.1 we show that P^* is the faithful representation of (G, o) such that $\zeta = \zeta_{P^*}$ and $\chi = \chi_{P^*}$. To prove that $P^*(g)$ is a reflexive $(n+1)$–relation, it is sufficient to observe that $P_a(g)$ is a reflexive relation on H_a. But this is obvious since $x \leqslant y[x^n]$ for all $x, y \in G$ and $a \sqsubset x \leqslant g[x^n]$ imply $(x_a, \ldots, x_a) \in P_a(g)$ for all $x_a \in H_a$. □

From the above theorem for $\chi = G \times G$, we obtain that *every stable ordered Menger algebra of rank* $n \geqslant 2$, *which satisfies the condition* $x \leqslant y[x^n]$, *is isomorphic to some f.o. Menger algebra of reflexive* $(n+1)$*–relations.* Remark that in Theorem 3.1.2 the assumption $n \geqslant 2$ is essential because for $n = 1$ (i.e., for semigroups) it is not difficult to see that $\chi = G \times G$ must be assumed as well. Moreover, as it follows from [263], besides $x \leqslant y[x^n]$ we should assume one more inequality.

Let ρ be a binary relation on a Menger algebra (G, o) of rank n. We define the relation $\zeta(\rho) \subset G \times G$ putting $(g_1, g_2) \in \zeta(\rho)$ if and only if there exist polynomials $t_i \in T_n(G)$, vectors $\bar{z}_i \in B = G^n \cup \{\bar{e}\}$ and pairs $(x_i, y_i) \in \rho \cup \{(e_1, e_1), \ldots, (e_n, e_n)\}$, where $\bar{e} = (e_1, \ldots, e_n)$ are selectors from (G^*, o^*), such that

$$g_1 = t_1(x_1[\bar{z}_1]) \wedge \bigwedge_{i=1}^{m} (t_i(y_i[\bar{z}_i]) = t_{i+1}(x_{i+1}[\bar{z}_{i+1}])) \wedge t_{m+1}(y_{m+1}[\bar{z}_{m+1}]) = g_2$$

for some natural number m. It is not difficult to verify that $\zeta(\rho)$ is the least stable quasi-order on (G, o), which contains ρ.

Theorem 3.1.3. *Let (G, o) be a Menger algebra of rank $n \geqslant 2$, $\chi \subset G \times G$, $\sigma = \{(x, g[x \cdots x]) \,|\, x, g \in G\}$. An algebraic system (G, o, χ) is isomorphic to some p.q-o. Menger algebra of reflexive $(n+1)$-relations if and only if $\zeta(\sigma)$ is an antisymmetric relation, χ is an l-regular v-negative quasi-order and the condition*

$$t(x[y_1 \cdots y_n]) \sqsubset t(g[x^n][y_1 \cdots y_n]) \tag{3.1.3}$$

is satisfied for all $t \in T_n(G)$, $x, g \in G$, $(y_1, \ldots, y_n) \in B$.

Proof. Let P be an isomorphism of (G, o, χ) onto some p.q-o. Menger algebra of reflexive $(n+1)$–relations. Obviously ζ_P contains $\zeta(\sigma)$, hence $\zeta(\sigma) \cap (\zeta(\sigma))^{-1} \subset \zeta_P \cap \zeta_P^{-1} \subset \triangle_G$, which proves that $\zeta(\sigma)$ is antisymmetric. Because ζ_P is stable and $\sigma \subset \zeta(\sigma) \subset \zeta_P$, then $(u, v) \in \sigma$ implies $(t(u[\bar{x}]), t(v[\bar{x}])) \in \zeta_P$ for all $t \in T_n(G)$ and $\bar{x} \in B$. But $\zeta_P \subset \chi_P = \chi$. Then $t(u[\bar{x}]) \sqsubset t(v[\bar{x}])$, which gives (3.1.3). The necessary part is proved.

To prove the sufficiency assume that (G, o, χ) satisfies all the conditions given in the theorem. Consider $(g_1, g_2) \in \zeta(\sigma)$, then $g_1 = t_1(x_1[\bar{z}_1])$, $g_2 = t_m(y_m[x_m \cdots x_m][\bar{z}_m])$ and $t_i(y_i[x_i \cdots x_i][\bar{z}_i]) = t_{i+1}(x_{i+1}[\bar{z}_{i+1}])$ for all $i = 1, \ldots, m-1$ and some $m \in \mathbb{N}$, $t \in T_n(G)$, $x_i, y_i \in G$, $\bar{z}_i \in B$. Hence, according to (3.1.3), we have

$$\begin{aligned} g_1 = t_1(x_1[\bar{z}_1]) \sqsubset t_1(y_1[x_1 \cdots x_1][\bar{z}_1]) = t_2(x_2[\bar{z}_2]) \sqsubset t_2(y_2[x_2 \cdots x_2][\bar{z}_2]) \\ = \cdots = t_m(x_m[\bar{z}_m]) \sqsubset t_m(y_m[x_m \cdots x_m][\bar{z}_m]) = g_2. \end{aligned}$$

Thus $g_1 \sqsubset g_2$, i.e., $\zeta(\sigma) \subset \chi$. So, $(G, o, \zeta(\sigma), \chi)$ satisfies all the conditions of the previous theorem. Hence (G, o, χ) is isomorphic to a p.q-o. Menger algebra of reflexive $(n+1)$-relations. □

Putting in the above theorem $\chi = G \times G$ we see that *a Menger algebra of rank $n \geqslant 1$ is isomorphically represented by the reflexive $(n+1)$–relations if and only if the relation $\zeta(\sigma)$ is antisymmetric.* It is not difficult to see that the antisymmetry of $\zeta(\sigma)$ is equivalent to the system of conditions $(A_m)_{m \in \mathbb{N}}$, where

$$A_m : A'_m \longrightarrow t_1(y_1[x_1 \cdots x_1][\bar{z}_1]) = t_2(y_2[x_2 \cdots x_2][\bar{z}_2]$$

and

$$A'_m : \begin{cases} t_1(y_1[x_1 \cdots x_1][\bar{z}_1]) = t_{m+1}(y_{m+1}[x_{m+1} \cdots x_{m+1}][\bar{z}_{m+1}]) \\ \bigwedge\limits_{i=1}^{m} t_i(y_i[x_i \cdots x_i][\bar{z}_i]) = t_{i+1}(x_{i+1}[\bar{z}_{i+1}]) \end{cases}$$

for all $t_i \in T_n(G)$, $\bar{z}_i \in B$, $x_i \in G$, $y_i \in G \cup \{e_i\}$.

Now we consider Menger algebras of the transitive $(n+1)$–relations. Let (G, o, ζ, χ) be the system, where (G, o) is a Menger algebra of rank n and

$\zeta, \chi \subset G \times G$ satisfy all the conditions of Theorem 3.1.1. Then (G, o, ζ, χ) *can be isomorphically represented by transitive* $(n+1)$*-relations if and only if for each* $g \in G$ *the inequality* $g[g^n] \leqslant g$ *is satisfied.* Indeed, if P is the isomorphism of (G, o, ζ, χ) onto some f.o.p. Menger algebra of $(n+1)$-relations, then, according to Theorem 3.1.1, we have $\zeta = \zeta_P$. Thus, the condition $g[g^n] \leqslant g$ is equivalent to $P(g)[P(g) \cdots P(g)] \subset P(g)$, which means the transitivity of an $(n+1)$–relation $P(g)$.

Analogously we can prove that *each stable ordered Menger algebra of rank* n*, which satisfies the inequality* $g[g^n] \leqslant g$*, is isomorphic to some f.o. Menger algebra of transitive* $(n+1)$*-relations.*

For $n = 1$, i.e., for a semigroup of binary relations, this result was firstly proved by K.A. Zaretsky [263].

Theorem 3.1.4. *An algebraic system* (G, o, χ)*, where* (G, o) *is a Menger algebra of rank* n *and* $\chi \subset G \times G$*, is isomorphic to some p.q-o. Menger algebra of transitive* $(n+1)$*-relations if and only if*
(1) χ *is an l-regular v-negative quasi-order,*
(2) $t(x[x^n][\bar{y}]) \sqsubset t(x[\bar{y}])$ *for all* $t \in T_n(G)$, $x \in G$, $\bar{y} \in B$,
(3) *for every natural* m *the condition*

$$B_m : P_m \longrightarrow t_1(x_1[\bar{y}_1]) = t_2(x_2[\bar{y}_2])$$

where

$$P_m : \begin{cases} \bigwedge\limits_{i=1}^{m} t_i(x_i[\bar{y}_i]) = t_{i+1}(x_{i+1}[x_{i+1} \cdots x_{i+1}][\bar{y}_{i+1}]) \\ t_1(x_1[\bar{y}_1]) = t_{m+1}(x_{m+1}[\bar{y}_{m+1}]) \end{cases}$$

for all $t \in T_n(G)$, $x \in G \cup \{e_i\}$, $\bar{y} \in B$, *is satisfied.*

Proof. Necessity. Let (Φ, O, χ_Φ) be some p.q-o. Menger algebra of transitive $(n+1)$-relations. Then $\rho[\rho^n] \subset \rho$ for every $\rho \in \Phi$ and, consequently, $\rho[\rho^n][\bar{\sigma}] \subset \rho[\bar{\sigma}]$ for every $\bar{\sigma} \in \Phi^n$. From the stability of ζ_Φ it follows that $t(\rho[\rho^n][\bar{\sigma}]) \subset t(\rho[\bar{\sigma}])$ for every $t \in T_n(\Phi)$. Hence $\mathrm{pr}_1 t(\rho[\rho^n][\bar{\sigma}]) \subset \mathrm{pr}_1 t(\rho\bar{\sigma}])$, which proves the necessity of (2).

Now let $t_i(\rho_i[\bar{\sigma}_i]) = t_{i+1}(\rho_{i+1}[\rho_{i+1} \cdots \rho_{i+1}][\bar{\sigma}_{i+1}])$ for $i = 1, \ldots, m$, and $t_1(\rho_1[\bar{\sigma}_1]) = t_{m+1}(\rho_{m+1}[\bar{\sigma}_{m+1}])$, where $\rho_i \in \Phi$, $\bar{\sigma}_i \in \Phi^n$, $t_i \in T_n(\Phi)$. Because $t_i(\rho_i[\rho_i \cdots \rho_i][\bar{\sigma}_i]) \subset t_i(\rho_i[\bar{\sigma}_i])$, then

$$t_1(\rho_1[\bar{\sigma}_1]) = t_2(\rho_2[\rho_2 \cdots \rho_2][\bar{\sigma}_2]) \subset t_2(\rho_2[\bar{\sigma}_2]) = t_3(\rho_3[\rho_3 \cdots \rho_3][\bar{\sigma}_3]) \subset \cdots$$
$$\cdots \subset t_{m+1}(\rho_{m+1}[\bar{\sigma}_{m+1}]) = t_1(\rho_1[\bar{\sigma}_1]).$$

Thus $t_1(\rho_1[\bar{\sigma}_1]) = t_2(\rho_2[\bar{\sigma}_2])$, which proves the necessity of the condition (3).

Sufficiency. Let the conditions (1), (2) and (3) be satisfied and let P_m denote the premise of condition B_m. It is not difficult to see that the system of conditions $(B_m)_{m\in\mathbb{N}}$ is equivalent to the system $(B'_m)_{m\in\mathbb{N}}$, where

$$B'_m : P_m \longrightarrow t_1(x_1[\bar{y}_1]) = t_2(x_2[\bar{y}_2]) = \cdots = t_m(x_m[\bar{y}_m]),$$

and, consequently, to the system $(B_{k,s})_{k,s\in\mathbb{N}}$, where $k+s=m$,

$$B_{k,s} : \; P_m \longrightarrow t_1(x_1[\bar{y}_1]) = t_k(x_k[\bar{y}_k]).$$

For $\rho = \{(x[x^n], x) \,|\, x \in G\}$, the antisymmetry of $\zeta(\rho)$ is equivalent to the system $(B_{k,s})_{k,s\in\mathbb{N}}$. Moreover, the assumption (2) guarantees $\zeta(\rho) \subset \chi$. Hence the system $(G, o, \zeta(\rho), \chi)$ satisfies all the conditions of Theorem 3.1.1 and, additionally, the inequality $g[g^n] \leqslant g$. Therefore, as it was mentioned above, it can be isomorphically represented by transitive $(n+1)$–relations. In this representation the relation χ corresponds to the inclusion of the first projections. □

Putting in the last theorem $\chi = G \times G$ we obtain the statement that says that *a Menger algebra of rank n is isomorphically represented by transitive* $(n+1)$*-relations if and only if it satisfies the system of conditions* $(B_m)_{m\in\mathbb{N}}$.

Based on the results obtained in this section, it is not difficult to give a characterization of Menger algebras of n-quasi-orders. In particular, from Theorem 3.1.2 and the inequality $g[g^n] \leqslant g$, it follows that *the system* (G, o, ζ, χ), *where* (G, o) *is a Menger algebra of rank* $n \geqslant 2$ *and* $\zeta, \chi \subset G \times G$, *is isomorphic to some f.o.p Menger algebra of n-quasi-orders if and only if it satisfies all the conditions formulated in Theorem* 3.1.2 *and the identity* $x[x^n] = x$ *holds.* This characterization also means that *every stable ordered idempotent Menger algebra of rank* $n \geqslant 2$ *satisfying the inequality* $x \leqslant y[x^n]$ *is isomorphic to some f.o. Menger algebra of n-quasi-orders.*

Moreover, using Theorem 3.1.3 it is not difficult to prove that (G, o, χ), *where* (G, o) *is a Menger algebra of rank* $n \geqslant 2$ *and* $\chi \subset G \times G$, *is isomorphic to some p.q-o. Menger algebra of n-quasi-orders if and only if* (G, o) *is an idempotent Menger algebra satisfying the system of conditions* $(A_m)_{m\in\mathbb{N}}$ *and* χ *has all the properties mentioned in the Theorem* 3.1.3. This, in the case $\chi = G \times G$, means that *every Menger algebra of rank* $n \geqslant 2$ *is isomorphic to some Menger algebra of n-quasi-orders if and only if it is idempotent and satisfies all the conditions* $(A_m)_{m\in\mathbb{N}}$.

3.2 F.o. and p.q-o. Menger algebras

Let (G, o) be a Menger algebra of rank n, ω — an order (quasi-order) defined on G. We say that ω is a (*strong*) *fundamental order* (respectively: *projection quasi-order*), if there exists a faithful representation of (G, o) by (reversive)

n-place functions, under which $\omega = \zeta_P$ (respectively: $\omega = \chi_P$). In this case we say also that (G, o, ω) is a (*strongly*) *fundamentally ordered* (respectively: *projection quasi-ordered*) *Menger algebra of rank n*.

If on a Menger algebra (G, o) of rank n we defined a binary order ω_1 and a binary quasi-order ω_2 such that $\omega_1 = \zeta_P$ and $\omega_2 = \chi_P$ for some faithful representation of (G, o) by (reversive) n-place functions, then the system $(G, o, \omega_1, \omega_2)$ is called a (*strongly*) *fundamentally ordered projection Menger algebra of rank n*. Of course, every f.o.p. Menger algebra is f.o. and p.q-o. Menger algebra. The converse is not true in general. The system $(G, o, \omega_1, \omega_2)$, where (G, o, ω_1) is a f.o. Menger algebra, (G, o, ω_2) — a p.q-o. Menger algebra, may not be a f.o.p. Menger algebra.

The abstract characterization of f.o.p. Menger algebras is given by the following theorem.

Theorem 3.2.1. *An algebraic system (G, o, ζ, χ), where o is an $(n+1)$-operation on G and $\zeta, \chi \subset G \times G$, is a f.o.p. Menger algebra of rank n if and only if o is a superassociative operation, ζ is a stable order, χ — an l-regular v-negative quasi-order containing ζ, and for all $i = 1, \ldots, n$, $u, g, g_1, g_2 \in G$, $\bar{w} \in G^n$ the following two implications hold:*

$$g_1 \leqslant g \;\wedge\; g_2 \leqslant g \;\wedge\; g_1 \sqsubset g_2 \longrightarrow g_1 \leqslant g_2, \tag{3.2.1}$$

$$g_1 \leqslant g_2 \;\wedge\; g \sqsubset g_1 \;\wedge\; g \sqsubset u[\bar{w}|_i g_2] \longrightarrow g \sqsubset u[\bar{w}|_i g_1], \tag{3.2.2}$$

where $x \leqslant y \longleftrightarrow (x, y) \in \zeta$, $x \sqsubset y \longleftrightarrow (x, y) \in \chi$.

Proof. Let $(\Phi, O, \zeta_\Phi, \chi_\Phi)$ be a f.o.p. Menger algebra of n-place functions. If $\varphi_1 \subset \psi$, $\varphi_2 \subset \psi$, $\mathrm{pr}_1\varphi_1 \subset \mathrm{pr}_1\varphi_2$ for some $\varphi_1, \varphi_2, \psi \in \Phi$, then, from the first two inclusions we get $\varphi_1 = \psi \circ \triangle_{\mathrm{pr}_1\varphi_1}$, $\varphi_2 = \psi \circ \triangle_{\mathrm{pr}_1\varphi_2}$, from the third — $\varphi_1 \circ \triangle_{\mathrm{pr}_1\varphi_2} = \varphi_1$. Thus

$$\varphi_2 \circ \triangle_{\mathrm{pr}_1\varphi_1} = \psi \circ \triangle_{\mathrm{pr}_1\varphi_2} \circ \triangle_{\mathrm{pr}_1\varphi_1} = \psi \circ \triangle_{\mathrm{pr}_1\varphi_1} \circ \triangle_{\mathrm{pr}_1\varphi_2} = \varphi_1 \circ \triangle_{\mathrm{pr}_1\varphi_2} = \varphi_1.$$

So, $\varphi_1 \subset \varphi_2$, which proves (3.2.1).

Now, let

$$\varphi_1 \subset \varphi_2, \quad \mathrm{pr}_1\psi \subset \mathrm{pr}_1\varphi_1, \quad \mathrm{pr}_1\psi \subset \mathrm{pr}_1 f[\bar{\chi}|_i\varphi_2],$$

where $\varphi_1, \varphi_2, \psi, f \in \Phi$, $\bar{\chi} \in \Phi^n$. Since $\varphi_1 = \varphi_2 \circ \triangle_{\mathrm{pr}_1\varphi_1}$, the last two implications give

$$\mathrm{pr}_1\psi = \triangle_{\mathrm{pr}_1\varphi_1}(\mathrm{pr}_1\psi) \subset \triangle_{\mathrm{pr}_1\varphi_1}(\mathrm{pr}_1 f[\bar{\chi}|_i\varphi_2]).$$

Therefore

$$\mathrm{pr}_1\psi \subset \mathrm{pr}_1(f[\bar{\chi}\,|_i\varphi_2] \circ \triangle_{\mathrm{pr}_1\varphi_1}) = \mathrm{pr}_1 f[\bar{\chi}\,|_i\varphi_2 \circ \triangle_{\mathrm{pr}_1\varphi_1}] = \mathrm{pr}_1 f[\bar{\chi}\,|_i\varphi_1].$$

This shows that $(\Phi, O, \zeta_\Phi, \chi_\Phi)$ satisfies the condition (3.2.2).

On the other hand, if the system (G, o, ζ, χ) satisfies all the conditions given in Theorem 3.2.1, then the implications

$$g_1 \leqslant g_2 \wedge g \sqsubset t_1(g_1) \wedge g \sqsubset t_2(g_2) \longrightarrow g \sqsubset t_2(g_1), \tag{3.2.3}$$

$$g_1 \leqslant g_2 \wedge g \sqsubset t_1(g_1) \wedge g \leqslant t_2(g_2) \longrightarrow g \leqslant t_2(g_1) \tag{3.2.4}$$

are true for all $g_1, g_2, g \in G$ and $t_1, t_2 \in T_n(G)$. If the premise of the implication (3.2.3) is satisfied, then from $g \sqsubset t_1(g_1)$ we obtain $g \sqsubset g_1$ because the relation χ is v-negative. Suppose that $t_1(g_2) = t_3(u[\bar{w}|_i g_2])$ for some $t_3 \in T_n(G)$, $u \in G$, $\bar{w} \in G^n$, $i \in \{1, \ldots, n\}$. Then $g \sqsubset t_3(u[\bar{w}|_i g_2])$ and, consequently $g \sqsubset u[\bar{w}|_i g_2]$, since χ is v-negative. Thus $g_1 \leqslant g_2$, $g \sqsubset g_1$ and $g \sqsubset u[\bar{w}|_i g_2]$, which, by (3.2.2), gives $g \sqsubset u[\bar{w}|_i g_1]$. Applying the stability of ζ to $g_1 \leqslant g_2$ we get $u[\bar{w}|_i g_1] \leqslant u[\bar{w}|_i g_2]$. Now suppose that $t_3(u[\bar{w}|_i g_2]) = t_4(x[\bar{z}|_k u[\bar{w}|_i g_2]])$ for some $t_4 \in T_n(G)$, $x \in G$, $\bar{z} \in G^n$ and $k \in \{1, \ldots, n\}$. Similarly as in the previous case $g \sqsubset t_3(x[\bar{z}|_k u[\bar{w}|_i g_2]])$ implies $g \sqsubset x[\bar{z}|_k u[\bar{w}|_i g_2]]$. In this way, we have

$$u[\bar{w}|_i g_1] \leqslant u[\bar{w}|_i g_2], \quad g \sqsubset u[\bar{w}|_i g_1], \quad g \sqsubset x[\bar{z}|_k u[\bar{w}|_i g_2]],$$

which, by (3.2.2), gives $g \sqsubset x[\bar{z}|_k u[\bar{w}|_i g_1]]$. Continuing this procedure, after a finite number of steps, we obtain $g \sqsubset t_3(g_1)$. So condition (3.2.3) is proved.

Now let the premise of the condition (3.2.4) be satisfied. Since $\zeta \subset \chi$, from $g \sqsubset t_2(g_2)$, by (3.2.3), we conclude $g \sqsubset t_2(g_1)$. Further, applying the stability of ζ to $g_1 \leqslant g_2$ we obtain $t_2(g_1) \leqslant t_2(g_2)$. Hence $g \leqslant t_2(g_2)$, $t_2(g_1) \leqslant t_2(g_2)$ and $g \sqsubset t_2(g_1)$, which, according to (3.2.1), gives $g \leqslant t_2(g_1)$. Condition (3.2.4) is proved.

Let us consider the family of the determining pairs

$$(\,(\varepsilon_g^*, W_v(\chi\langle g\rangle))\,)_{g \in G}\,,$$

where $\varepsilon_g = \varepsilon_v(\zeta\langle g\rangle) \cap \varepsilon_v(\chi\langle g\rangle)$, $\varepsilon_g^* = \varepsilon_g \cup \{(e_1, e_1), \ldots, (e_n, e_n)\}$, $e_1, \ldots, e_n$ — selectors in (G^*, o^*). Obviously $\zeta \subset \chi$. Hence $W_v(\chi\langle g\rangle) \subset W_v(\zeta\langle g\rangle)$. So, any nonempty $W_v(\chi\langle g\rangle)$ is a ε_g-class. Therefore $(\varepsilon_g^*, W_v(\chi\langle g\rangle))$ is the determining pair for every $g \in G$. Let P be the sum of the simplest representations of a Menger algebra (G, o), which corresponds to the determining pairs $((\varepsilon_g^*, W_v(\chi\langle g\rangle)))_{g \in G}$. Of course, P is a representation of (G, o) by n-place functions. We must check that $\zeta = \zeta_P$ and $\chi = \chi_P$.

For this, let $(g_1, g_2) \in \zeta_P$, i.e., $(g_1, g_2) \in \zeta_{(\varepsilon_g^*, W_v(\chi\langle g\rangle))}$ for each $g \in G$. Then

$$(\forall g \in G)(\forall \bar{x} \in B)\ (g_1[\bar{x}] \notin W_v(\chi\langle g\rangle) \longrightarrow g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon_g)).$$

Putting $\bar{x} = (e_1, \ldots, e_n)$ in this implication, we get

$$(\forall g \in G)\ (g_1 \notin W_v(\chi\langle g\rangle) \longrightarrow g_1 \equiv g_2(\varepsilon_g)).$$

This shows that $g_1 \equiv g_2(\varepsilon_{g_1})$ because $g_1 \notin W_v(\chi\langle g_1\rangle)$. Thus, as a consequence, $g_1 \equiv g_2(\varepsilon_v(\zeta\langle g_1\rangle))$, which means that

$$(\forall t \in T_n(G))\ (g_1 \leqslant t(g_1) \longleftrightarrow g_1 \leqslant t(g_2)).$$

Since for $t = \triangle_G$ the above equivalence gives $g_1 \leqslant g_2$, we proved $\zeta_P \subset \zeta$.

Conversely, let $(g_1, g_2) \in \zeta$ and $g_1[\bar{x}] \notin W_v(\chi\langle g\rangle)$ for some $g \in G$, where $\bar{x} \in B$. The last inclusion means that there exists a polynomial $t_1 \in T_n(G)$ such that $g \sqsubset t_1(g_1[\bar{x}])$. Since the stability of ζ and $g_1 \leqslant g_2$ imply $g_1[\bar{x}] \leqslant g_2[\bar{x}]$, we have: (a) $g_1[\bar{x}] \leqslant g_2[\bar{x}]$, (b) $g \sqsubset t_1(g_1[\bar{x}])$. If $g \leqslant t(g_1[\bar{x}])$ for some $t \in T_n(G)$, then (a) implies $t(g_1[\bar{x}]) \leqslant t(g_2[\bar{x}])$, thus $g \leqslant t(g_2[\bar{x}])$. If for some $t \in T_n(G)$ we have $g \leqslant t(g_2[\bar{x}])$, then $g \leqslant t(g_1[\bar{x}])$, by (a), (b) and (3.2.4). Hence $g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon_v(\zeta\langle g\rangle))$. Now let $g \sqsubset t(g_1[\bar{x}])$ for some $t \in T_n(G)$. As $t(g_1[\bar{x}]) \leqslant t(g_2[\bar{x}])$ and $\zeta \subset \chi$, then, obviously, $g \sqsubset t(g_2[\bar{x}])$. In the case when $g \sqsubset t(g_2[\bar{x}])$ for some $t \in T_n(G)$, according to (a), (b) and (3.2.3), we must have $g \sqsubset t(g_1[\bar{x}])$. So, we showed that $g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon_v(\chi\langle g\rangle))$. Thus, $g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon_g)$. This means that $(g_1, g_2) \in \zeta_P$. So, $\zeta \subset \zeta_P$, and consequently $\zeta = \zeta_P$.

Further, let $(g_1, g_2) \in \chi_P$. Then

$$(\forall g \in G)(\forall \bar{x} \in B)\ (g_1[\bar{x}] \notin W_v(\chi\langle g\rangle) \longrightarrow g_2[\bar{x}] \notin W_v(\chi\langle g\rangle),$$

which for $\bar{x} = (e_1, \ldots, e_n)$ and $g = g_1$ gives $g_2 \notin W_v(\chi\langle g_1\rangle)$. This means that $g_1 \sqsubset t(g_2)$ for some $t \in T_n(G)$. As χ is v-negative, then $t(g_2) \sqsubset g_2$. Hence $g_1 \sqsubset g_2$. So, $\chi_P \subset \chi$. Conversely, if $(g_1, g_2) \in \chi$, then, according to the l-regularity of χ we have $g_1[\bar{x}] \sqsubset g_2[\bar{x}]$ for every vector $\bar{x} \in B$. If $g_1[\bar{x}] \notin W_v(\chi\langle g\rangle)$ for some $g \in G$, i.e., $g \sqsubset t(g_1[\bar{x}])$ for some $g \in G$ and $t \in T_n(G)$, then $g \sqsubset g_1[\bar{x}]$, by the v-negativity of χ. Therefore $g \sqsubset g_2[\bar{x}]$, i.e., $g_2[\bar{x}] \notin W_v(\chi\langle g\rangle)$. So, $(g_1, g_2) \in \chi_P$, and consequently $\chi = \chi_P$.

Finally, let $P(g_1) = P(g_2)$, i.e., $P(g_1) \subset P(g_2)$ and $P(g_2) \subset P(g_1)$. Since $\zeta = \zeta_P$, from the above we obtain $g_1 \leqslant g_2$ and $g_2 \leqslant g_1$. So, $g_1 = g_2$, because ζ is antisymmetric. This shows that P is a faithful representation of (G, o). □

Corollary 3.2.2. *An algebraic system (G, o, χ) is a p.q-o. Menger algebra of rank n if and only if o is a superassociative $(n+1)$-operation and χ is an l-regular v-negative quasi-order on G.*

For the proof it is enough to put in the last theorem $\zeta = \triangle_G$.

Theorem 3.2.3. *The necessary and sufficient condition for an algebraic system (G, o, ζ) to be a f.o. Menger algebra is that o is a superassociative $(n+1)$-operation and ζ is a stable weakly steady order on G.*

Proof. If (G, o, ζ) is a f.o. Menger algebra of rank n, then the operation o is superassociative and there is a faithful representation of (G, o) by n-place functions such that $\zeta = \zeta_P$. This shows that ζ is stable. Since (G, o, ζ, χ_P) is a f.o.p. Menger algebra, it satisfies condition (3.2.4) too. Now let $g_1 \leqslant g_2$, $g \leqslant t_1(g_1)$ and $g \leqslant t_2(g_2)$ for some $g, g_1, g_2 \in G$ and $t_1, t_2 \in T_n(G)$. But $\zeta \subset \chi_P$. So, $g \sqsubset t_1(g_1) \sqsubset g_1$, where $\sqsubset$ means χ_P. Thus from $g_1 \leqslant g_2$, $g \sqsubset g_1$, $g \leqslant t_2(g_2)$, according to (3.2.4), we conclude $g \leqslant t_2(g_1)$, which proves that ζ is weakly steady.

Conversely, let (G, o, ζ) be an algebraic system satisfying all the conditions mentioned in the theorem. Consider the relation $\chi(\zeta) = \delta \circ \zeta$, where $\delta = \{(g_1, g_2) \,|\, (\exists t \in T_n(G))\; g_1 = t(g_2)\}$. Since ζ, δ are reflexive and l-regular, $\delta \circ \zeta$ is reflexive and l-regular too. To prove that the relation $\delta \circ \zeta$ is transitive, let $(g_1, g_2), (g_2, g_3) \in \delta \circ \zeta$, i.e., let $g_1 \leqslant t_1(g_2)$, $g_2 \leqslant t_2(g_3)$ for some $t_1, t_2 \in T_n(G)$. According to the stability of ζ, the last inequality implies $t_1(g_2) \leqslant t_1(t_2(g_3))$. Hence $g_1 \leqslant t_1(t_2(g_3)) = (t_1 \circ t_2)(g_3)$. But $t_1 \circ t_2 \in T_n(G)$. So, $(g_1, g_3) \in \delta \circ \zeta$, which proves the transitivity of $\delta \circ \zeta$. This, together with $\delta \subset \chi(\zeta)$, shows that $\chi(\zeta)$ is an l-regular v-negative quasi-order. Because ζ is weakly steady, $\chi(\zeta)$ satisfies the conditions (3.2.1) and (3.2.2). This means that $(G, o, \zeta, \chi(\zeta))$ satisfies all the conditions of Theorem 3.2.1, hence (G, o, ζ) is a f.o. Menger algebra. □

Definition 3.2.4. The inverse Menger algebra (G, o) of rank n is called *fundamental* if there exists a faithful representation P of it by n-place functions such that

$$g_1 \leqslant g_2 \longleftrightarrow P(g_1) \subset P(g_2), \tag{3.2.5}$$

$$g_1 \sqsubset g_2 \longleftrightarrow \mathrm{pr}_1 P(g_1) \subset \mathrm{pr}_1 P(g_2) \tag{3.2.6}$$

for all $g_1, g_2 \in G$, where the relations $\leqslant$ and $\sqsubset$ are defined by (2.3.7) and (2.3.8) respectively.

Theorem 3.2.5. *The inverse Menger algebra (G, o) of rank n with the diagonal semigroup $(G, \cdot)$ is fundamental if and only if it satisfies the conditions:*

$$u[\bar{w}|_i(g \cdot e)] \leqslant u[\bar{w}|_i g], \tag{3.2.7}$$

$$u[\bar{w}|_i g] \cdot e \leqslant u[\bar{w}|_i(g \cdot e)], \tag{3.2.8}$$

$$x[\bar{y}] \leqslant x[\bar{y}] \cdot y_i^{-1} \cdot y_i \tag{3.2.9}$$

for all $i = 1, \ldots, n$, $e, x, y_i, u, g \in G$, $\bar{w} \in G^n$, where e is idempotent and y^{-1} is the inverse element of y in $(G, \cdot)$.

Proof. Let (G, o) be a fundamental inverse Menger algebra of rank n. By Proposition 2.3.10, $g \cdot e \leqslant g$ for every $g \in G$ and every idempotent $e \in G$. Since $\leqslant$ is stable, it is also v-regular and i-regular for every $i = 1, \ldots, n$. So, applying the i-regularity to $g \cdot e \leqslant g$ we obtain $u[\bar{w}|_i(g \cdot e)] \leqslant u[\bar{w}|_i g]$ for all $i = 1, \ldots, n$,

$u \in G$, $\bar{w} \in G^n$. This proves (3.2.7). Further, according to Lemma 2.1.3, we have

$$u[\bar{w}|_i g] \cdot e = u[(w_1 \cdot e) \cdots (g \cdot e) \cdots (w_n \cdot e)].$$

But, by Proposition 2.3.10, $w_i \cdot e \leqslant w_i$, $i = 1, \ldots, n$, and $g \cdot e \leqslant g \cdot e$ for each idempotent $e \in G$. This, together with the v-regularity of $\leqslant$ implies

$$u[(w_1 \cdot e) \cdots (g \cdot e) \cdots (w_n \cdot e)] \leqslant u[w_1 \cdots (g \cdot e) \cdots w_n] = u[\bar{w}|_i (g \cdot e)].$$

Hence $u[\bar{w}|_i g] \cdot e \leqslant u[\bar{w}|_i (g \cdot e)]$, which proves (3.2.8). To prove (3.2.9) observe that $\sqsubset$ is a v-negative quasi-order. Therefore $x[\bar{y}] \sqsubset y_i$ for all $i = 1, \ldots, n$. This implies $x[\bar{y}] \cdot y_i^{-1} \cdot y_i = x[\bar{y}]$ and, consequently, (3.2.9).

Conversely, let (G, o) be an inverse Menger algebra of rank n satisfying the conditions (3.2.7)–(3.2.9). Suppose that $x \leqslant y$, i.e., $x = y \cdot x^{-1} \cdot x$. Then $u[\bar{w}|_i x] = u[\bar{w}|_i (y \cdot x^{-1} \cdot x)]$. But $x^{-1} \cdot x$ is an idempotent of $(G, \cdot)$, so, according to (3.2.7), $u[\bar{w}|_i x] \leqslant [\bar{w}|_i y]$, which proves that $\leqslant$ is i-regular for all $i = 1, \ldots, n$, hence it is v-regular (see § 2.1). By Proposition 2.3.10, it is also l-regular. Thus $\leqslant$ is stable on (G, o). Then, according to (3.2.8) and (3.2.9),

$$x[\bar{y}] \leqslant x[\bar{y}] \cdot y_i^{-1} \cdot y_i \leqslant x[\bar{y}|_i (y_i \cdot y_i^{-1} \cdot y_i)] = x[\bar{y}].$$

Therefore $x[\bar{y}] \cdot y_i^{-1} \cdot y_i = x[\bar{y}]$, i.e., $x[\bar{y}] \sqsubset y_i$ for all $i = 1, \ldots, n$. This means that $\sqsubset$ is v-negative.

Now let $x \leqslant z$, $y \leqslant z$ and $x \sqsubset y$, i.e., $x = z \cdot x^{-1} \cdot x$, $y = z \cdot y^{-1} \cdot y$ and $x^{-1} \cdot x = x^{-1} \cdot x \cdot y^{-1} \cdot y$. From the second inequality we have

$$y \cdot x^{-1} \cdot x = z \cdot y^{-1} \cdot y \cdot x^{-1} \cdot x = z \cdot x^{-1} \cdot x \cdot y^{-1} \cdot y = z \cdot x^{-1} \cdot x = x.$$

So, $x \leqslant y$. This proves (3.2.1).

To prove (3.2.2), let $x \leqslant y$, $z \sqsubset x$ and $z \sqsubset u[\bar{w}|_i y]$. The first two conditions can be written in the form $x = y \cdot x^{-1} \cdot x$, $z = z \cdot x^{-1} \cdot x$. This, together with the l-regularity of $\sqsubset$ and the third condition, gives

$$z \cdot x^{-1} \cdot x \sqsubset u[\bar{w}|_i y] \cdot x^{-1} \cdot x.$$

Hence, by (3.2.8), we have

$$u[\bar{w}|_i y] \cdot x^{-1} \cdot x \leqslant u[\bar{w}|_i (y \cdot x^{-1} \cdot x)] = u[\bar{w}|_i x]$$

and $u[\bar{w}|_i y] \cdot x^{-1} \cdot x \sqsubset u[\bar{w}|_i x]$, by Proposition 2.3.11. Therefore

$$z = z \cdot x^{-1} \cdot x \sqsubset u[\bar{w}|_i y] \cdot x^{-1} \cdot x \sqsubset u[\bar{w}|_i y],$$

which proves (3.2.2).

So, all the conditions of Theorem 3.2.1 are fulfilled. This means that there exists a faithful representation of an inverse Menger algebra (G, o) by n-place functions satisfying the conditions (3.2.5) and (3.2.6). Thus (G, o) is fundamental. □

3.3 Algebras of reversive functions

In this section, we will consider *strong* (f.o.p., f.o., and p.q-o.) *Menger algebras* of rank $n > 1$, i.e., such (f.o.p., f.o., and p.q-o.) Menger algebras that can be isomorphically represented by reversive n-place functions.

Let (Φ, O) be some Menger algebra of reversive n-place functions and H — a subset of the set Φ. By induction we can show that for $\psi_1, \psi_2 \in \Phi$ the implication

$$(\psi_1, \psi_2) \in \varepsilon_m \langle \zeta_\Phi^{-1} \circ \zeta_\Phi \, , H \rangle \longrightarrow \psi_1 \circ \triangle_{\mathrm{pr}_1 H} = \psi_2 \circ \triangle_{\mathrm{pr}_1 H}, \tag{3.3.1}$$

where $\mathrm{pr}_1 H = \bigcap \{ \mathrm{pr}_1 \varphi \,|\, \varphi \in H \}$, is valid.

Theorem 3.3.1. *An algebraic system (G, o, ζ, χ), where (G, o) is a Menger algebra of rank $n \geqslant 2$, $\zeta, \chi \subset G \times G$, is a strong f.o.p. Menger algebra if and only if ζ is an l-regular order, χ — an l-regular v-negative quasi-order containing ζ, and the following conditions are satisfied:*

$$(g_1, g_2) \in \varepsilon_m \langle \zeta^{-1} \circ \zeta \, , \chi \langle g_1 \rangle \rangle \longrightarrow g_1 \leqslant g_2, \tag{3.3.2}$$

$$(g_1, g_2) \in \varepsilon_m \langle \zeta^{-1} \circ \zeta \, , \chi \langle g \rangle \rangle \wedge g \sqsubset x[\bar{y}|_i g_2] \longrightarrow g \sqsubset x[\bar{y}|_i g_1] \tag{3.3.3}$$

for all $m \in \mathbb{N}$, $i = 1, \ldots, n$, $g, g_1, g_2, x \in G$, $\bar{y} \in G^n$.

Proof. Let (G, o, ζ, χ) be a strong f.o.p. Menger algebra $n > 1$ isomorphic to some f.o.p. Menger algebra $(\Phi, O, \zeta_\Phi, \chi_\Phi)$ of reversive n-place functions. Let $P : G \to \Phi$ be the corresponding isomorphism. If for $g_1, g_2 \in G$ we have $(g_1, g_2) \in \varepsilon_m \langle \zeta^{-1} \circ \zeta \, , \chi \langle g_1 \rangle \rangle$, then $(\psi_1, \psi_2) \in \varepsilon_m \langle \zeta_\Phi^{-1} \circ \zeta_\Phi, \chi_\Phi \langle \psi_1 \rangle \rangle$, where $P : g_1 \mapsto \psi_1$, $g_2 \mapsto \psi_2$. Applying (3.3.1) we get $\psi_1 \circ \triangle_{\mathrm{pr}_1 \psi_1} = \psi_2 \circ \triangle_{\mathrm{pr}_1 \psi_1}$, because $\mathrm{pr}_1 \psi_1 = \mathrm{pr}_1 \chi_\Phi \langle \psi_1 \rangle$. In this way, $\psi_1 = \psi_2 \circ \triangle_{\mathrm{pr}_1 \psi_1}$, i.e., $\psi_1 \subset \psi_2$. Thus $g_1 \leqslant g_2$. This proves (3.3.2). The proof of (3.3.3) is similar.

Conversely, let the system (G, o, ζ, χ) satisfy all the conditions of the above theorem and let

$$\varepsilon_g^* = \varepsilon \langle \zeta^{-1} \circ \zeta \, , \chi \langle g \rangle \rangle \cup \{ (e_1, e_1), \ldots, (e_n, e_n) \}, \quad W_g = G \setminus \chi \langle g \rangle,$$

where $g \in G$, $e_1, \ldots, e_n$ — selectors in (G^*, o^*). Since, by assumption, χ is v-negative, W_g is an l-ideal of (G, o). This, by (3.3.3), means that (ε_g^*, W_g) is the determining pair of (G, o) and satisfies condition (2.7.1). Obviously, this pair determines the simplest representation of (G, o) by reversive n-place functions. Let $P = \sum\limits_{g \in G} P_{(\varepsilon_g^*, W_g)}$. Of course, P is a representation of (G, o) by reversive n-place functions. We will show that $\zeta = \zeta_P$, $\chi = \chi_P$.

Let $(g_1, g_2) \in \zeta$, $\bar{x} \in B = G^n \cup \{ (e_1, \ldots, e_n) \}$ and $g_1[\bar{x}] \in \chi \langle g \rangle$ for some $g \in G$. From $g_1[\bar{x}] \leqslant g_2[\bar{x}]$ it follows that $g_2[\bar{x}] \in \chi \langle g \rangle$. Hence

$$(g_1[\bar{x}], g_2[\bar{x}]) \in \varepsilon_0 \langle \zeta^{-1} \circ \zeta \, , \chi \langle g \rangle \rangle \subset \varepsilon_g^*.$$

Consequently, $(g_1, g_2) \in \bigcap_{g\in G} \zeta_{(\varepsilon_g^*, W_g)} = \zeta_P$. Thus $\zeta \subset \zeta_P$. Conversely, if $(g_1, g_2) \in \zeta_P$, then

$$(\forall g \in G)(\forall \bar{x} \in B) \left(g_1 \sqsubset g_1[\bar{x}] \longrightarrow g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon_g^*)\right),$$

which, for $g = g_1,\ \bar{x} = (e_1, \ldots, e_n)$, gives

$$(g_1, g_2) \in \varepsilon_m\langle \zeta^{-1} \circ \zeta \,, \chi\langle g_1\rangle\rangle$$

for some $m \in \mathbb{N}$. This, according to (3.3.2), implies $g_1 \leqslant g_2$. So, $\zeta = \zeta_P$.

Now, let $(g_1, g_2) \in \chi$, $\bar{x} \in B$ and $g \sqsubset g_1[\bar{x}]$ for some $g \in G$. From $g_1[\bar{x}] \sqsubset g_2[\bar{x}]$ it follows $g \sqsubset g_2[\bar{x}]$. Thus

$$(g_1, g_2) \in \bigcap_{g\in G} \chi_{(\varepsilon_g^*, W_g)} = \chi_P.$$

Consequently, $\chi \subset \chi_P$. On the other hand, if $(g_1, g_2) \in \chi_P$, then

$$(\forall g \in G)(\forall \bar{x} \in B)\ (g \sqsubset g_1[\bar{x}] \longrightarrow g \sqsubset g_2[\bar{x}]),$$

which for $g = g_1,\ \bar{x} = (e_1, \ldots, e_n)$ gives $g_1 \sqsubset g_2$. This proves $\chi_P \subset \chi$ and, consequently, $\chi = \chi_P$.

By the antisymmetry of ζ, P is one-to-one. So, P is an isomorphism. This completes our proof. □

Note that in Theorem 3.3.1 is assumed only the l-regularity of ζ, but, as it is not difficult to show, its v-regularity follows from (3.3.2) and (3.3.3). Moreover, each of these conditions is equivalent to some elementary formula (see the end of Section 2.1). Namely, condition (3.3.2) is equivalent to the formula

$$A_m : A_m' \longrightarrow u_1[\bar{w}_1|_{q_1} c_1] \leqslant u_1[\bar{w}_1|_{q_1} d_1],$$

the condition (3.3.3) — to the formula

$$B_m : B_m' \longrightarrow g \sqsubset x[\bar{y}|_j g_1],$$

where

$$A_m' : \begin{cases} \displaystyle\bigwedge_{i=1}^{2^{m-1}-1} \begin{pmatrix} u_1[\bar{w}_1|_{q_1} c_1] \sqsubset u_{2i}[\bar{w}_{2i}|_{q_{2i}} c_{2i}] = s_i[\bar{v}_i|_{r_i} c_i], \\ u_1[\bar{w}_1|_{q_1} c_1] \sqsubset u_{2i+1}[\bar{w}_{2i+1}|_{q_{2i+1}} c_{2i+1}] = u_{2i}[\bar{w}_{2i}|_{q_{2i}} d_{2i}], \\ u_1[\bar{w}_1|_{q_1} c_1] \sqsubset u_{2i+1}[\bar{w}_{2i+1}|_{q_{2i+1}} d_{2i+1}] = s_i[\bar{v}_i|_{r_i} d_i] \end{pmatrix}, \\ \displaystyle\bigwedge_{i=2^{m-1}}^{2^m-1} \begin{pmatrix} u_1[\bar{w}_1|_{q_1} c_1] \sqsubset s_i[\bar{v}_i|_{r_i} c_i] \leqslant l_i, \\ u_1[\bar{w}_1|_{q_1} c_1] \sqsubset z_i \leqslant l_i, \\ u_1[\bar{w}_1|_{q_1} c_1] \sqsubset s_i[\bar{v}_i|_{r_i} d_i] \leqslant f_i, \\ u_1[\bar{w}_1|_{q_1} c_1] \sqsubset z_i \leqslant f_i \end{pmatrix}, \\ u_1[\bar{w}_1|_{q_1} c_1] \sqsubset u_1[\bar{w}_1|_{q_1} d_1], \end{cases}$$

$$
B'_m : \begin{cases}
g \sqsubset g_1 = u_1[\bar{w}_1|_{q_1} c_1], \\
g \sqsubset g_2 = u_1[\bar{w}_1|_{q_1} d_1], \\
\bigwedge\limits_{i=1}^{2^{m-1}-1} \begin{pmatrix} g \sqsubset u_{2i}[\bar{w}_{2i}|_{q_{2i}} c_{2i}] = s_i[\bar{v}_i|_{r_i} c_i], \\ g \sqsubset u_{2i+1}[\bar{w}_{2i+1}|_{q_{2i+1}} c_{2i+1}] = u_{2i}[\bar{w}_{2i}|_{q_{2i}} d_{2i}], \\ g \sqsubset u_{2i+1}[\bar{w}_{2i+1}|_{q_{2i+1}} d_{2i+1}] = s_i[\bar{v}_i|_{r_i} d_i] \end{pmatrix}, \\
\bigwedge\limits_{i=2^{m-1}}^{2^m-1} \begin{pmatrix} g \sqsubset s_i[\bar{v}_i|_{r_i} c_i] \leqslant l_i, \\ g \sqsubset z_i \leqslant l_i, \\ g \sqsubset s_i[\bar{v}_i|_{r_i} d_i] \leqslant f_i, \\ g \sqsubset z_i \leqslant f_i \end{pmatrix}, \\
g \sqsubset x[\bar{y}|_j g_2],
\end{cases}
$$

$u_i \in G \cup \{e_{q_i}\}$, $s_i \in G \cup \{e_{r_i}\}$, $q_i, r_i, j \in \{1, \ldots, n\}$, $i = 1, 2, \ldots, 2^m - 1$, and other elements belong to G.

We will now characterize strong f.o. and p.q-o. Menger algebras.

Theorem 3.3.2. *An algebraic system (G, o, ζ), where (G, o) is a Menger algebra of rank $n > 1$, $\zeta \subset G \times G$, is a strong f.o. Menger algebra if and only if ζ is stable and steady order.*

Proof. Let (Φ, O, ζ_Φ) be some f.o. Menger algebra of reversive n-place functions. Assume that

$$\varphi \subset t_1(\psi_1), \quad \varphi \subset t_1(\psi_2), \quad \varphi \subset t_2(\psi_2),$$

where $\varphi, \psi_1, \psi_2 \in \Phi$, $t_1, t_2 \in T_n(\Phi)$. Then

$$\varphi = t_1(\psi_1) \circ \triangle_{\mathrm{pr}_1 \varphi}, \quad \varphi = t_1(\psi_2) \circ \triangle_{\mathrm{pr}_1 \varphi},$$

and, as a consequence,

$$t_1(\psi_1 \circ \triangle_{\mathrm{pr}_1 \varphi}) = t_1(\psi_2 \circ \triangle_{\mathrm{pr}_1 \varphi}).$$

All functions from Φ are reversive, so, from the previous equality we conclude

$$\psi_1 \circ \triangle_{\mathrm{pr}_1 \varphi} = \psi_2 \circ \triangle_{\mathrm{pr}_1 \varphi},$$

which together with $\varphi \subset t_2(\psi_2)$ gives

$$\varphi = t_2(\psi_2) \circ \triangle_{\mathrm{pr}_1 \varphi} = t_2(\psi_2 \circ \triangle_{\mathrm{pr}_1 \varphi}) = t_2(\psi_1 \circ \triangle_{\mathrm{pr}_1 \varphi}) \subset t_2(\psi_1).$$

This proves the necessity.

To prove the sufficiency, let ζ be a stable and steady order on a Menger algebra (G, o) of rank $n > 1$. We must show that the system $(G, o, \zeta, \chi(\zeta))$

satisfies all the conditions of Theorem 3.3.1. As it is known, $\chi(\zeta) = \delta \circ \zeta$ is an l-regular v-negative quasi-order containing ζ. Moreover

$$W_v(\zeta\langle g\rangle) = G \setminus \chi(\zeta)\langle g\rangle$$

for any $g \in G$. Now let $g_1 \leqslant g_2$ and $g \leqslant t_1(g_1)$, where $g_1, g_2 \in G,\ t \in T_n(G)$. If $g \leqslant t(g_1)$ for some $t \in T_n(G)$, then from $t(g_1) \leqslant t(g_2)$ it follows $g \leqslant t(g_2)$. As $t(g_1) \leqslant t(g_2)$, then $g \leqslant t_1(g_2)$. Thus, if $g \leqslant t(g_2)$, then from $g \leqslant t_1(g_1)$, $g \leqslant t_1(g_2)$, $g \leqslant t(g_2)$ we conclude $g \leqslant t(g_1)$, because ζ is a steady order. Consequently, $g_1 \equiv g_2(\varepsilon_v(\zeta\langle g\rangle))$. In this way, we have proved that

$$\zeta \circ \triangle_{\chi(\zeta)\langle g\rangle} \subset \varepsilon_v(\zeta\langle g\rangle),$$

which implies

$$\triangle_{\chi(\zeta)\langle g\rangle} \circ \zeta^{-1} \circ \zeta \circ \triangle_{\chi(\zeta)\langle g\rangle} \subset \varepsilon_v(\zeta\langle g\rangle).$$

This last inclusion means that

$$\varepsilon_0\langle \zeta^{-1} \circ \zeta\,, \chi(\zeta)\langle g\rangle\rangle \subset \varepsilon_v(\zeta\langle g\rangle).$$

Assume that $\varepsilon_m\langle \zeta^{-1} \circ \zeta\,, \chi(\zeta)\langle g\rangle\rangle \subset \varepsilon_v(\zeta\langle g\rangle)$ and consider

$$(g_1, g_2) \in \varepsilon_{m+1}\langle \zeta^{-1} \circ \zeta\,, \chi(\zeta)\langle g\rangle\rangle.$$

Then

$$g_1 = u[\bar{w}|_q c] \in \chi(\zeta)\langle g\rangle, \quad g_2 = u[\bar{w}|_q d] \in \chi(\zeta)\langle g\rangle,$$
$$(s[\bar{v}|_r c], s[\bar{v}|_r d]) \in \varepsilon_m\langle \zeta^{-1} \circ \zeta\,, \chi(\zeta)\langle g\rangle\rangle \circ \varepsilon_m\langle \zeta^{-1} \circ \zeta\,, \chi(\zeta)\langle g\rangle\rangle,$$

which, according to the above assumption, gives

$$(s[\bar{v}|_r c], s[\bar{v}|_r d]) \in \varepsilon_v(\zeta\langle g\rangle) \circ \varepsilon_v(\zeta\langle g\rangle) \subset \varepsilon_v(\zeta\langle g\rangle),$$

where $s[\bar{v}|_r c] \in \chi(\zeta)\langle g\rangle$. But $\zeta\langle g\rangle$ is a strong subset, so, the restriction of $\varepsilon_v(\zeta\langle g\rangle)$ to $W_v'(\zeta\langle g\rangle)$ is v-cancellative. Therefore

$$(s[\bar{v}|_r c], s[\bar{v}|_r d]) \in \varepsilon_v(\zeta\langle g\rangle) \quad \text{and} \quad s[\bar{v}|_r c] \in \chi(\zeta)\langle g\rangle$$

imply $c \equiv d(\varepsilon_v(\zeta\langle g\rangle)$. Hence

$$u[\bar{w}|_q c] \equiv u[\bar{w}|_q d](\varepsilon_v(\zeta\langle g\rangle)),$$

i.e., $g_1 \equiv g_2(\varepsilon_v(\zeta\langle g\rangle))$ and consequently,

$$\varepsilon_{m+1}\langle \zeta^{-1} \circ \zeta\,, \chi(\zeta)\langle g\rangle\rangle \subset \varepsilon_v(\zeta\langle g\rangle).$$

Thus, by induction

$$\varepsilon_m\langle \zeta^{-1} \circ \zeta\,, \chi(\zeta)\langle g\rangle\rangle \subset \varepsilon_v(\zeta\langle g\rangle).$$

for every $m \in \mathbb{N}$.

Now, let

$$(g_1, g_2) \in \varepsilon_m \langle \zeta^{-1} \circ \zeta \,, \chi(\zeta)\langle g_1 \rangle\rangle.$$

Then $(g_1, g_2) \in \varepsilon_v(\zeta\langle g_1\rangle)$, i.e.,

$$(\forall t \in T_n(G))\,(g_1 \leqslant t(g_1) \longleftrightarrow g_1 \leqslant t(g_2)),$$

which for $t = \triangle_G$ gives $g_1 \leqslant g_2$. This proves (3.3.2).

Further, let

$$(g_1, g_2) \in \varepsilon_m \langle \zeta^{-1} \circ \zeta \,, \chi(\zeta)\langle g\rangle\rangle \quad \text{and} \quad (g, x[\bar{y}|_i g_2]) \in \chi(\zeta),$$

then

$$(g_1, g_2) \in \varepsilon_v(\zeta\langle g\rangle) \quad \text{and} \quad g \leqslant t_1(x[\bar{y}|_i g_2])$$

for some $t_1 \in T_n(G)$. Consequently, $g \leqslant t_1(x[\bar{y}|_i g_1])$, i.e., $(g, x[\bar{y}|_i g_1]) \in \chi(\zeta)$. This means that also (3.3.3) is satisfied. Thus we have proved that $(G, o, \zeta, \chi(\zeta))$ is a strong f.o.p. Menger algebra. Therefore (G, o, ζ) is a strong f.o. Menger algebra. □

Another proof of this theorem is given in [215].

Theorem 3.3.3. *An algebraic system (G, o, χ), where (G, o) is a Menger algebra of rank $n > 1$, $\chi \subset G \times G$, is a strong p.q-o. Menger algebra if and only if χ is an l-regular v-negative quasi-order and for all $i = 1, \dots, n$, $x, g, g_1, g_2 \in G$, $\bar{y} \in G^n$ and any natural m, the following two implication hold:*

$$(g_1, g_2) \in \varepsilon_m \langle \triangle_G \,, \chi\langle g\rangle\rangle \;\wedge\; g \sqsubset x[\bar{y}|_i g_2] \longrightarrow g \sqsubset x[\bar{y}|_i g_1], \tag{3.3.4}$$

$$(g_1, g_2) \in \varepsilon_m \langle \triangle_G \,, \chi\langle g_1\rangle\rangle \;\wedge\; g_2 \sqsubset g_1 \longrightarrow g_1 = g_2. \tag{3.3.5}$$

Proof. The necessity of the conditions given can be proved in the same manner as in the proof of Theorem 3.3.1. We prove the sufficiency. For this, assume that all the conditions of the theorem are satisfied. Denote by ε_g the relation $\varepsilon\langle \triangle_G \,, \chi\langle g\rangle\rangle$, and by W_g the set $G \setminus \chi\langle g\rangle$. Then $(\varepsilon_g^* \,, W_g)$ is the determining pair of (G, o), which corresponds to the simplest representation of (G, o) by reversive n-place functions. Let P be the sum of the family representations $(P_{(\varepsilon_g^* \,, W_g)})_{g \in G}$. Analogously as in the proof of Theorem 3.3.1, we show $\chi = \chi_P$. So, we must only prove that P is one-to-one. Indeed, if $P(g_1) = P(g_2)$ for some $g_1, g_2 \in G$, then: (a) $\mathrm{pr}_1 P(g_1) \subset \mathrm{pr}_1 P(g_2)$, (b) $\mathrm{pr}_1 P(g_2) \subset \mathrm{pr}_1 P(g_1)$, (c) $P_{(\varepsilon_g^*, W_g)}(g_1) = P_{(\varepsilon_g^*, W_g)}(g_2)$ for every $g \in G$. As $\chi = \chi_P$, from (a) and (b) we get $g_1 \sqsubset g_2$ and $g_2 \sqsubset g_1$. Condition (c) implies

$$(\forall y)\;(\, g_1 \equiv y(\varepsilon_{g_1}) \longleftrightarrow (g_1 \sqsubset g_2 \longrightarrow g_2 \equiv y(\varepsilon_{g_1}))\,),$$

which, as it is not difficult to see, gives

$$(\forall y)\ (g_1 \sqsubset g_2 \longrightarrow (g_1 \equiv y(\varepsilon_{g_1}) \longrightarrow g_2 \equiv y(\varepsilon_{g_1}))).$$

Since $g_1 \sqsubset g_2$, for $y = g_1$ from the last condition we obtain $g_1 \equiv g_2(\varepsilon_{g_1})$. Thus, $(g_1, g_2) \in \varepsilon_m\langle\triangle_G, \chi\langle g_1\rangle\rangle$ for some $m \in \mathbb{N}$ and $g_2 \sqsubset g_1$. This, together with (3.3.5), proves $g_1 = g_2$. So, P is one-to-one. □

Remark that according to Section 2.1, the condition (3.3.4) is equivalent to the elementary formula

$$C_m : C'_m \longrightarrow g \sqsubset x[\bar{y}|_j g_1],$$

where

$$C'_m : \begin{cases} g \sqsubset g_1 = u_1[\bar{w}_1|_{q_1} c_1] \wedge g \sqsubset g_2 = u_1[\bar{w}_1|_{q_1} d_1], \\ \bigwedge\limits_{i=1}^{2^{m-1}-1} \begin{pmatrix} g \sqsubset u_{2i}[\bar{w}_{2i}|_{q_{2i}} c_{2i}] = s_i[\bar{v}_i|_{r_i} c_i], \\ g \sqsubset u_{2i+1}[\bar{w}_{2i+1}|_{q_{2i+1}} c_{2i+1}] = u_{2i}[\bar{w}_{2i}|_{q_{2i}} d_{2i}], \\ g \sqsubset u_{2i+1}[\bar{w}_{2i+1}|_{q_{2i+1}} d_{2i+1}] = s_i[\bar{v}_i|_{r_i} d_i] \end{pmatrix}, \\ \bigwedge\limits_{i=2^{m-1}}^{2^m-1} (g \sqsubset s_i[\bar{v}_i|_{r_i} c_i] = s_i[\bar{v}_i|_{r_i} d_i]) \wedge g \sqsubset x[\bar{y}|_j g_2]. \end{cases}$$

The condition (3.3.5) is equivalent to the formula

$$D_m : D'_m \longrightarrow u_1[\bar{w}_1|_{q_1} c_1] = u_1[\bar{w}_1|_{q_1} d_1],$$

where

$$D'_m : \begin{cases} u_1[\bar{w}_1|_{q_1} c_1] \sqsubset u_1[\bar{w}_1|_{q_1} d_1] \wedge u_1[\bar{w}_1|_{q_1} d_1] \sqsubset u_1[\bar{w}_1|_{c_1} d_1], \\ \bigwedge\limits_{i=1}^{2^{m-1}-1} \begin{pmatrix} u_1[\bar{w}_1|_{q_1} c_1] \sqsubset u_{2i}[\bar{w}_{2i}|_{q_{2i}} c_{2i}] = s_i[\bar{v}_i|_{r_i} c_i], \\ u_1[\bar{w}_1|_{q_1} c_1] \sqsubset u_{2i+1}[\bar{w}_{2i+1}|_{q_{2i+1}} c_{2i+1}] = u_{2i}[\bar{w}_{2i}|_{q_{2i}} d_{2i}], \\ u_1[\bar{w}_1|_{q_1} c_1] \sqsubset u_{2i+1}[\bar{w}_{2i+1}|_{q_{2i+1}} d_{2i+1}] = s_i[\bar{v}_i|_{r_i} d_i] \end{pmatrix}, \\ \bigwedge\limits_{i=2^{m-1}}^{2^m-1} (\ u_1[\bar{w}_1|_{q_1} c_1] \sqsubset s_i[\bar{v}_i|_{r_i} c_i] = s_i[\bar{v}_i|_{r_i} d_i]\ ), \end{cases}$$

$u_i \in G \cup \{e_{q_i}\}$, $s_i \in G \cup \{e_{r_i}\}$, $q_i, r_i, j \in \{1, \ldots, n\}$ for all $i = 1, 2, \ldots, 2^m - 1$, and the rest of the elements belong to G.

Let (G, o) be a Menger algebra of rank $n > 1$. Consider the binary relation

$$\overset{\wedge}{\zeta} = \{(g_1, g_2) \mid \overset{\wedge}{g_2} \subset \overset{\wedge}{g_1}\}, \tag{3.3.6}$$

where $\hat{g}$ is the strong closure of the set $\{g\}$. It is not difficult to show (see [215]) that the relation $\overset{\wedge}{\zeta}$ is a stable and steady quasi-order, which is contained in any reflexive steady binary relation defined on (G, o). The relation $\overset{\wedge}{\zeta}$ will be called a *strong quasi-order*.

Based on Theorem 3.3.2, we can prove that *a Menger algebra* (G, o) *of rank* $n > 1$ *is isomorphic to some Menger algebra of reversive* n*-place functions if and only if its strong quasi-order is antisymmetric*. Indeed, if P is a faithful representation of (G, o) by reversive n-place functions, then ζ_P is a stable steady order. So $\overset{\wedge}{\zeta} \subset \zeta_P$ and

$$\overset{\wedge}{\zeta} \cap \overset{\wedge}{\zeta}^{-1} \subset \zeta_P \cap \zeta_P^{-1} \subset \triangle_G,$$

i.e., $\overset{\wedge}{\zeta}$ is antisymmetric. On the other hand, if $\overset{\wedge}{\zeta}$ is antisymmetric, then the system $(G, o, \overset{\wedge}{\zeta})$ is a strong f.o. Menger algebra. Thus (G, o) is represented by reversive n-place functions.

The fact that $\overset{\wedge}{\zeta}$ is antisymmetric means that

$$g_1 \in \overset{\wedge}{g_2} \;\wedge\; g_2 \in \overset{\wedge}{g_1} \longrightarrow g_1 = g_2 \tag{3.3.7}$$

for all $g_1, g_2 \in G$. According to § 2.1, the implication (3.3.7) is equivalent to the system $(E'_{m,n})_{m,n \in \mathbb{N}}$, where

$$E'_{m,n} : \; g_1 \in \overset{m}{D}(\{g_2\}) \;\wedge\; g_2 \in \overset{n}{D}(\{g_1\}) \longrightarrow g_1 = g_2.$$

Since $\overset{m}{D}(X) \subset \overset{n}{D}(X)$ for $m < n$, the system $(E'_{m,n})_{m,n \in \mathbb{N}}$ is equivalent to its subsystem $(E'_m)_{m \in \mathbb{N}}$ such that

$$E'_m : \; g_1 \in \overset{m}{D}(\{g_2\}) \wedge g_2 \in \overset{m}{D}(\{g_1\}) \longrightarrow g_1 = g_2.$$

Then, according to § 2.1, condition E'_m is equivalent to the elementary formula

$$E_m : E''_m \longrightarrow t''_1(u_1) = t''_{-1}(u_{-1}), \tag{3.3.8}$$

where

$$E''_m : \begin{cases} \displaystyle\bigwedge_{i=1}^{\frac{3^{m-1}-1}{2}} \begin{pmatrix} t''_{3i-1}(u_{3i-1}) = t'_i(u_i) \wedge t''_{1-3i}(u_{1-3i}) = t'_{-i}(u_{-i}), \\ t''_{3i}(u_{3i}) = t'_i(v_i) \qquad \wedge t''_{-3i}(u_{-3i}) = t'_{-i}(v_{-i}), \\ t''_{3i+1}(u_{3i+1}) = t''_i(v_i) \wedge t''_{-1-3i}(u_{-1-3i}) = t''_{-i}(v_{-i}) \end{pmatrix}, \\ \displaystyle\bigwedge_{i=\frac{3^{m-1}+1}{2}}^{\frac{3^m-1}{2}} \begin{pmatrix} t'_i(u_i) = t'_i(v_i) = t''_i(v_i) = t''_{-1}(u_{-1}), \\ t'_{-i}(u_{-i}) = t'_{-i}(v_{-i}) = t''_{-i}(v_{-i}) = t''_1(u_1) \end{pmatrix} \end{cases}$$

In this way we proved the following theorem.

Theorem 3.3.4. *A Menger algebra of rank $n > 1$ is strong if and only if for every natural m it satisfies* (3.3.8)*.*

3.4 (⅄)-, (Y)-, (⅄, Y)-Menger algebras

Many authors investigate sets of multiplace functions closed with respect to the Menger composition of functions and other naturally defined operations such as, for example, the set-theoretic intersection of functions considered as subsets of the corresponding Cartesian product.

At first, we describe the set Φ of n-place functions closed with respect to the Menger composition O, set-theoretic intersection $\cap$ and the inclusion of domains χ_Φ. The systems $(\Phi, O, \cap)$ and $(\Phi, O, \cap, \chi_\Phi)$ will be called respectively: a $\cap$-*Menger algebra* and a *projection quasi-ordered* $\cap$*-Menger algebra of n-place functions.* The abstract analog of such systems will be called respectively: a $\curlywedge$*-Menger algebra of rank n*[1] and a *p.q-o.* $\curlywedge$*-Menger algebra of rank n.* In the case of reversive n-place functions, the corresponding algebraic system will be called *strong.*

Theorem 3.4.1. *For an algebra $(G, o, \curlywedge)$ of type $(n+1, 2)$, the following statements are true:*
(a) *$(G, o, \curlywedge)$ is a $\curlywedge$-Menger algebra of rank n, if and only if*
 (i) *o is a superassociative operation,*
 (ii) *$(G, \curlywedge)$ is a semilattice, and the following two conditions*

$$(x \curlywedge y)[\bar{z}] = x[\bar{z}] \curlywedge y[\bar{z}], \tag{3.4.1}$$

$$t_1(x \curlywedge y \curlywedge z) \curlywedge t_2(y) = t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z) \tag{3.4.2}$$

 hold for all $x, y, z \in G$, $\bar{z} \in G^n$ and $t_1, t_2 \in T_n(G)$,
(b) *if $n > 1$, then $(G, o, \curlywedge)$ is a strong $\curlywedge$-Menger algebra of rank n if and only if the conditions* (i) *and* (ii) *above,* (3.4.1) *and*

$$u[\bar{w}|_i(x \curlywedge y)] = u[\bar{w}|_i x] \curlywedge u[\bar{w}|_i y], \tag{3.4.3}$$

$$t_1(x \curlywedge y) \curlywedge t_2(y) = t_1(x \curlywedge y) \curlywedge t_2(x) \tag{3.4.4}$$

 are true for all $i = 1, \ldots, n$, $u, x, y \in G$, $\bar{w} \in G^n$, $t_1, t_2 \in T_n(G)$.

Proof. Necessity. Let $(G, o, \curlywedge)$ be a (strong) $\curlywedge$-Menger algebra of rank n. Then it is isomorphic to some $\cap$-Menger algebra of (reversive) n-place functions, for example to $(\Phi, O, \cap)$. It is clear that (Φ, O) is a Menger algebra and $(\Phi, \cap)$ is a semilattice. So, the conditions (i) and (ii) are satisfied.

[1] In the literature $\curlywedge$-Menger algebras also are called Menger $\mathcal{P}$-algebras.

To prove (3.4.1), let $\varphi_1, \varphi_2, \psi_1, \ldots, \psi_n \in \mathbf{\Phi}$. Then, obviously,

$$(\varphi_1 \cap \varphi_2)[\psi_1 \cdots \psi_n] \subset \varphi_1[\psi_1 \cdots \psi_n] \cap \varphi_2[\psi_1 \cdots \psi_n].$$

Conversely, if $(\bar{a}, d) \in \varphi_1[\psi_1 \cdots \psi_n] \cap \varphi_2[\psi_1 \cdots \psi_n]$, then one can find $\bar{b}$ and $\bar{c}$ such that $(\bar{a}, b_i) \in \psi_i$, $(\bar{b}, d) \in \varphi_1$, $(\bar{a}, c_j) \in \psi_j$, $(\bar{c}, d) \in \varphi_2$ for all $i, j = 1, \ldots, n$. Because all ψ_i are n-place functions, from the above, it follows that $b_i = c_i$, $i = 1, \ldots, n$, i.e., $\bar{b} = \bar{c}$. Hence $(\bar{b}, d) \in \varphi_1 \cap \varphi_2$ and, consequently, $(\bar{a}, d) \in (\varphi_1 \cap \varphi_2)[\psi_1 \cdots \psi_n]$. Therefore

$$\varphi_1[\psi_1 \cdots \psi_n] \cap \varphi_2[\psi_1 \cdots \psi_n] \subset (\varphi_1 \cap \varphi_2)[\psi_1 \cdots \psi_n],$$

which, together with the previous inclusion, proves (3.4.1).

To prove (3.4.2), observe first that for any binary relations $\rho, \sigma \subset A \times B$ and every subset $H \subset A$ the following equality takes place:

$$\rho \circ \triangle_H \cap \sigma = (\rho \cap \sigma) \circ \triangle_H. \tag{3.4.5}$$

Let $t_1, t_2 \in T_n(\mathbf{\Phi})$ and $\varphi, \psi, \chi \in \mathbf{\Phi}$. Then, as it is not difficult to see,

$$t_1(\varphi \cap \psi \cap \chi) \cap t_2(\psi) \subset t_1(\varphi \cap \psi) \cap t_2(\psi)$$

and

$$\mathrm{pr}_1(t_1(\varphi \cap \psi \cap \chi) \cap t_2(\psi)) \subset \mathrm{pr}_1(\psi \cap \chi).$$

Thus, using (3.4.5), we get

$$\begin{aligned} & t_1(\varphi \cap \psi \cap \chi) \cap t_2(\psi) \subset (t_1(\varphi \cap \psi) \cap t_2(\psi)) \circ \triangle_{\mathrm{pr}_1(\psi \cap \chi)} \\ & = t_1(\varphi \cap \psi) \cap t_2(\psi) \circ \triangle_{\mathrm{pr}\ _1(\varphi \cap \chi)} = t_1(\varphi \cap \psi) \cap t_2(\psi \circ \triangle_{\mathrm{pr}_1(\psi \cap \chi)}) \\ & = t_1(\varphi \cap \psi) \cap t_2(\psi \cap \chi). \end{aligned}$$

So, $t_1(\varphi \cap \psi \cap \chi) \cap t_2(\psi) \subset t_1(\varphi \cap \psi) \cap t_2(\psi \cap \chi)$. It is easy to check that for any functions φ and ψ satisfying the condition $\varphi \circ \triangle_{\mathrm{pr}_1 \psi} = \psi \circ \triangle_{\mathrm{pr}_1 \varphi}$, the following equality holds

$$\varphi \circ \triangle_{\mathrm{pr}_1 \psi} = \varphi \cap \psi. \tag{3.4.6}$$

Hence, using $(\varphi \cap \psi) \circ \triangle_{\mathrm{pr}_1(\psi \cap \chi)} = (\psi \cap \chi) \circ \triangle_{\mathrm{pr}_1(\varphi \cap \psi)}$ together with (3.4.5) and (3.4.6), we obtain

$$\begin{aligned} & t_1(\varphi \cap \psi) \cap t_2(\psi \cap \chi) = t_1(\varphi \cap \psi) \cap (t_2(\psi \cap \chi) \circ \triangle_{\mathrm{pr}_1(\psi \cap \chi)}) \\ & = (t_1(\varphi \cap \psi) \circ \triangle_{\mathrm{pr}_1(\psi \cap \chi)}) \cap t_2(\psi \cap \chi) \\ & = t_1((\varphi \cap \psi) \circ \triangle_{\mathrm{pr}_1(\psi \cap \chi)}) \cap t_2(\psi \cap \chi) \\ & = t_1(\varphi \cap \psi \cap \chi) \cap t_2(\psi \cap \chi) \subset t_1(\varphi \cap \psi \cap \chi) \cap t_2(\psi). \end{aligned}$$

So, $t_1(\varphi \cap \psi \cap \chi) \cap t_2(\psi) = t_1(\varphi \cap \psi) \cap t_2(\psi \cap \chi)$, which proves (3.4.2).

Now, we prove (3.4.3). Let $\mathbf{\Phi}$ be the set of reversive n-place functions. It is clear that

$$f[\bar{\chi}|_i\varphi \cap \psi] \subset f[\bar{\chi}|_i\varphi] \cap f[\bar{\chi}|_i\psi]$$

for every $i = 1,\ldots,n$, $\varphi,\psi,f \in \mathbf{\Phi}$ and $\bar{\chi} \in \mathbf{\Phi}^n$. To prove the converse inclusion, let $(\bar{a},d) \in f[\bar{\chi}|_i\varphi] \cap f[\bar{\chi}|_i\psi]$. Then there are $\bar{b}$ and $\bar{c}$ for which $(\bar{a},b_j) \in \chi_j$, $j \in \{1,\ldots,n\} \setminus \{i\}$, $(\bar{a},b_i) \in \varphi$, $(\bar{b},d) \in f$, $(\bar{a},c_j) \in \chi_j$, $j \in \{1,\ldots,n\} \setminus \{i\}$, $(\bar{a},c_i) \in \psi$ and $(\bar{c},d) \in f$. The fact that χ_j is a function implies $b_j = c_j$ for $j \in \{1,\ldots,n\} \setminus \{i\}$. By assumption f is a reversive function, so, from $f(\bar{b}) = f(\bar{b}|_i c_i)$ it follows $b_i = c_i$. Thus $(\bar{a},b_i) \in \varphi \cap \psi$, and, consequently $(\bar{a},d) \in f[\bar{\chi}|_i\varphi \cap \psi]$. Hence

$$f[\bar{\chi}|_i\varphi] \cap f[\bar{\chi}|_i\psi] \subset f[\bar{\chi}|_i\varphi \cap \psi].$$

This proves (3.4.3).

Finally, according to (3.4.2), for $t_1,t_2 \in T_n(\mathbf{\Phi})$, $\varphi,\psi \in \mathbf{\Phi}$ we have

$$t_1(\varphi\cap\psi)\cap t_2(\psi) = t_1(\varphi\cap\psi\cap\varphi)\cap t_2(\psi) = t_1(\varphi\cap\psi)\cap t_2(\varphi\cap\psi),$$

and

$$t_1(\varphi\cap\psi)\cap t_2(\varphi) = t_1(\psi\cap\varphi\cap\psi)\cap t_2(\varphi) = t_1(\varphi\cap\psi)\cap t_2(\varphi\cap\psi).$$

Therefore $t_1(\varphi\cap\psi)\cap t_2(\psi) = t_1(\varphi\cap\psi)\cap t_2(\varphi)$, which proves (3.4.4).

The necessity is proved.

Sufficiency. For the proof of this part of our theorem we shall need three propositions.

Proposition 3.4.2. *Assume that $(G,o,\curlywedge)$ satisfies all the conditions of Theorem* 3.4.1 (a), *maybe with the exception of* (3.4.2). *If (ε^*,W) is the determining pair of a Menger algebra (G,o), where $\varepsilon \subset G\times G$, $W \subset G$, $\varepsilon^* = \varepsilon \cup \{(e_1,e_1),\ldots,(e_n,e_n)\}$, $e_1,\ldots,e_n$ — selectors in (G^*,o^*), then the equality*

$$P_{(\varepsilon^*,W)}(g_1 \curlywedge g_2) = P_{(\varepsilon^*,W)}(g_1) \cap P_{(\varepsilon^*,W)}(g_2) \tag{3.4.7}$$

holds for all $g_1,g_2 \in G$ if and only if for all $g_1,g_2 \in G$ the pair (ε,W) satisfies the following three implications:

$$g_1 \in W \longrightarrow g_1 \curlywedge g_2 \in W, \tag{3.4.8}$$

$$g_1 \curlywedge g_2 \notin W \longrightarrow g_1 \equiv g_2(\varepsilon), \tag{3.4.9}$$

$$g_1 \notin W \wedge g_1 \equiv g_2(\varepsilon) \longrightarrow g_1 \curlywedge g_2 \equiv g_1(\varepsilon). \tag{3.4.10}$$

Proof. From the fact that equality (3.4.7) holds for all $g_1,g_2 \in G$, we obtain

$$(\forall g_1)(\forall g_2)(\forall y \notin W)\ (g_1 \curlywedge g_2 \equiv y(\varepsilon) \longleftrightarrow g_1 \equiv g_2 \equiv y(\varepsilon)).$$

This condition is, of course, equivalent to the following two implications:

(1) $(\forall g_1)(\forall g_2)(\forall y)\ (y \notin W \wedge g_1 \curlywedge g_2 \equiv y(\varepsilon) \longrightarrow g_1 \equiv g_2 \equiv y(\varepsilon))$,

(2) $(\forall g_1)(\forall g_2)(\forall y)\ (y \notin W \wedge g_1 \equiv g_2 \equiv y(\varepsilon) \longrightarrow g_1 \curlywedge g_2 \equiv y(\varepsilon))$.

Now let $g_1 \curlywedge g_2 \notin W$. By replacing in (1) y with $g_1 \curlywedge g_2$ we get $g_1 \equiv g_2 \equiv g_1 \curlywedge g_2(\varepsilon)$. Thus $g_1 \equiv g_2(\varepsilon)$ and $g_1 \notin W$. This proves (3.4.8) and (3.4.9). We obtain the condition (3.4.10) by putting $y = g_1$ in (2).

Conversely, if the implications (3.4.8)–(3.4.10) are satisfied and $y \notin W$, $g_1 \curlywedge g_2 \equiv y(\varepsilon)$, then $g_1 \curlywedge g_2 \notin W$, which, by (3.4.8) and (3.4.9), gives $g_1 \notin W$ and $g_1 \equiv g_2(\varepsilon)$. From this, according to (3.4.10), we get $g_1 \curlywedge g_2 \equiv g_1(\varepsilon)$. But $g_1 \curlywedge g_2 \equiv y(\varepsilon)$, so, $g_1 \equiv g_2 \equiv y(\varepsilon)$. This proves (1). If $y \notin W$ and $g_1 \equiv g_2 \equiv y(\varepsilon)$, then $g_1 \notin W$, which, according to (3.4.10), gives $g_1 \curlywedge g_2 \equiv g_1 \equiv y(\varepsilon)$, so, (2) is proved. □

It is not difficult to show that if the determining pair (ε^*, W) satisfies the conditions (3.4.8)–(3.4.10), then the relation ε is defined in the following way:

$$\varepsilon = \{(g_1, g_2) \mid (g_1 \curlywedge g_2) \notin W \vee g_1, g_2 \in W\}. \tag{3.4.11}$$

Now assume that an algebra $(G, o, \curlywedge)$ of type $(n+1, 2)$ satisfies all the conditions mentioned in Theorem 3.4.1 (a) and consider the binary relation $\zeta = \{(g_1, g_2) \mid g_1 \curlywedge g_2 = g_1\}$. Instead of $(g_1, g_2) \in \zeta$ we shall write $g_1 \leqslant g_2$.

Proposition 3.4.3. *The relation ζ is a stable order on $(G, o, \curlywedge)$. Moreover, with respect to the operation o it is weakly steady.*

Proof. By assumption $(G, \curlywedge)$ is a semilattice, so ζ is a stable order on $(G, \curlywedge)$. To prove that ζ is stable on (G, o), let $x \leqslant y$, i.e., $x \curlywedge y = x$. Then, according to (3.4.1), we have $x[\bar{z}] \curlywedge y[\bar{z}] = x[\bar{z}]$, i.e., $x[\bar{z}] \leqslant y[\bar{z}]$. So, ζ is l-regular. Further, from $x \leqslant y$ we obtain $u[\bar{w}|_i(x \curlywedge y)] = u[\bar{w}|_i x]$, $i = 1, \ldots, n$. Hence $u[\bar{w}|_i(x \curlywedge y)] \curlywedge u[\bar{w}|_i y] = u[\bar{w}|_i x] \curlywedge u[\bar{w}|_i y]$. Putting $z = x$ in (3.4.2) we get $t_1(x \curlywedge y) \curlywedge t_2(y) = t_1(x \curlywedge y) \curlywedge t_2(x \curlywedge y)$ for all $t_1, t_2 \in T_n(G)$. Thus

$$\begin{aligned} u[\bar{w}|_i(x \curlywedge y)] \curlywedge u[\bar{w}|_i y] &= u[\bar{w}|_i(x \curlywedge y)] \curlywedge u[\bar{w}|_i(x \curlywedge y)] \\ &= u[\bar{w}|_i(x \curlywedge y)] = u[\bar{w}|_i x]. \end{aligned}$$

So, $u[\bar{w}|_i x] \leqslant u[\bar{w}|_i y]$, i.e., ζ is i-regular for every $i = 1, \ldots, n$. This proves that ζ is stable on (G, o).

Now let $g_1 \leqslant g_2$, $g \leqslant t_1(g_1)$ and $g \leqslant t_2(g_2)$ for some $g, g_1, g_2 \in G$, $t_1, t_2 \in T_n(G)$. Then

$$g \curlywedge t_1(g_1 \curlywedge g_2 \curlywedge g_1) \curlywedge t_2(g_2) = g,$$

which, according to (3.4.2), gives us

$$g \curlywedge t_1(g_1 \curlywedge g_2 \curlywedge g_1) \curlywedge t_2(g_2 \curlywedge g_1) = g,$$

i.e., $g \curlywedge t_1(g_1) \curlywedge t_2(g_1) = g$. Applying $g \curlywedge t_1(g_1) = g$ to this equality, we obtain $g \curlywedge t_2(g_1) = g$, i.e., $g \leqslant t_2(g)$. So, ζ is weakly steady. □

Note that in the case when the algebra $(G, o, \curlywedge)$ from the Proposition 3.4.3 satisfies all the conditions of Theorem 3.4.1 (b), the relation ζ is steady on $(G, o, \curlywedge)$.

Now we consider the relation

$$\varepsilon_g = \{(g_1, g_2) \,|\, g_1 \curlywedge g_2 \notin W_v(\zeta\langle g\rangle) \ \vee \ g_1, g_2 \in W_v(\zeta\langle g\rangle)\}.$$

Proposition 3.4.4. *For every $g \in G$, the pair $(\varepsilon_g^*, W_v(\zeta\langle g\rangle))$, where $\varepsilon_g^* = \varepsilon_g \cup \{(e_1, e_1), \ldots, (e_n, e_n)\}$, $e_1, \ldots, e_n$ — selectors in (G^*, o^*), is the determining pair of (G, o) and satisfies the conditions* (3.4.8) – (3.4.10).

Proof. Let $g_1 \curlywedge g_2 \notin W_v(\zeta\langle g\rangle)$. Then $(g_1, g_2) \in \varepsilon_g$ and $g \leqslant t(g_1 \curlywedge g_2)$ for some $t \in T_n(G)$. Since $t(g_1 \curlywedge g_2) \leqslant t(g_1)$, we have $g \leqslant t(g_1)$, which gives $g_1 \notin W_v(\zeta\langle g\rangle)$. So, the pair $(\varepsilon_g^*, W_v(\zeta\langle g\rangle))$ satisfies (3.4.8) and (3.4.9). If $g_1 \notin W_v(\zeta\langle g\rangle)$ and $(g_1, g_2) \in \varepsilon_g$, then $g_1 \curlywedge g_2 \notin W_v(\zeta\langle g\rangle)$, and, consequently $(g_1 \curlywedge g_2, g_1) \in \varepsilon_g$. This proves (3.4.10).

Of course, ε_g is a reflexive and symmetric relation. Let $g_1 \equiv g_2(\varepsilon_g)$ and $g_2 \equiv g_3(\varepsilon_g)$. If at least one of g_1, g_2, g_3 is in $W_v(\zeta\langle g\rangle)$, then obviously all of them are in $W_v(\zeta\langle g\rangle)$. Thus $g_1 \equiv g_3(\varepsilon_g)$. If any of g_1, g_2, g_3 is not in $W_v(\zeta\langle g\rangle)$, then $g_1 \curlywedge g_2 \notin W_v(\zeta\langle g\rangle)$ and $g_2 \curlywedge g_3 \notin W_v(\zeta\langle g\rangle)$. Consequently, $g \leqslant t_1(g_1 \curlywedge g_2)$ and $g \leqslant t_2(g_2 \curlywedge g_3)$ for some $t_1, t_2 \in T_n(G)$. Therefore $g \leqslant t_1(g_1 \curlywedge g_2) \curlywedge t_2(g_2 \curlywedge g_3)$. This, according to (3.4.2), gives

$$g \leqslant t_1(g_1 \curlywedge g_2 \curlywedge g_3) \curlywedge t_2(g_3) \leqslant t_1(g_1 \curlywedge g_3).$$

Hence $g_1 \curlywedge g_3 \notin W_v(\zeta\langle g\rangle)$, i.e., $g_1 \equiv g_3(\varepsilon_g)$. So, ε_g is transitive in any case. This proves that ε_g is an equivalence on G.

Now we prove the v-regularity of ε_g. For this let $g_1 \equiv g_2(\varepsilon_g)$. If g_1, g_2 are in $W_v(\zeta\langle g\rangle)$, then also $u[\bar{w}|_i g_1]$, $u[\bar{w}|_i g_2]$ are in $W_v(\zeta\langle g\rangle)$, because $W_v(\zeta\langle g\rangle)$ is an l-ideal. Thus $u[\bar{w}|_i g_1] \equiv u[\bar{w}|_i g_2](\varepsilon_g)$. If $g_1, g_2 \notin W_v(\zeta\langle g\rangle)$, then also $g_1 \curlywedge g_2 \notin W_v(\zeta\langle g\rangle)$, i.e., $g \leqslant t(g_1 \curlywedge g_2)$ for some $t \in T_n(G)$. If $u[\bar{w}|_i g_1]$ is not in $W_v(\zeta\langle g\rangle)$, then obviously $g \leqslant t_1(u[\bar{w}|_i g_1])$ for some $t_1 \in T_n(G)$. This, together with $g_1 \curlywedge g_2 \leqslant g_1$, $g \leqslant t(g_1 \curlywedge g_2)$ and the weak stability, implies $g \leqslant t_1(u[\bar{w}|_i(g_1 \curlywedge g_2)])$. Since $u[\bar{w}|_i(g_1 \curlywedge g_2)] \leqslant u[\bar{w}|_i g_k]$ for $k = 1, 2$, the above proves $u[\bar{w}|_i(g_1 \curlywedge g_2)] \leqslant u[\bar{w}|_i g_1] \curlywedge u[\bar{w}|_i g_2]$. Thus

$$t_1(u[\bar{w}|_i(g_1 \curlywedge g_2)]) \leqslant t_1(u[\bar{w}|_i g_1] \curlywedge u[\bar{w}|_i g_2]).$$

Hence $g \leqslant t_1(u[\bar{w}|_i g_1] \curlywedge u[\bar{w}|_i g_2])$, i.e., $u[\bar{w}|_i g_1] \curlywedge u[\bar{w}|_i g_2] \notin W_v(\zeta\langle g\rangle)$. Therefore $u[\bar{w}|_i g_1] \equiv u[\bar{w}|_i g_2](\varepsilon_g)$, which proves that ε_g is i-regular for every $i = 1, \ldots, n$. By Proposition 2.1.11, it is v-regular. □

From the stability of ζ it follows that for every $g \in G$ the determining pair $(\varepsilon_g^*, W_v(\zeta\langle g\rangle))$ satisfies the condition

$$\zeta \subset \zeta_{(\varepsilon_g^*, W_v(\zeta\langle g\rangle))}. \tag{3.4.12}$$

If $(G, o, \curlywedge)$, besides it, satisfies condition (3.4.3), then the relation ε_g is v-cancellative on a Menger algebra (G, o) and the above determining pair determines the simplest representation of (G, o) by reversive n-place functions.

Now we are in a position to finish the proof of Theorem 3.4.1.

Assume that the algebra $(G, o, \curlywedge)$ satisfies all the conditions of the item (a). According to the Proposition 3.4.4, we can consider the family of the simplest representations $\big(P_{(\varepsilon_g^*, W_v(\zeta\langle g\rangle))}\big)_{g\in G}$ of a Menger algebra (G, o), the sum of which will be denoted by P. From the Propositions 3.4.2 and 3.4.4 it follows that for any $g, g_1, g_2 \in G$ we have

$$P_{(\varepsilon_g^*, W_v(\zeta\langle g\rangle))}(g_1 \curlywedge g_2) = P_{(\varepsilon_g^*, W_v(\zeta\langle g\rangle))}(g_1) \cap P_{(\varepsilon_g^*, W_v(\zeta\langle g\rangle))}(g_2).$$

Hence $P(g_1 \curlywedge g_2) = P(g_1) \cap P(g_2)$ for all $g_1, g_2 \in G$. So, P is a homomorphism of $(G, o, \curlywedge)$ onto some $\cap$-Menger algebra of n-place functions. We prove that this homomorphism is one-to-one. Indeed, since $\zeta_P = \bigcap\limits_{g\in G} \zeta_{(\varepsilon_g^*, W_v(\zeta\langle g\rangle))}$, condition (3.4.12) implies $\zeta \subset \zeta_P$. Conversely, if $(g_1, g_2) \in \zeta_P$, then

$$(\forall g \in G)(\forall \bar{x} \in B)\ (g_1[\bar{x}] \notin W_v(\zeta\langle g\rangle) \longrightarrow g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon_g)\,),$$

where $B = G^n \cup \{\bar{e}\}$, $\bar{e} = (e_1, \ldots, e_n)$ — selectors in $(G^*; o^*)$. This for $\bar{x} = \bar{e}$ and $g = g_1$ gives $g_1 \curlywedge g_2 \in W_v(\zeta\langle g_1\rangle)$. Hence $g_1 \leqslant t(g_1 \curlywedge g_2)$ for some $t \in T_n(G)$. Consequently, $g_1 \curlywedge g_2 \leqslant g_1,\ g_1 \leqslant t(g_1 \curlywedge g_2)$ and $g_1 \leqslant g_1$. But ζ is weakly steady, so, $g_1 \leqslant g_1 \curlywedge g_2 \leqslant g_2$, i.e., $\zeta_P \subset \zeta$. Thus $\zeta = \zeta_P$. Now, using the antisymmetry of ζ we have

$$\begin{aligned} P(g_1) = P(g_2) &\longleftrightarrow P(g_1) \subset P(g_2)\ \wedge\ P(g_2) \subset P(g_1) \\ &\longleftrightarrow g_1 \leqslant g_2\ \wedge\ g_2 \leqslant g_1 \longrightarrow g_1 = g_2, \end{aligned}$$

which proves that P is one-to-one. So, P is an isomorphism.

Now assume that the algebra $(G, o, \curlywedge)$ satisfies all the conditions of the item (b). In this case, (3.4.3) and (3.4.4) imply (3.4.2). Indeed,

$$\begin{aligned} t_1(x \curlywedge y \curlywedge z) \curlywedge t_2(y) &= t_1(x \curlywedge y \curlywedge z) \curlywedge t_2(y) \curlywedge t_2(y) \\ &= t_1(x \curlywedge y \curlywedge z) \curlywedge t_2(x \curlywedge z) \curlywedge t_2(y) \\ &= t_1(x \curlywedge y \curlywedge z) \curlywedge t_2(x \curlywedge y \curlywedge z) \leqslant t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z). \end{aligned}$$

We have further

$$\begin{aligned} t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z) &= t_1(x \curlywedge y) \curlywedge t_2(y) \curlywedge t_2(y) \curlywedge t_2(z) \\ &= t_1(x \curlywedge y) \curlywedge t_2(x) \curlywedge t_2(y) \curlywedge t_2(z) \\ &= t_1(x \curlywedge y) \curlywedge t_2(x \curlywedge y \curlywedge z) \leqslant u, \end{aligned}$$

where $u \in \{t_2(z),\, t_2(x \curlywedge y),\, t(x \curlywedge y)\}$. From this, in view of the steadiness of ζ, we conclude $t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z) \leqslant t_1(z)$. Thus

$$t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z) \leqslant t_1(x \curlywedge y) \curlywedge t_1(z) \curlywedge t_2(y) = t_1(x \curlywedge y \curlywedge z) \curlywedge t_2(y),$$

which together with the previous inequality proves (3.4.2). So, all the demands of item (a) are satisfied. Therefore, basing on the just proved item (a), we conclude that the algebra $(G, o, \curlywedge)$ satisfying all the conditions of the item (b) is isomorphic to a $\cap$-Menger algebra of reversive n-place functions. □

Theorem 3.4.5. *An algebraic system* $(G, o, \curlywedge, \chi)$*, where* (G, o) *is a Menger algebra of rank* n*,* $(G, \curlywedge)$ *— a semilattice,* χ *— a binary relation defined on* G*, is isomorphic to some projection quasi-ordered* $\cap$*-Menger algebra of* n*-place functions if and only if* χ *is an* l*-regular* v*-negative quasi-order containing the semilattice order* ζ *of* $(G, \curlywedge)$*, the identity* (3.4.1) *and the conditions*

$$x \sqsubset x \curlywedge y \longrightarrow x \leqslant y, \tag{3.4.13}$$

$$u[(x_1 \curlywedge y_1) \cdots (x_n \curlywedge y_n)] \leqslant u[y_1 \cdots y_n], \tag{3.4.14}$$

$$x \sqsubset y \curlywedge z \,\wedge\, x \sqsubset u[\bar{w}|_i(z \curlywedge v)] \longrightarrow x \sqsubset u[\bar{w}|_i(y \curlywedge z \curlywedge v)] \tag{3.4.15}$$

hold for all $i = 1, \ldots, n,\ u \in G \cup \{e_i\},\ x, y, z, x_k, y_k \in G,\ k = 1, \ldots, n,\ \bar{w} \in G^n$*, where* $x \leqslant y \longleftrightarrow (x, y) \in \zeta,\ \ x \sqsubset y \longleftrightarrow (x, y) \in \chi$*.*

Proof. We prove only the sufficiency of these conditions because the proof of their necessity is not difficult. First of all, observe that (3.4.1) and (3.4.14) imply the stability of ζ. Next, for every $g \in G$ we define the relation ε_g letting

$$\varepsilon_g = \{(g_1, g_2) \,|\, g \sqsubset g_1 \curlywedge g_2 \ \ or \ \ g_1, g_2 \in W_g\},$$

where $W_g = G \setminus \chi\langle g\rangle$. From the v-negativity of χ we conclude that W_g is an l-ideal. Analogously, from condition (3.4.15) and the properties of ζ we deduce that ε_g is a v-congruence on a Menger algebra (G, o). Thus, (ε_g^*, W_g) is the determining pair of (G, o), which, as it is easy to see, satisfies all the conditions of (3.4.8)–(3.4.10), where $\varepsilon_g^* = \varepsilon_g \cup \{(e_1, e_1), \ldots, (e_n, e_n)\}$, $e_1, \ldots, e_n$ — selectors in (G^*, o^*).

Let P be such representation of (G, o), which is the sum of the family of representations $\big(P_{(\varepsilon_g^*, W_g)}\big)_{g\in G}$. In view of Proposition 3.4.2, this representation satisfies the identity

$$P(g_1 \curlywedge g_2) = P(g_1) \cap P(g_2),$$

where $g_1, g_2 \in G$. Moreover, in this case $\zeta = \zeta_P$ and $\chi = \chi_P$. Indeed, let $g_1 \leqslant g_2$ and $g \sqsubset g_1[\bar{x}]$ for some $g \in G$ and $\bar{x} \in B = G^n \cup \{(e_1, \ldots, e_n)\}$. Since $g_1 \curlywedge g_2 = g_1$, implies $g \sqsubset (g_1 \curlywedge g_2)[\bar{x}] = g_1[\bar{x}] \curlywedge g_2[\bar{x}]$, from the above we obtain $g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon_g)$. So, $(g_1, g_2) \in \zeta_P$.

Conversely, if $(g_1, g_2) \in \zeta_P$, then

$$(\forall g)(\forall \bar{x})\ \left(g \sqsubset g_1[\bar{x}] \longrightarrow g \sqsubset g_1[\bar{x}] \curlywedge g_2[\bar{x}]\right),$$

which for $g = g_1,\ \bar{x} = (e_1, \ldots, e_n)$ gives $g_1 \sqsubset g_1 \curlywedge g_2$. Hence $g_1 \leqslant g_2$, by (3.4.13). This proves $\zeta = \zeta_P$. The proof of $\chi = \chi_P$ is very similar.

We complete this proof by the simple observation that $P(g_1) = P(g_2)$ is equivalent to $P(g_1) \subset P(g_2)$ and $P(g_2) \subset P(g_1)$. Hence we have $g_1 \leqslant g_2$ and $g_2 \leqslant g_1$, which implies $g_1 = g_2$. This means that P is a faithful representation of (G, o). □

Let (Φ, O) be a Menger algebra of n-place functions defined on the set A and let $\overset{n+1}{\triangle}_A$ be an n-place function belonging to Φ. In this algebra the intersection of functions can be expressed by the composition, namely, for any functions $\varphi_1, \ldots, \varphi_n \in \Phi$ we have

$$\overset{n+1}{\triangle}_A[\varphi_1 \cdots \varphi_n] = \varphi_1 \cap \cdots \cap \varphi_n.$$

Hence $\varphi_1 \cap \varphi_2 = \overset{n+1}{\triangle}_A[\varphi_1\varphi_2 \cdots \varphi_2]$, which means that $\overset{n+1}{\triangle}_A$ can be treatment as a nullary operation. Such an algebra $(\Phi, O, \overset{n+1}{\triangle}_A)$ will be called a *$\mathcal{D}$-Menger algebra of n-place functions* and can be characterized by some abstract algebra (G, o, e) of type $(n+1, 0)$, where $n \geqslant 2$, with the added binary operation $\curlywedge$ defined by the formula $g_1 \curlywedge g_2 = e[g_1 g_2 \cdots g_2]$.

Theorem 3.4.6. *An algebra (G, o, e) of type $(n+1, 0)$, where $n \geqslant 2$, is isomorphic to some $\mathcal{D}$-Menger algebra of n-place functions if and only if the operation o is superassociative, $(G, \curlywedge)$ is a semilattice,* (3.4.4) *holds and for all $g, g_1, g_2, \ldots, g_n \in G$, $t \in T_n(G)$ the following conditions take place:*

$$e[g_1 g_2 \cdots g_n] = g_1 \curlywedge g_2 \curlywedge \cdots \curlywedge g_n, \tag{3.4.16}$$

$$g[e^n] \curlywedge g = g[e^n], \tag{3.4.17}$$

$$g \curlywedge t(e) = g \longrightarrow g[e^n] = g. \tag{3.4.18}$$

Proof. The necessity part is obvious. To prove the sufficiency observe first that $(G, o, \curlywedge)$ is a $\curlywedge$-Menger algebra. Indeed, the identity (3.4.1) is an immediate consequence of the superassociativity and the definition of $\curlywedge$. Next, using (3.4.4) we obtain

$$t_1(g_1 \curlywedge g_2) \curlywedge t_2(g_1 \curlywedge g_2) = t_1(g_1 \curlywedge g_2 \curlywedge g_2) \curlywedge t_2(g_1 \curlywedge g_2) = t_1(g_1 \curlywedge g_2) \curlywedge t_1(g_2),$$

which together with

$$t_1(g_1 \curlywedge g_2) \curlywedge t_2(g_2 \curlywedge g_3) = t_1(e[g_2 g_1 \cdots g_1]) \curlywedge t_2(g_2 \curlywedge g_3)$$

gives us

$$\begin{aligned} & t_1(g_1 \curlywedge g_2) \curlywedge t_2(g_2 \curlywedge g_3) = t_1(e[(g_2 \curlywedge g_3) g_1 \cdots g_1]) \curlywedge t_2(g_2 \curlywedge g_3) \\ & = t_1(g_1 \curlywedge g_2 \curlywedge g_3) \curlywedge t_2(g_2 \curlywedge g_3) = t_1(g_1 \curlywedge g_2 \curlywedge g_3) \curlywedge t_2(g_1 \curlywedge g_2 \curlywedge g_3) \\ & = t_1((g_1 \curlywedge g_3) \curlywedge g_2) \curlywedge t_2((g_1 \curlywedge g_3) \curlywedge g_2) = t_1(g_1 \curlywedge g_2 \curlywedge g_3) \curlywedge t_2(g_2). \end{aligned}$$

This proves (3.4.2). Thus, $(G, o, \curlywedge)$ is a $\curlywedge$-Menger algebra and all the arguments presented in the proof above are valid in this case too.

It is not difficult to see that condition (3.4.17) implies $g[e^n] \leqslant g$. The condition (3.4.18) is equivalent to the implication $g \leqslant t(e) \longrightarrow g[e^n] = g$, i.e., to

$$e \notin W_v(\zeta\langle g\rangle) \longrightarrow g[e^n] = g.$$

Consider the following two subsets of G :

$$G_1 = \{g \mid e \in W_v(\zeta\langle g\rangle)\}, \quad G_2 = \{g \mid e \notin W_v(\zeta\langle g\rangle)\}.$$

It is clear that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \varnothing$.

Let P be a representation of a Menger algebra (G, o) such that we have

$$P = \sum_{g \in G_1} P_{(\varepsilon_g^*, W_v(\zeta\langle g\rangle))} + \sum_{g \in G_2} P_{(\varepsilon_g, W_v(\zeta\langle g\rangle))}, \tag{3.4.19}$$

where, according to the definition, for every $g_1 \in G$

$$P_{(\varepsilon_g, W_v(\zeta\langle g\rangle))}(g_1) = P_{(\varepsilon_g^*, W_v(\zeta\langle g\rangle))}(g_1) \setminus \{(\bar{b}, a)\},$$

if ε^*-classes $\varepsilon_g^*\langle g_1\rangle$, $\varepsilon_g^*\langle e_i\rangle$, $i = 1, \ldots, n$, are indexed by a, b_i, $i = 1, \ldots, n$. We show that $\zeta = \zeta_P$.

The inclusion $\zeta \subset \zeta_P$ is a consequence of condition (3.4.12), which is true also for the pairs $(\varepsilon_g, W_v(\zeta\langle g\rangle))$. To prove the converse inclusion, assume that $(g_1, g_2) \in \zeta_P$. This is equivalent to the following two conditions:

$$(\forall g \in G_1)(\forall \bar{x} \in B)\ (g_1[\bar{x}]) \notin W_v(\zeta\langle g\rangle) \longrightarrow g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon_g)),$$

$$(\forall g \in G_2)(\forall \bar{x} \in G^n)\ (g_1[\bar{x}]) \notin W_v(\zeta\langle g\rangle) \longrightarrow g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon_g)),$$

where $B = G^n \cup \{(e_1, \ldots, e_n)\}$.

If $g_1 \in G_1$, then from the first condition for $g = g_1$, $\bar{x} = (e_1, \ldots, e_n)$ we obtain $g_1 \curlywedge g_2 \notin W_v(\zeta\langle g_1\rangle)$, i.e., $g_1 \leqslant t(g_1 \curlywedge g_2)$ for some $t \in T_n(G)$. But $g_1 \curlywedge g_2 \leqslant g_1$ and $g_1 \leqslant g_1$, so, $g_1 \leqslant g_1 \curlywedge g_2 \leqslant g_2$ because ζ is a weakly steady relation.

If $g_1 \in G_2$, then the second condition implies $g_2[e^n] \curlywedge g_1 \notin W_v(\zeta\langle g_1\rangle)$, since $g_1[e^n] = g_1$. Thus $g_1 \leqslant t(g_2[e^n] \curlywedge g_1)$ for some $t \in T_n(G)$. Then, from $g_2[e^n] \curlywedge g_1 \leqslant g_1$, $g_1 \leqslant t(g_2[e^n] \curlywedge g_1)$, $g_1 \leqslant g_1$ and the properties of ζ we deduce $g_1 \leqslant g_2[e^n] \curlywedge g_1$. Hence $g_1 \leqslant g_2[e^n]$, and $g_1 \leqslant g_2[e^n] \leqslant g_2$ by (3.4.17). So, $\zeta_P \subset \zeta$. This proves $\zeta = \zeta_P$. Therefore P is a faithful representation of (G, o) by n-place functions.

Further on, from condition (3.4.16) we get $e[g^n] = g$ for every $g \in G$. Besides, from $x_1 \curlywedge x_2 \curlywedge \cdots \curlywedge x_n \notin W_v(\zeta\langle g\rangle)$, we conclude that for every $g \in G$ we also have $x_1 \curlywedge x_2 \curlywedge \cdots \curlywedge x_n \equiv x_1 \equiv x_2 \equiv \cdots \equiv x_n(\varepsilon_g)$. This, according to (3.4.19) gives $P(e) = \overset{n+1}{\triangle}_{A_0}$, where A_0 is the set of elements used for indexation of the equivalence classes $\varepsilon_g\langle g_1\rangle$ such that $g \in G$ and $g_1 \notin W_v(\zeta\langle g\rangle)$. Since P is a representation of (G, o) by n-place functions defined on the set A, where $A_0 \subset A$, instead of P we consider P^* defined on G in the following way:

$$P^*(g) = \begin{cases} P(g) \cup \overset{n+1}{\triangle}_{A \setminus A_0} & \text{if } g \neq e, \\ \overset{n+1}{\triangle}_{A} & \text{if } g = e. \end{cases}$$

If $g \in G_1$, then $\mathrm{pr}_1 P(g) \subset A_0^n \cup \{(b_{1g}, \ldots, b_{ng}) \,|\, g \in G_1\}$, where b_{ig} is the index of the class $\varepsilon_g^*\langle e_i\rangle$, $i = 1, \ldots, n$. If $g \in G_2$, then $\mathrm{pr}_1 P(g) \subset A_0^n$. Besides, for every $g \in G$ we have $\mathrm{pr}_2 P(g) \subset A_0$. This means that the algebra $(P(G), O, \overset{n+1}{\triangle}_{A_0})$ is isomorphic to the algebra $(P^*(G), O, \overset{n+1}{\triangle}_{A})$. □

We shall now consider the set Φ of n-place functions closed with respect to the Menger composition O and the set-theoretic union $\cup$. The algebra $(\Phi, O, \cup)$ will be called a $\cup$-*Menger algebra of n-place functions*, their abstract analog — a $\curlyvee$-*Menger algebra of rank n*.

Let $(G, o, \curlyvee)$ be an arbitrary algebra of type $(n + 1, 2)$, where (G, o) is a Menger algebra, $(G, \curlyvee)$ — a semilattice satisfying the identity:

$$(x \curlyvee y)[\bar{z}] = x[\bar{z}] \curlyvee y[\bar{z}], \tag{3.4.20}$$

(ε^*, W) — the determining pair of (G, o), $\varepsilon^* = \varepsilon \cup \{(e_1, e_1), \ldots, (e_n, e_n)\}$, $e_1, \ldots, e_n$ — selectors in (G^*, o^*). Then the following proposition takes place.

Proposition 3.4.7. *The determining pair* (ε^*, W) *satisfies the identity*

$$P_{(\varepsilon^*,W)}(g_1 \curlyvee g_2) = P_{(\varepsilon^*,W)}(g_1) \cup P_{(\varepsilon^*,W)}(g_2), \tag{3.4.21}$$

if and only if for all $g_1, g_2 \in G$ *the following three conditions hold:*

$$g_1 \in W \ \wedge \ g_2 \in W \longrightarrow g_1 \curlyvee g_2 \in W, \tag{3.4.22}$$

$$g_1 \leqslant g_2 \ \wedge \ g_2 \in W \longrightarrow g_1 \in W, \tag{3.4.23}$$

$$\varepsilon = (W' \times W') \cup (W \times W), \tag{3.4.24}$$

where $W' = G \setminus W$ *and* $x \leqslant y \longleftrightarrow x \curlyvee y = y$.

Proof. It is not difficult to see that the identity (3.4.21) is equivalent to the following system of three conditions:

$$y \notin W \ \wedge \ (g_1 \curlyvee g_2) \equiv y(\varepsilon) \longrightarrow g_1 \equiv y(\varepsilon) \ \vee \ g_2 \equiv y(\varepsilon), \tag{3.4.25}$$

$$y \notin W \ \wedge \ g_1 \equiv y(\varepsilon) \longrightarrow (g_1 \curlyvee g_2) \equiv y(\varepsilon), \tag{3.4.26}$$

$$y \notin W \ \wedge \ g_2 \equiv y(\varepsilon) \longrightarrow (g_1 \curlyvee g_2) \equiv y(\varepsilon), \tag{3.4.27}$$

which must be satisfied for all $g_1, g_2, y \in G$. From condition (3.4.25) for $y = g_1 \curlyvee g_2$ we get $g_1 \curlyvee g_2 \notin W \longrightarrow g_1 \notin W \vee g_2 \notin W$, which implies (3.4.22). Putting $y = g_1$ in (3.4.26) we obtain $g_1 \notin W \longrightarrow g_1 \curlyvee g_2 \notin W$. This means that $g_1 \leqslant g_2$ implies $g_2 \notin W$. So, (3.4.23) is proved. Further, replacing y in (3.4.26) by g_1 and by g_2 in (3.4.27), we get $g_i \notin W \longrightarrow g_1 \curlyvee g_2 \equiv g_i(\varepsilon)$, $i = 1, 2$. Therefore $g_1 \equiv g_2(\varepsilon)$ for $g_1, g_2 \in W'$, which proves (3.4.24), because W is a ε-class.

Conversely, let the conditions (3.4.22)–(3.4.24) be satisfied. If the premise of (3.4.25) is fulfilled, then, according to (3.4.22), we have $g_1 \in W'$ or $g_2 \in W'$, which, by (3.4.24), gives $g_1 \equiv y(\varepsilon)$ or $g_2 \equiv y(\varepsilon)$. So, (3.4.25) is proved. Now, applying (3.4.23) to the premise of (3.4.26), we obtain $g_1 \curlyvee g_2 \in W'$, which proves (3.4.24). The proof of (3.4.27) is analogous. □

Theorem 3.4.8. *An algebra* $(G, o, \curlyvee)$ *of type* $(n+1, 2)$ *is a* $\curlyvee$*-Menger algebra of rank* n *if and only if the operation* o *is superassociative,* $(G, \curlyvee)$ *is a semilattice, the identity* (3.4.20) *holds and for all* $x, y, y_i, z \in G$, $\bar{w} \in G^n$, $u \in G \cup \{e_i\}$, $i = 1, \ldots, n$, *the conditions*

$$u[\bar{w}|_i(x \curlyvee y)] = u[\bar{w}|_i x] \curlyvee u[\bar{w}|_i y], \tag{3.4.28}$$

$$x[y_1 \cdots y_n] \leqslant y_i, \tag{3.4.29}$$

$$x \leqslant y \curlyvee u[\bar{w}|_i z] \longrightarrow x \leqslant y \curlyvee u[\bar{w}|_i x], \tag{3.4.30}$$

where $x \leqslant y \longleftrightarrow x \curlyvee y = y$, *are satisfied.*

Proof. Necessity. Let $(\mathbf{\Phi}, O, \cup)$ be some $\cup$-Menger algebra of n-place functions. We check only the conditions (3.4.28)–(3.4.30), as the rest is obvious. Let $\alpha, \beta, \beta_i, \gamma, \varphi \in \mathbf{\Phi}$, $i = 1, \ldots, n$, $\bar{\chi} \in \mathbf{\Phi}^n$ be arbitrary. We have $\varphi[\bar{\chi}|_i\alpha] \cup \varphi[\bar{\chi}|_i\beta] \subset \varphi[\bar{\chi}|_i(\alpha \cup \beta)]$. Conversely, if $(\bar{a}, c) \in \varphi[\bar{\chi}|_i(\alpha \cup \beta)]$, then

$$(\bar{a}, b_i) \in \alpha \cup \beta,\ (\bar{a}, b_j) \in \chi_j,\ j = 1, \ldots, n,\ j \neq i,\ (\bar{b}, c) \in \varphi$$

for some $\bar{b}$. Thus $(\bar{a}, b_i) \in \alpha$ or $(\bar{a}, b_i) \in \beta$, $(\bar{a}, b_j) \in \chi_j$, $j = 1, \ldots, n$, $j \neq i$, $(\bar{b}, c) \in \varphi$. Therefore, $(\bar{a}, c) \in \varphi[\bar{\chi}|_i\alpha]$ or $(\bar{a}, c) \in \varphi[\bar{\chi}|_i\beta]$, i.e.,

$$(\bar{a}, c) \in \varphi[\bar{\chi}|_i\alpha] \cup \varphi[\bar{\chi}|_i\beta].$$

So, $\varphi[\bar{\chi}|_i(\alpha \cup \beta)] \subset \varphi[\bar{\chi}|_i\alpha] \cup \varphi[\bar{\chi}|_i\beta]$. This proves (3.4.28).

The inclusion $\mathrm{pr}_1\alpha[\beta_1 \cdots \beta_n] \subset \mathrm{pr}_1\beta_i$ is true for every $i = 1, \ldots, n$. Thus $\alpha[\beta_1 \cdots \beta_n] \circ \triangle_{\mathrm{pr}_1\beta_i} = \alpha[\beta_1 \cdots \beta_n]$. As $\alpha[\beta_1 \cdots \beta_n] \cup \beta_i \in \mathbf{\Phi}$, then $(\alpha[\beta_1 \cdots \beta_n],\, \beta_i) \in \xi_{\mathbf{\Phi}}$, i.e.,

$$\alpha[\beta_1 \cdots \beta_n] \circ \triangle_{\mathrm{pr}\ {}_1\beta_i} = \beta_i \circ \triangle_{\mathrm{pr}_1\alpha[\beta_1 \cdots \beta_n]}.$$

Hence $\alpha[\beta_1 \cdots \beta_n] = \beta_i \circ \triangle_{\mathrm{pr}_1\alpha[\beta_1 \cdots \beta_n]}$. So, $\alpha[\beta_1 \cdots \beta_n] \subset \beta_i$, which proves (3.4.29).

Finally, let $\alpha \subset \beta \cup \varphi[\bar{\chi}|_i\, \gamma]$. As $\alpha \cup \gamma \in \mathbf{\Phi}$, then, evidently, $(\alpha, \gamma) \in \xi_{\mathbf{\Phi}}$, i.e., $\alpha \circ \triangle_{\mathrm{pr}_1\gamma} = \gamma \circ \triangle_{\mathrm{pr}_1\alpha}$. Therefore

$$\begin{aligned}
\alpha = \alpha \circ \triangle_{\mathrm{pr}_1\alpha} &\subset (\beta \cup \varphi[\bar{\chi}|_i\, \gamma]) \circ \triangle_{\mathrm{pr}_1\alpha} \\
&= (\beta \circ \triangle_{\mathrm{pr}_1\alpha}) \cup (\varphi[\bar{\chi}|_i\, \gamma] \circ \triangle_{\mathrm{pr}_1\alpha}) \\
&= (\beta \circ \triangle_{\mathrm{pr}_1\alpha}) \cup \varphi[\bar{\chi}|_i(\gamma \circ \triangle_{\mathrm{pr}_1\alpha})] \\
&= (\beta \circ \triangle_{\mathrm{pr}_1\alpha}) \cup \varphi[\bar{\chi}|_i(\alpha \circ \triangle_{\mathrm{pr}_1\gamma})] \subset \beta \cup \varphi[\bar{\chi}|_i\, \alpha].
\end{aligned}$$

Thus (3.4.30) is proved.

Sufficiency. Let $(G, o, \curlyvee)$ satisfy all the conditions of the theorem. From the conditions (3.4.20) and (3.4.28), it follows that the order $\leqslant$ is stable. Moreover, for all $h, g_1, g_2, z \in G$, $u \in G \cup \{e_i\}$, $\bar{w} \in G^n$, $i = 1, \ldots, n$ the following condition is true

$$h \leqslant g_1 \ \wedge\ h \leqslant z \curlyvee u[\bar{w}|_i g_2] \longrightarrow h \leqslant z \curlyvee u[\bar{w}|_i g_1]. \tag{3.4.31}$$

Indeed, if the premise of (3.4.31) is satisfied, then, applying (3.4.30) to the inequality $h \leqslant z \curlyvee u[\bar{w}|_i g_2]$, we obtain $h \leqslant z \curlyvee u[\bar{w}|_i h]$. Next, using the stability of the relation $\leqslant$ and $h \leqslant g_1$ we get $u[\bar{w}|_i h] \leqslant u[\bar{w}|_i g_1]$. Therefore, $z \curlyvee u[\bar{w}|_i h] \leqslant z \curlyvee u[\bar{w}|_i g_1]$, and, as a consequence, $h \leqslant z \curlyvee u[\bar{w}|_i g_1]$, which proves (3.4.31).

Let $L_{(h,g)}$, where $h, g \in G$ and $h \nleqslant g$, be the family of all l-ideals of (G, o), which contain the element g but do not contain h and which satisfy the conditions (3.4.22) and (3.4.23). The family $L_{(h,g)}$ is nonempty because

$\{x \in G \,|\, x \leqslant g\} \in L_{(h,g)}$. According to Zorn's Lemma in $L_{(h,g)}$, there is a maximal (with respect to the inclusion) element $W_{(h,g)}$. We will show that for every $g_1 \notin W_{(h,g)}$ there exists $z \in W_{(h,g)}$ such that $h \leqslant z \curlyvee g_1$. In order to do this, we consider the set W of all $h_1 \in G$ such that $h_1 \leqslant z \curlyvee g_1$ for some $z \in W_{(h,g)}$. Obviously $g \in W$, because $g \in W_{(h,g)}$ and $g \leqslant g \curlyvee g_1$. Let $h_1 \leqslant h_2$. If $h_2 \in W$, then $h_2 \leqslant z \curlyvee g_1$ for some $z \in W_{(h,g)}$. Thus $h_1 \leqslant z \curlyvee g_1$, i.e., $h_1 \in W$. If $h_1, h_2 \in W$, then $h_1 \leqslant z_1 \curlyvee g_1$ and $h_2 \leqslant z_2 \curlyvee g_1$ for some $z_1, z_2 \in W_{(h,g)}$. Hence $h_1 \curlyvee h_2 \leqslant (z_1 \curlyvee z_2) \curlyvee g_1$. But $z_1 \curlyvee z_2 \in W_{(h,g)}$, so $h_1 \curlyvee h_2 \in W$. If $h_1 \in W$, then there exists $z \in W_{(h,g)}$ such that $h_1 \leqslant z \curlyvee g_1$. Thus $u[\bar{w}|_i h_1] \leqslant u[\bar{w}|_i z \curlyvee g_1] \leqslant z \curlyvee g_1$. Consequently $u[\bar{w}|_i h_1] \in W$. In this way, we have proved that W is an l-ideal satisfying (3.4.22), (3.4.23) and containing g. It is obvious that $W_{(h,g)} \subset W$. But, by assumption, $W_{(h,g)}$ is maximal in $L_{(h,g)}$, so $h \in W$. Therefore $h \leqslant z \curlyvee g_1$ for some $z \in W_{(h,g)}$.

Let us consider now the equivalence $\varepsilon_{(h,g)}$ defined on (G, o) and having only two equivalence classes $W_{(h,g)}$ and $W'_{(h,g)}$. We show that $\varepsilon_{(h,g)}$ is a v-congruence. Let $g_1 \equiv g_2(\varepsilon_{(h,g)})$. Assume that $g_1, g_2, u[\bar{w}|_i g_1] \in W'_{(h,g)}$. Then $h \leqslant z_1 \curlyvee g_1$, $h \leqslant z_2 \curlyvee g_2$ and $h \leqslant z_3 \curlyvee u[\bar{w}|_i g_1]$ for some $z_1, z_2, z_3 \in W_{(h,g)}$. From the last two inequalities and (3.4.31) we get

$$h \leqslant z_3 \curlyvee u[\bar{w}|_i(z_2 \curlyvee g_2)] = z_3 \curlyvee u[\bar{w}|_i z_2] \curlyvee u[\bar{w}|_i g_2].$$

From $z_3 \curlyvee u[\bar{w}|_i z_2] \in W_{(h,g)}$ it follows $u[\bar{w}|_i z_2] \notin W_{(h,g)}$, because in the other case $h \in W_{(h,g)}$ which is impossible by construction of $W_{(h,g)}$. Similarly we prove that $g_1, g_2, u[\bar{w}|_i g_2] \in W'_{(h,g)}$ implies $u[\bar{w}|_i g_1] \notin W'_{(h,g)}$. This means that $u[\bar{w}|_i g_1] \equiv u[\bar{w}|_i g_2](\varepsilon_{(h,g)})$. Since $W_{(h,g)}$ is an l-ideal, the same condition holds in the case when $g_1, g_2 \in W_{(h,g)}$. From the above, it follows that $\varepsilon_{(h,g)}$ is an i-regular for every $i = 1, \ldots, n$, i.e., it is a v-congruence. Therefore $(\varepsilon^*_{(h,g)}, W_{(h,g)})$, where $\varepsilon^*_{(h,g)} = \varepsilon_{(h,g)} \cup \{(e_1, e_1), \ldots, (e_n, e_n)\}$, $e_1, \ldots, e_n$ — selectors in (G^*, o^*), is the determining pair of (G, o), for which the conditions (3.4.22)–(3.4.24) are satisfied. With any such determining pair is connected the simplest representation $P_{(\varepsilon^*_{(h,g)}, W_{(h,g)})}$.

Let P be the sum of the family of representations $(P_{(\varepsilon^*_{(h,g)}, W_{(h,g)})})_{h,g \in G}$, where $h \nleqslant g$. According to the Proposition 3.4.7, P satisfies the identity $P(g_1 \curlyvee g_2) = P(g_1) \cup P(g_2)$. Moreover, $g_1 \leqslant g_2 \longleftrightarrow P(g_1) \subset P(g_2)$. Indeed, if $g_1 \leqslant g_2$, then $g_2 = g_1 \curlyvee g_2$, i.e., $P(g_2) = P(g_1 \curlyvee g_2) = P(g_1) \cup P(g_2)$. Hence $P(g_1) \subset P(g_2)$. Conversely, from $P(g_1) \subset P(g_2)$ we obtain $(g_1, g_2) \in \zeta_P$. So, $g_1 \leqslant g_2$ or $g_1 \nleqslant g_2$. If $g_1 \nleqslant g_2$, then, evidently, $(g_1, g_2) \in \zeta_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,g_2)})}$, which means that

$$(\forall \bar{x} \in B)\ \big(g_1[\bar{x}] \notin W_{(g_1,g_2)} \longrightarrow g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon_{(g_1,g_2)})\big)\,.$$

Putting $\bar{x} = (e_1, \ldots, e_n)$ in this condition and applying (3.4.24) we obtain $g_2 \notin W_{(g_1,g_2)}$, which is impossible. Thus, $g_1 \leqslant g_2$. This completes the proof of the equivalence $g_1 \leqslant g_2 \longleftrightarrow P(g_1) \subset P(g_2)$.

From the above $P(g_1) = P(g_2) \longleftrightarrow g_1 = g_2$. So, the representation P is faithful.

The proof is complete. □

A Menger algebra (Φ, O) of n-place functions closed with respect to the set-theoretic intersection and union of functions will be denoted by $(\Phi, O, \cap, \cup)$ and will be called a $(\cap, \cup)$-*Menger algebra of n-place functions*. Abstract algebras of type $(n+1, 2, 2)$ isomorphic to some $(\cap, \cup)$-Menger algebras of n-place functions will be called $(\curlywedge, \curlyvee)$-*Menger algebras of rank n*. Such algebras are characterized by the following theorem.

Theorem 3.4.9. *An abstract algebra $(G, o, \curlywedge, \curlyvee)$ of type $(n+1, 2, 2)$ is a $(\curlywedge, \curlyvee)$-Menger algebra of rank n if and only if the operation o is superassociative, $(G, \curlywedge, \curlyvee)$ is a distributive lattice, the identities* (3.4.1), (3.4.20), (3.4.28) *and*

$$x[(y_1 \curlywedge z_1) \cdots (y_n \curlywedge z_n)] = x[y_1 \cdots y_n] \curlywedge z_1 \curlywedge \cdots \curlywedge z_n \tag{3.4.32}$$

are satisfied.

Proof. The necessity of the above conditions is obvious. To prove the sufficiency assume that an algebra $(G, o, \curlywedge, \curlyvee)$ satisfies all of these conditions. Then it also satisfies (3.4.29) and (3.4.30). Indeed, according to (3.4.32), we have

$$\begin{aligned} x[y_1 \cdots y_n] &= x[(y_1 \curlywedge y_1) \cdots (y_i \curlywedge (y_i \curlywedge y_i)) \cdots (y_n \curlywedge y_n)] \\ &= x[y_1 \cdots y_n] \curlywedge y_1 \curlywedge \cdots \curlywedge (y_i \curlywedge y_i) \curlywedge \cdots \curlywedge y_n \\ &= (x[y_1 \cdots y_n] \curlywedge y_1 \curlywedge \cdots \curlywedge y_i \curlywedge \cdots \curlywedge y_n) \curlywedge y_i = x[y_1 \cdots y_n] \curlywedge y_i \end{aligned}$$

which implies (3.4.29). If $x \leqslant y \curlyvee u[\bar{w}|_i z]$ holds, then $x \curlywedge (y \curlyvee u[\bar{w}|_i z]) = x$, and consequently

$$x = (x \curlywedge y) \curlyvee (x \curlywedge u[\bar{w}|_i\, z]) = (x \curlywedge y) \curlyvee u[\bar{w}|_i (x \curlywedge z)] \leqslant y \curlyvee u[\bar{w}|_i x].$$

This proves (3.4.30). So, $(G, o, \curlyvee)$ satisfies all the conditions of Theorem 3.4.8. In the proof of this theorem we constructed a faithful representation P of (G, o) satisfying the identity $P(g_1 \curlyvee g_2) = P(g_1) \cup P(g_2)$. It satisfies also the identity $P(g_1 \curlywedge g_2) = P(g_1) \cap P(g_2)$. To prove this fact observe that, according to (3.4.22) and (3.4.23), $W_{(h,g)}$ is an ideal in the lattice $(G, \curlywedge, \curlyvee)$. From the construction of $W_{(h,g)}$, it follows that this ideal is maximal. But from the general theory of lattices (see for example [15]) we known that in a distributive lattice, every maximal ideal is simple, so, $x \curlywedge y \in W_{(h,g)}$ implies either $x \in W_{(h,g)}$ or $y \in W_{(h,g)}$. Moreover, from this theory follows also that in a distributive lattice the complement of a simple ideal is a simple filter, i.e., for $W'_{(h,g)}$ the following conditions must be satisfied:

$$x, y \in W'_{(h,g)} \longrightarrow x \curlywedge y \in W'_{(h,g)},$$
$$x \leqslant y \wedge x \in W'_{(h,g)} \longrightarrow y \in W'_{(h,g)},$$

and for $x \curlyvee y \in W'_{(h,g)}$ we must have either $x \in W'_{(h,g)}$ or $y \in W'_{(h,g)}$. This means that the determining pair $(\varepsilon^*_{(h,g)}, W_{(h,g)})$ satisfies the conditions (3.4.8) –(3.4.10), and the corresponding simplest representation satisfies (3.4.7). This proves the identity $P(g_1 \curlywedge g_2) = P(g_1) \cap P(g_2)$. So, we have proved that $(G, \curlywedge, \curlyvee)$ is isomorphic to a $(\cap, \cup)$-Menger algebra $(P(G), O, \cap, \cup)$ of n-place functions. □

We considered above $\cup$– and $(\cap, \cup)$–Menger algebras of n-place functions, i.e., algebras $(\Phi, O, \cup)$ and $(\Phi, O, \cap, \cup)$. Since for any functions $f, g \in \Phi$ the union $f \cup g$ belongs again to Φ, any two functions $f, h \in \Phi$ are *semi-compatible*, i.e., $(f, g) \in \xi_\Phi$, which means that they coincide on $\mathrm{pr}_1 f \cap \mathrm{pr}_1 g$. Thus, we can study Menger algebras (Φ, O) with the relation of universal semi-compatibility ξ_P, i.e., with the relation $\xi_P = \Phi \times \Phi$. Menger algebras isomorphic to such algebras are called *compatible*. Of course, $\curlyvee$- and $(\curlywedge, \curlyvee)$-Menger algebras are compatible.

Theorem 3.4.10. *An algebra* $(G, o, \curlywedge)$ *of type* $(n+1, 2)$ *is a compatible* $\curlywedge$*-Menger algebra if and only if the operation* o *is superassociative,* $(G, \curlywedge)$ *is a semilattice and the identities* (3.4.1), (3.4.32) *hold.*

Proof. The necessity is obvious, so only the sufficiency will be proved. If $(G, o, \curlywedge)$ satisfies all the conditions of the theorem, then, clearly,

$$\zeta = \{(x, y) \mid x \curlywedge y = x\}$$

is an order. The condition (3.4.1) shows that ζ is l-regular. The condition (3.4.32) guarantees the v-regularity. Indeed, for $x \leqslant y$, where $\leqslant$ denotes ζ, we have $x \curlywedge y = x$ and $u[\bar{w}|_i(x \curlywedge y)] = u[\bar{w}|_i x]$. Thus

$$\begin{aligned} u[\bar{w}|_i(x \curlywedge y)] &= u[\bar{w}|_i(x \curlywedge y)] \curlywedge u[\bar{w}|_i(x \curlywedge y)] \\ &= u[\bar{w}|_i x] \curlywedge y \curlywedge u[\bar{w}|_i y] \curlywedge x = u[\bar{w}|_i x] \curlywedge x \curlywedge u[\bar{w}|_i y] \curlywedge y \\ &= u[\bar{w}|_i(x \curlywedge x)] \curlywedge u[\bar{w}|_i(y \curlywedge y)] = u[\bar{w}|_i x] \curlywedge u[\bar{w}|_i y], \end{aligned}$$

which implies $u[\bar{w}|_i x] \curlywedge u[\bar{w}|_i y] = u[\bar{w}|_i x]$. Hence $u[\bar{w}|_i x] \leqslant u[\bar{w}|_i y]$. So, ζ is i-regular for every $i = 1, \ldots, n$, i.e., it is v-regular. Consequently, ζ is a stable order. It is not difficult to see that from (3.4.32) we can deduce the v-negativeness of ζ and the conditions (3.4.13)–(3.4.15). Thus, the system $(G, o, \curlywedge, \zeta)$ satisfies all the conditions of Theorem 3.4.5. Therefore, it is isomorphic to a p.q-o. $\cap$-Menger algebra $(\Phi, O, \cap, \chi_\Phi)$ of n-place functions, where $\Phi = P(G)$ and P is the faithful representation considered in the proof of Theorem 3.4.5. Moreover, the functions $P(g_1)$ and $P(g_2)$ coincide on the common part of their domains. To verify this fact we must prove that for every $g \in G$ we have $(g_1, g_2) \in \xi_{(\varepsilon^*_g, W_g)}$, where ε_g and W_g are as in the proof of Theorem 3.4.5.

Indeed, if $g_1[\bar{x}] \notin W_g$ and $g_2[\bar{x}] \notin W_g$ for some $\bar{x} \in G^n \cup \{(e_1, \dots, e_n)\}$, then $g \leqslant g_1[\bar{x}]$ and $g \leqslant g_2[\bar{x}]$. Thus $g \leqslant g_1[\bar{x}] \curlywedge g_2[\bar{x}]$, i.e., $g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon_g)$, which was to be proved. □

Theorem 3.4.11. *A Menger algebra of rank n is compatible if and only if for all $i, j = 1, \dots, n$ it satisfies the identities:*

$$x[\bar{y}|_i(x[\bar{y}])\,] = x[\bar{y}], \tag{3.4.33}$$

$$x[\bar{y}|_i(z[\bar{u}|_j g])\,] = z[\bar{u}|_j(x[\bar{y}|_i g])\,]. \tag{3.4.34}$$

Proof. *Necessity.* Let (G, o) be a compatible Menger algebra, P — an isomorphism of (G, o) onto such Menger algebra (Φ, O) of n-place functions for which $\xi_\Phi = \Phi \times \Phi$. Of course, any two functions from Φ are semi-compatible. Thus $\zeta_\Phi = \chi_\Phi$, and, consequently, $\zeta_P = \chi_P$. Therefore ζ_P is a stable, weakly steady and v-negative order. So, for any $x, y_i \in G$, $i = 1, \dots, n$, we have $x[\bar{y}|_i(x[\bar{y}])\,] \leqslant x[\bar{y}]$, where $x \leqslant y \longleftrightarrow (x, y) \in \zeta_P$.

To prove the converse inequality let $f, g_i \in \Phi$ and $P(x) = f$, $P(y_i) = g_i$, for $i = 1, \dots, n$. Obviously, $f[\bar{g}](\bar{a}) = f(g_1(\bar{a}), \dots, g_n(\bar{a}))$ and $f[\bar{g}](\bar{a}) = g_i(\bar{a})$ for all $\bar{a} \in \mathrm{pr}_1 f[\bar{g}]$. Therefore

$$\begin{aligned} f[\bar{g}](\bar{a}) &= f(g_1(\bar{a}), \dots, g_{i-1}(\bar{a}), f[\bar{g}](\bar{a}), g_{i+1}(\bar{a}), \dots, g_n(\bar{a})) \\ &= f[g_1 \cdots g_{i-1}\; f[\bar{g}]\; g_{i+1} \cdots g_n](\bar{a}) = f[\bar{g}|_i(f[\bar{g}])\,]. \end{aligned}$$

Hence $f[\bar{g}] \subset f[\bar{g}|_i(f[\bar{g}])\,]$, which implies $x[\bar{y}] \leqslant x[\bar{y}|_i(x[\bar{y}])\,]$. This completes the proof of (3.4.34).

Sufficiency. Let (G, o) be a Menger algebra of rank n satisfying all the conditions of the theorem. It is easy to see that from (3.4.33) and (3.4.34) we can deduce the identities $t(t(g)) = t(g)$ and $t_1(t_2(g)) = t_2(t_1(g))$, where $g \in G$, $t, t_1, t_2 \in T_n(G)$. Consider on (G, o) the relation

$$\delta = \{(g_1, g_2) \,|\, g_1 = t(g_2) \text{ for some } t \in T_n(G)\}.$$

Of course, δ is an l-regular v-negative quasi-order. It is also antisymmetric and i-regular for every $i = 1, \dots, n$. Indeed, if $x < y$ and $y < x$ (here $x < y$ means $(x, y) \in \delta$), then $x = t_1(y)$ and $y = t_2(x)$ for some $t_1, t_2 \in T_n(G)$. Hence $x = t_1(t_2(x)) = t_2(t_1(x))$ and $t_1(x) = t_1(t_1(y)) = t_1(y) = x$, which gives $x = t_2(x) = y$. This proves the antisymmetry of δ. Further, if $x < y$, i.e., $x = t(y)$ for some $t \in T_n(G)$, then $u[\bar{w}|_i x] = u[\bar{w}|_i t(y)] = t(u[\bar{w}|_i y])$. Thus $u[\bar{w}|_i x] < u[\bar{w}|_i y]$, which proves the i-regularity. Summarizing, δ is a stable v-negative order.

Now let $x < y$ and $x < u[\bar{w}|_i z]$, i.e., $x = t_1(y)$ and $x = t_2(u[\bar{w}|_i z])$ for some $t_1, t_2 \in T_n(G)$. Since $u[\bar{w}|_i x] = u[\bar{w}|_i t_2(u[\bar{w}|_i z])] = t_2(u[\bar{w}|_i u[\bar{w}|_i z]]) = t_2(u[\bar{w}|_i z]) = x$, we have $x = u[\bar{w}|_i x] = u[\bar{w}|_i t_1(y)] = t_1(u[\bar{w}|_i y])$. Therefore

$x < u[\bar{w}|_i y]$. In this way we have proved that for all $x, y, z, u \in G$, $\bar{w} \in G^n$, $i = 1, \ldots, n$, the following implication holds:

$$x < y \wedge x < u[\bar{w}|_i z] \longrightarrow x < u[\bar{w}|_i y]. \tag{3.4.35}$$

Finally, we consider the representation P of (G, o), where

$$P = \sum_{g \in G} P_{(\varepsilon_g^*, W_g)}, \tag{3.4.36}$$

$\varepsilon_g^* = \varepsilon_v(\delta\langle g\rangle) \cup \{(e_1, e_1), \ldots, (e_n, e_n)\}$, $e_1, \ldots, e_n$ — selectors in (G^*, o^*), $W_g = W_v(\delta\langle g\rangle)$. Because δ is v-negative, then, evidently, $W_g = G \setminus \delta\langle g\rangle$. Also χ_P is v-negative, so $\delta \subset \chi_P$. In fact, we have $\delta = \chi_P$. Indeed, if $(g_1, g_2) \in \chi_P$, then

$$(\forall g \in G)(\forall \bar{x} \in B)\ (g < g_1[\bar{x}] \longrightarrow g < g_2[\bar{x}]),$$

where $B = G^n \cup \{(e_1, \ldots, e_n)\}$. This, for $g = g_1$ and $\bar{x} = (e_1, \ldots, e_n)$, gives $g_1 < g_2$. Thus $\chi_P \subset \delta$, which proves $\delta = \chi_P$. Moreover, from $P(g_1) = P(g_2)$, we get $(g_1, g_2) \in \chi_P$ and $(g_2, g_1) \in \chi_P$, i.e., $g_1 < g_2$ and $g_2 < g_1$. Hence $g_1 = g_2$. In this way we have proved that the representation P is faithful.

To complete this proof we must show $\xi_P = G \times G$. For this it is sufficient to prove that

$$g < g_1[\bar{x}] \wedge g < g_2[\bar{x}] \longrightarrow g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon_v(\delta\langle g\rangle))$$

for all $g, g_1, g_2 \in G$ and $\bar{x} \in B$. Indeed, if the premise of this implication is satisfied and $g < t(g_1[\bar{x}])$ for some $t \in T_n(G)$, then from $g < g_2[\bar{x}]$ and $g < t(g_1[\bar{x}])$, according to (3.4.35), we get $g < t(g_2[\bar{x}])$. Analogously from $g < g_1[\bar{x}]$ and $g < t(g_2[\bar{x}])$, we conclude $g < t(g_1[\bar{x}])$. So, $g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon_v(\delta\langle g\rangle))$. This completes the proof. □

3.5 Subtraction Menger algebras

As is well known, the set-theoretic inclusion $\subset$ and the operations $\cap$, $\cup$ can be expressed by the set-theoretic difference (subtraction) in the following way:

$$A \subset B \longleftrightarrow A \setminus B = \varnothing, \quad A \cap B = A \setminus (A \setminus B),$$
$$A \cup B = C \setminus ((C \setminus A) \cap (C \setminus B)),$$

where A, B, C are arbitrary sets such that $A \subset C$ and $B \subset C$.

Therefore, it makes sense to examine sets of functions closed with respect to the subtraction of functions. Such sets of functions are called *difference semigroups*, their abstract analogs — *subtraction semigroups*. Properties of subtraction semigroups were found in [1]. The investigation of difference semigroups was initiated by B. M. Schein in [188].

Below we present a generalization of Schein's results to the case of Menger algebras of n-place functions, i.e., to the case of algebras $(\mathbf{\Phi}, \mathbf{O}, \setminus, \varnothing)$, where $\mathbf{\Phi} \subset \mathcal{F}(A^n, A)$, $\varnothing \in \mathbf{\Phi}$. Such algebras will be called *difference Menger algebras*.

Definition 3.5.1. An algebra $(G, -, 0)$ of type $(2, 0)$ is called a *subtraction algebra* if it satisfies the following identities:

$$x - (y - x) = x, \tag{3.5.1}$$

$$x - (x - y) = y - (y - x), \tag{3.5.2}$$

$$(x - y) - z = (x - z) - y, \tag{3.5.3}$$

$$0 - 0 = 0 \tag{3.5.4}$$

for all $x, y, z \in G$.

Proposition 3.5.2. *Any subtraction algebra satisfies the identity:*

$$0 = x - x. \tag{3.5.5}$$

Proof. Below we give a short proof of this identity:

$$\begin{aligned} 0 &= 0 - ((0 - (x - x)) - 0) = 0 - ((0 - 0) - (x - x)) = 0 - (0 - (x - x)) \\ &= (x - x) - ((x - x) - 0) = (x - x) - ((x - 0) - x) \\ &= (x - ((x - 0) - x)) - x = x - x, \end{aligned}$$

which is what was required to show. □

From (3.5.5), by using (3.5.1), we obtain the following two identities:

$$x - 0 = x, \quad 0 - x = 0. \tag{3.5.6}$$

Similarly, from (3.5.2), (3.5.3), (3.5.5) and (3.5.6) we can deduce:

$$((x - y) - (x - z)) - (z - y) = 0, \tag{3.5.7}$$

$$(x - (x - y)) - y = 0. \tag{3.5.8}$$

This means that a subtraction algebra $(G, -, 0)$ is a special case of BCK-algebras studied by many mathematicians.

Definition 3.5.3. An algebra $(G, o, -, 0)$ of type $(n + 1, 2, 0)$ is called a *subtraction Menger algebra* of rank n, if (G, o) is a Menger algebra of rank n, $(G, -, 0)$ is a subtraction algebra and the following conditions:

$$(x - y)[z_1 \cdots z_n] = x[z_1 \cdots z_n] - y[z_1 \cdots z_n], \tag{3.5.9}$$

$$u[\bar{w}|_i(x - (x - y))] = u[\bar{w}|_i\, x] - u[\bar{w}|_i(x - y)], \tag{3.5.10}$$

$$x - y = z - t_1(x) = z - t_2(y) = 0 \longrightarrow z - t_2(x) = 0 \tag{3.5.11}$$

are satisfied for all $x, y, z, u, z_1, \ldots, z_n \in G$, $\bar{w} \in G^n$, $i = 1, \ldots, n$ and $t_1, t_2 \in T_n(G)$.

By putting $n = 1$ in the above definition, we obtain a *weak subtraction semigroup* [2] studied by B. M. Schein (cf. [188]). Such semigroups are isomorphic to some subtraction semigroups of the form $(\mathbf{\Phi}, \circ, \setminus)$.

Theorem 3.5.4. *Each difference Menger algebra of n-place functions is a subtraction Menger algebra of rank n.*

Proof. Let $(\mathbf{\Phi}, \mathrm{O}, \setminus, \varnothing)$ be a difference Menger algebra of n-place functions defined on A. The algebra $(\mathbf{\Phi}, \mathrm{O})$ is a Menger algebra of rank n. The operation $\setminus$ satisfies (3.5.1), (3.5.2), and (3.5.3). Hence $(\mathbf{\Phi}, \setminus, \varnothing)$ is a subtraction algebra. Thus, $(\mathbf{\Phi}, \mathrm{O}, \setminus, \varnothing)$ will be a subtraction Menger algebra if (3.5.9), (3.5.10), and (3.5.11) will be satisfied.

To verify (3.5.9), observe that for each $(\bar{a}, c) \in (f \setminus g)[h_1 \cdots h_n]$, where $f, g, h_1, \ldots, h_n \in \mathbf{\Phi}$, $\bar{a} \in A^n$, $c \in A$ there exists $\bar{b} = (b_1, \ldots, b_n) \in A^n$ such that $(\bar{b}, c) \in f \setminus g$ and $(\bar{a}, b_i) \in h_i$ for each $i = 1, \ldots, n$. Consequently, $(\bar{b}, c) \in f$ and $(\bar{b}, c) \notin g$. Thus, $(\bar{a}, c) \in f[h_1 \cdots h_n]$. If $(\bar{a}, c) \in g[h_1 \cdots h_n]$, then there exists $\bar{d} = (d_1, \ldots, d_n) \in A^n$ such that $(\bar{d}, c) \in g$ and $(\bar{a}, d_i) \in h_i$ for every $i = 1, \ldots, n$. Since $h_1, \ldots, h_n$ are functions, we obtain $b_i = d_i$ for all $i = 1, \ldots, n$. Thus $\bar{b} = \bar{d}$. Therefore $(\bar{b}, c) \in g$, which is impossible. Hence $(\bar{a}, c) \notin g[h_1 \cdots h_n]$. This means that $(\bar{a}, c) \in f[h_1 \cdots h_n] \setminus g[h_1 \cdots h_n]$. So, the following implication:

$$(\bar{a}, c) \in (f \setminus g)[h_1 \cdots h_n] \longrightarrow (\bar{a}, c) \in f[h_1 \cdots h_n] \setminus g[h_1 \cdots h_n]$$

is valid for any $\bar{a} \in A^n$, $c \in A$, i.e.,

$$(f \setminus g)[h_1 \cdots h_n] \subset f[h_1 \cdots h_n] \setminus g[h_1 \cdots h_n].$$

Conversely, let $(\bar{a}, c) \in f[h_1 \cdots h_n] \setminus g[h_1 \cdots h_n]$. Then $(\bar{a}, c) \in f[h_1 \cdots h_n]$ and $(\bar{a}, c) \notin g[h_1 \cdots h_n]$. Thus, there exists $\bar{b} = (b_1, \ldots, b_n) \in A^n$ such that $(\bar{b}, c) \in f$, $(\bar{b}, c) \notin g$ and $(\bar{a}, b_i) \in h_i$ for each $i = 1, \ldots, n$. Hence, $(\bar{b}, c) \in f \setminus g$ and $(\bar{a}, c) \in (f \setminus g)[h_1 \cdots h_n]$. So,

$$(\bar{a}, c) \in f[h_1 \cdots h_n] \setminus g[h_1 \cdots h_n] \longrightarrow (\bar{a}, c) \in (f \setminus g)[h_1 \cdots h_n]$$

for any $\bar{a} \in A^n$, $c \in A$, i.e., $f[h_1 \cdots h_n] \setminus g[h_1 \cdots h_n] \subset (f \setminus g)[h_1 \cdots h_n]$. Thus,

$$(f \setminus g)[h_1 \cdots h_n] = f[h_1 \cdots h_n] \setminus g[h_1 \cdots h_n],$$

which proves (3.5.9).

Now, let $(\bar{a}, c) \in u[\bar{\omega}|_i(f \setminus (f \setminus g))] = u[\bar{\omega}|_i(f \cap g)]$, where $f, g, u \in \mathbf{\Phi}$, $\bar{\omega} \in \mathbf{\Phi}^n$, $\bar{a} \in A^n$, $c \in A$. Then there exists $\bar{b} = (b_1, \ldots, b_n) \in A^n$ such that $(\bar{a}, b_i) \in f \cap g$, $(\bar{a}, b_j) \in \omega_j$, $j \in \{1, \ldots, n\} \setminus \{i\}$ and $(\bar{b}, c) \in u$. Since $(\bar{a}, b_i) \in f \cap g$ implies

[2] A weak subtraction semigroup $(S, \cdot, -)$ is a semigroup $(S, \cdot)$ satisfying the identities (3.5.1), (3.5.2), (3.5.3), $x(y - z) = xy - xz$ and $(x - (x - y))z = xz - (x - y)z$.

$(\bar{a}, b_i) \notin f \setminus g$, we have $(\bar{a}, c) \in u[\bar{\omega}|_i f]$ and $(\bar{a}, c) \notin u[\bar{\omega}|_i(f \setminus g)]$. Therefore, $(\bar{a}, c) \in u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i(f \setminus g)]$. Thus, we have shown that for any $\bar{a} \in A^n$, $c \in A$ holds the implication

$$(\bar{a}, c) \in u[\bar{\omega}|_i(f \setminus (f \setminus g))] \longrightarrow (\bar{a}, c) \in u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i(f \setminus g)],$$

which is equivalent to the inclusion $u[\bar{\omega}|_i(f \setminus (f \setminus g))] \subset u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i(f \setminus g)]$.

Conversely, let $(\bar{a}, c) \in u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i(f \setminus g)]$. Then $(\bar{a}, c) \in u[\bar{\omega}|_i f]$ and $(\bar{a}, c) \notin u[\bar{\omega}|_i(f \setminus g)]$. The first of these two conditions means that there exists $\bar{b} = (b_1, \ldots, b_n) \in A^n$ such that $(\bar{a}, b_i) \in f$, $(\bar{a}, b_j) \in \omega_j$ for each $j \in \{1, \ldots, n\} \setminus \{i\}$ and $(\bar{b}, c) \in u$. It is easy to see that the second condition $(\bar{a}, c) \notin u[\bar{\omega}|_i(f \setminus g)]$ is equivalent to the implication

$$(\forall \bar{d})\Big((\bar{a}, d_i) \in f \wedge \bigwedge_{j=1, j\neq i}^{n} (\bar{a}, d_j) \in \omega_j \wedge (\bar{d}, c) \in u \longrightarrow (\bar{a}, d_i) \in g\Big), \qquad (3.5.12)$$

where $\bar{d} = (d_1, \ldots, d_n) \in A^n$. From this implication for $\bar{d} = \bar{b}$, we obtain

$$(\bar{a}, b_i) \in f \wedge \bigwedge_{j=1, j\neq i}^{n} (\bar{a}, b_j) \in \omega_j \wedge (\bar{b}, c) \in u \longrightarrow (\bar{a}, b_i) \in g,$$

which gives $(\bar{a}, b_i) \in g$. Therefore $(\bar{a}, b_i) \in f \cap g = f \setminus (f \setminus g)$. This means that $(\bar{a}, c) \in u[\bar{\omega}|_i(f \setminus (f \setminus g))]$. So, the implication

$$(\bar{a}, c) \in u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i(f \setminus g)] \longrightarrow (\bar{a}, c) \in u[\bar{\omega}|_i(f \setminus (f \setminus g))]$$

is true for all $\bar{a} \in A^n$, $c \in A$. Hence $u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i(f \setminus g)] \subset u[\bar{\omega}|_i(f \setminus (f \setminus g))]$. Thus

$$u[\bar{\omega}|_i(f \setminus (f \setminus g))] = u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i(f \setminus g)].$$

This proves (3.5.10).

To prove (3.5.11) suppose that for some $f, g, h \in \Phi$ and $t_1, t_2 \in T_n(\Phi)$ we have $f \setminus g = \varnothing$, $h \setminus t_1(f) = \varnothing$ and $h \setminus t_2(g) = \varnothing$. Then $f \subset g$, $h \subset t_1(f)$ and $h \subset t_2(g)$. Hence $f = g \circ \triangle_{\mathrm{pr}_1 f}$ and $\mathrm{pr}_1 h \subset \mathrm{pr}_1 f$, where $\mathrm{pr}_1 f$ denotes the domain of f and $\Delta_{\mathrm{pr}_1 f}$ is the identity binary relation on $\mathrm{pr}_1 f$.

From the inclusion $h \subset t_2(g)$ we obtain

$$h = h \circ \Delta_{\mathrm{pr}_1 f} \subset t_2(g) \circ \Delta_{\mathrm{pr}_1 f} = t_2(g \circ \Delta_{\mathrm{pr}_1 f}) = t_2(f),$$

which means that (3.5.11) is also satisfied. This completes the proof that $(\Phi, \mathrm{O}, \setminus, \varnothing)$ is a subtraction Menger algebra of rank n. □

To prove the converse statement, we need to consider a number of properties of a subtraction Menger algebra of rank n, introduce some definitions and prove some auxiliary propositions.

Let $(G, o, -, 0)$ be a subtraction Menger algebra of rank n.

Proposition 3.5.5. *In any subtraction Menger algebra of rank n we have*

$$0[x_1 \cdots x_n] = 0, \qquad x[x_1 \cdots x_{i-1} 0\, x_{i+1} \cdots x_n] = 0$$

for all $x, x_1, \ldots, x_n \in G$, $i = 1, \ldots, n$.

Proof. Indeed, using (3.5.5) and (3.5.9), we obtain

$$0[x_1 \cdots x_n] = (0 - 0)[x_1 \ldots x_n] = 0[x_1 \cdots x_n] - 0[x_1 \cdots x_n] = 0.$$

Similarly, applying (3.5.10) and (3.5.5), we get

$$u[\bar{w}|_i\, 0] = u[\bar{w}|_i (0 - (0 - 0))] = u[\bar{w}|_i\, 0] - u[\bar{w}|_i (0 - 0)] = u[\bar{w}|_i\, 0] - u[\bar{w}|_i\, 0] = 0,$$

which was to be showed. □

Let ω be a binary relation defined on $(G, o, -, 0)$ in the following way:

$$\omega = \{(x, y) \in G \times G \,|\, x - y = 0\}.$$

Using (3.5.5), (3.5.6), and (3.5.7) it is easy to see that this is an order. In connection with this fact we will sometimes write $x \leqslant y$ instead of $(x, y) \in \omega$. Using this notation it is not difficult to verify that

$$0 \leqslant x, \quad x - y \leqslant x, \tag{3.5.13}$$
$$x \leqslant y \longleftrightarrow x - (x - y) = x, \tag{3.5.14}$$
$$x \leqslant y \longrightarrow x - z \leqslant y - z, \tag{3.5.15}$$
$$x \leqslant y \longrightarrow z - y \leqslant z - x, \tag{3.5.16}$$
$$x \leqslant y \wedge u \leqslant v \longrightarrow x - v \leqslant y - u \tag{3.5.17}$$

holds for all $x, y, z, u, v \in G$.

Moreover, in a subtraction algebra the following two identities:

$$(x - y) - y = x - y, \tag{3.5.18}$$
$$(x - y) - z = (x - z) - (y - z) \tag{3.5.19}$$

are true. The first is a consequence of (3.5.1). Indeed,

$$x - y = (x - y) - (y - (x - y)) = (x - y) - y.$$

From (3.5.13), using (3.5.3) and (3.5.16), we obtain

$$(x - y) - z \leqslant (x - z) - (y - z).$$

On the other side, (3.5.7) implies $((x - z) - z) - (y - z) \leqslant (x - z) - y$, which together with (3.5.18) and (3.5.3) gives $((x - z) - (y - z)) - ((x - y) - z) = (((x - z) - z) - (y - z)) - ((x - y) - z) \leqslant ((x - z) - y) - ((x - y) - z) = 0$. So, $(x - z) - (y - z) \leqslant (x - y) - z$. This, by (3.5.14) and (3.5.2), proves (3.5.19).

Proposition 3.5.6. *On the algebra* $(G, o, -, 0)$ *the relation* ω *is stable and weakly steady.*

Proof. Let $x \leqslant y$ for some $x, y \in G$. Then $x - y = 0$ and

$$(x-y)[z_1 \cdots z_n] = 0[z_1 \cdots z_n] = (0-0)[z_1 \cdots z_n] = 0[z_1 \cdots z_n] - 0[z_1 \cdots z_n] = 0$$

for all $z_1, \ldots, z_n \in G$. This, by (3.5.9), implies

$$x[z_1 \cdots z_n] - y[z_1 \cdots z_n] = 0,$$

i.e., $x[z_1 \cdots z_n] \leqslant y[z_1 \cdots z_n]$. Thus, ω is l-regular.

Moreover, from $x \leqslant y$, using (3.5.6), we obtain $x-(x-y) = x$, which together with (3.5.2), gives $y - (y - x) = x$. Consequently, for any $u \in G$, $\bar{w} \in G^n$ we have $u[\bar{w}|_i (y-(y-x))] = u[\bar{w}|_i\, x]$. This and (3.5.9) give $u[\bar{w}|_i\, y] - u[\bar{w}|_i (y-x)] = u[\bar{w}|_i\, x]$. Hence, according to (3.5.13), we obtain $u[\bar{w}|_i\, x] \leqslant u[\bar{w}|_i\, y]$. Thus, ω is i-regular for every $i = 1, \ldots, n$. Since ω is a quasi-order, the last means that ω is v-regular. But ω also is l-regular, hence it is stable.

It is clear that ω is weakly steady if and only if it satisfies (3.5.11).[3] □

Proposition 3.5.7. *The axiom* (3.5.10) *is equivalent to each of the following conditions:*

$$x \leqslant y \longrightarrow u[\bar{w}|_i\, (y-x)] = u[\bar{w}|_i\, y] - u[\bar{w}|_i\, x], \tag{3.5.20}$$

$$x \leqslant y \longrightarrow t(y-x) = t(y) - t(x), \tag{3.5.21}$$

$$t(x-(x-y)) = t(x) - t(x-y) \tag{3.5.22}$$

for all $x, y, u \in G$, $\bar{w} \in G^n$, $i = 1, \ldots, n$, $t \in T_n(G)$.

Proof. (3.5.10) $\longrightarrow$ (3.5.20). Suppose that the condition (3.5.10) is satisfied and $x \leqslant y$ for some $x, y \in G$. Then, according to (3.5.14), we have

$$x - (x-y) = x.$$

Hence, by (3.5.2), we obtain $y - (y-x) = x$. Thus,

$$y - x = y - (y - (y-x)),$$

which, in view of (3.5.10), gives $u[\bar{w}|_i\, (y-x)] = u[\bar{w}|_i\, (y-(y-(y-x)))] = u[\bar{w}|_i y] - u[\bar{w}|_i (y-(y-x))] = u[\bar{w}|_i y] - u[\bar{w}|_i x]$. This means that (3.5.10) implies (3.5.20).

(3.5.20) $\longrightarrow$ (3.5.21). From (3.5.20), it follows that for $x \leqslant y$ and all polynomials $t \in T_n(G)$ of the form $t(x) = u[\bar{w}|_i x]$ the condition (3.5.21) is

[3] In the case of semigroups the fact that ω is weakly steady can be deduced directly from the axioms of a weak subtraction semigroup (cf. [188]).

satisfied. To prove that (3.5.21) is satisfied by an arbitrary polynomial from $T_n(G)$ suppose that it is satisfied by some $t' \in T_n(G)$. Since the relation ω is stable on the algebra $(G, o, -, 0)$, from $x \leqslant y$ it follows $t'(x) \leqslant t'(y)$, which in view of (3.5.20), implies

$$u[\bar{w}|_i\,(t'(y) - t'(x))] = u[\bar{w}|_i\,t'(y)] - u[\bar{w}|_i\,t'(x)].$$

But according to the assumption on t' for $x \leqslant y$, we have $t'(y)-t'(x) = t'(y-x)$, so the above equation can be written as

$$u[\bar{w}|_i\,t'(y-x)] = u[\bar{w}|_i\,t'(y)] - u[\bar{w}|_i\,t'(x)].$$

Thus, (3.5.21) is satisfied by polynomials of the form $t(x) = u[\bar{w}|_i t'(x)]$.

From the construction of $T_n(G)$, it follows that (3.5.21) is satisfied by all polynomials $t \in T_n(G)$. Therefore (3.5.20) implies (3.5.21).

(3.5.21) $\longrightarrow$ (3.5.22). Since, by means of (3.5.13), $x - y \leqslant x$ holds for all $x, y \in G$, from (3.5.21) follows $t(x-(x-y)) = t(x)-t(x-y)$ for any polynomial $t \in T_n(G)$. Thus, (3.5.21) implies (3.5.22).

(3.5.22) $\longrightarrow$ (3.5.10). By putting $t(x) = u[\bar{w}|_i\,x]$ we obtain (3.5.10). □

On a subtraction Menger algebra $(G, o, -, 0)$ of rank n, we can define a binary operation $\curlywedge$ by putting

$$x \curlywedge y = x - (x - y). \tag{3.5.23}$$

By using this operation, the conditions (3.5.9), (3.5.14), and (3.5.22) can be written in a more useful form:

$$u[\bar{w}|_i(x \curlywedge y)] = u[\bar{w}|_i\,x] - u[\bar{w}|_i(x - y)], \tag{3.5.24}$$

$$x \leqslant y \longleftrightarrow x \curlywedge y = x, \tag{3.5.25}$$

$$t(x \curlywedge y) = t(x) - t(x - y), \tag{3.5.26}$$

where $x, y, u \in G$, $\bar{w} \in G^n$, $i = 1, \ldots, n$, $t \in T_n(G)$. Moreover, from (3.5.9) and (3.5.23), we can deduce the identity:

$$(x \curlywedge y)[z_1 \cdots z_n] = x[z_1 \cdots z_n] \curlywedge y[z_1 \cdots z_n]. \tag{3.5.27}$$

The algebra $(G, \curlywedge)$ is a lower semilattice. Directly from the conditions (3.5.1) –(3.5.8) we obtain the following properties:

$$x \leqslant y \wedge x \leqslant z \longrightarrow x \leqslant y \curlywedge z, \tag{3.5.28}$$

$$x \leqslant y \longrightarrow x \curlywedge z \leqslant y \curlywedge z, \tag{3.5.29}$$

$$x \curlywedge y = 0 \longrightarrow x - y = x, \tag{3.5.30}$$

$$(x - y) \curlywedge y = 0, \tag{3.5.31}$$

$$x \curlywedge (y - z) = (x \curlywedge y) - (x \curlywedge z), \tag{3.5.32}$$

$$x - y = x - (x \curlywedge y), \tag{3.5.33}$$
$$(x \curlywedge y) - (y - z) = x \curlywedge y \curlywedge z, \tag{3.5.34}$$
$$(x \curlywedge y) - z = (x - z) \curlywedge (y - z), \tag{3.5.35}$$
$$(x \curlywedge y) - z = (x - z) \curlywedge y \tag{3.5.36}$$

for all $x, y, z \in G$.

Proposition 3.5.8. *In a subtraction Menger algebra $(G, o, -, 0)$ of rank n the following conditions*

$$t(x - y) = t(x) - t(x \curlywedge y), \tag{3.5.37}$$
$$t(x) - t(y) \leqslant t(x - y) \tag{3.5.38}$$

are true for each $t \in T_n(G)$ and $x, y \in G$.

Proof. From (3.5.33), we obtain $t(x - y) = t(x - (x \curlywedge y))$ for every $t \in T_n(G)$. (3.5.23) and (3.5.13) imply $x \curlywedge y \leqslant x$, which together with (3.5.21) gives $t(x - (x \curlywedge y)) = t(x) - t(x \curlywedge y)$. Hence, $t(x - y) = t(x) - t(x \curlywedge y)$. This proves (3.5.37).

Since $x \curlywedge y \leqslant y$, the stability of ω implies $t(x \curlywedge y) \leqslant t(y)$ for every $t \in T_n(G)$. From this, by applying (3.5.13) and (3.5.16), we obtain $t(x) - t(y) \leqslant t(x) - t(x \curlywedge y) = t(x - y)$, which proves (3.5.38). □

By $[0, a]$, we denote the *initial segment* of the algebra $(G, -, 0)$, i.e., the set of all $x \in G$ such that $0 \leqslant x \leqslant a$. On any $[0, a]$ we can define a binary operation $\curlyvee$ by letting:

$$x \curlyvee y = a - ((a - x) \curlywedge (a - y)) \tag{3.5.39}$$

for all $x, y \in [0, a]$. It is not difficult to see that this operation is idempotent and commutative, and 0 is its neutral element, i.e., $x \curlyvee x = x$, $x \curlyvee y = y \curlyvee x$, $x \curlyvee 0 = x$ for all $x, y \in [0, a]$.

Proposition 3.5.9. *For any $x, y \in [0, b] \subset [0, a]$, where $a, b \in G$, we have*

$$b - ((b - x) \curlywedge (b - y)) = a - ((a - x) \curlywedge (a - y)). \tag{3.5.40}$$

Proof. Note first that $b = b \curlywedge a$ because $b \leqslant a$. Moreover, from $x \leqslant b$ and $y \leqslant b$, according to (3.5.16), we obtain $a - b \leqslant a - x$ and $a - b \leqslant a - y$. This, together with (3.5.28), gives $a - b \leqslant (a - x) \curlywedge (a - y)$. Thus, $(a - b) - ((a - x) \curlywedge (a - y)) = 0$.

By (3.5.13), we have $b - ((a - x) \curlywedge (a - y)) \leqslant b$, which implies

$$b \curlywedge (b - ((a - x) \curlywedge (a - y))) = b - ((a - x) \curlywedge (a - y)). \tag{3.5.41}$$

Obviously $b = b \curlywedge b = b \curlywedge a$, $x = b \curlywedge x$, $y = b \curlywedge y$. Therefore:[4]

$$\begin{aligned} b - ((b-x) \curlywedge (b-y)) &= b \curlywedge b - ((b \curlywedge a - b \curlywedge x) \curlywedge (b \curlywedge a - b \curlywedge y)) \\ &\overset{(3.5.32)}{=} b \curlywedge b - (b \curlywedge (a-x) \curlywedge b \curlywedge (a-y)) \\ &= b \curlywedge b - b \curlywedge ((a-x) \curlywedge (a-y)) \\ &\overset{(3.5.32)}{=} b \curlywedge (b - ((a-x) \curlywedge (a-y))) \\ &\overset{(3.5.40)}{=} b - ((a-x) \curlywedge (a-y)) \\ &= a \curlywedge b - ((a-x) \curlywedge (a-y)) \\ &\overset{(3.5.23)}{=} (a-(a-b)) - ((a-x) \curlywedge (a-y)) \\ &\overset{(3.5.19)}{=} (a - ((a-x) \curlywedge (a-y))) \\ &\qquad - ((a-b) - ((a-x) \curlywedge (a-y))) \\ &= (a - ((a-x) \curlywedge (a-y))) - 0 \\ &\overset{(3.5.6)}{=} a - ((a-x) \curlywedge (a-y)), \end{aligned}$$

which completes the proof. □

Corollary 3.5.10. *The condition* (3.5.40) *is true for all* $x, y \in [0,a] \cap [0,b]$.

Proof. Since $[0,a] \cap [0,b] = [0, a \curlywedge b] \subset [0,a] \cup [0,b]$, by Proposition 3.5.9, for all $x, y \in [0,a] \cap [0,b]$ we have:

$$\begin{aligned} a - ((a-x) \curlywedge (a-y)) &= a \curlywedge b - ((a \curlywedge b - x) \curlywedge (a \curlywedge b - y)), \\ b - ((b-x) \curlywedge (b-y)) &= a \curlywedge b - ((a \curlywedge b - x) \curlywedge (a \curlywedge b - y)). \end{aligned}$$

This implies (3.5.40). □

From the above corollary, it follows that the value of $x \curlyvee y$, if it exists, does not depend on the choice of the interval $[0,a]$ containing the elements x and y. In [1] it is proved that for $x, y, z \in [0,a]$ we have

$$x \curlywedge (x \curlyvee y) = x, \tag{3.5.42}$$

$$x \curlyvee (x \curlywedge y) = x, \tag{3.5.43}$$

$$(x \curlyvee y) \curlyvee z = x \curlyvee (y \curlyvee z), \tag{3.5.44}$$

$$x \curlywedge (y \curlyvee z) = (x \curlywedge y) \curlyvee (x \curlywedge z), \tag{3.5.45}$$

$$x \curlyvee (y \curlywedge z) = (x \curlyvee y) \curlywedge (x \curlyvee z), \tag{3.5.46}$$

$$(x \curlyvee y) - z = (x - z) \curlyvee (y - z), \tag{3.5.47}$$

$$x \leqslant z \wedge y \leqslant z \longrightarrow x \curlyvee y \leqslant z, \tag{3.5.48}$$

[4] To reduce the number of brackets we will write $x \curlywedge y - z$ instead of $(x \curlywedge y) - z$.

$$y \leqslant x \longrightarrow x = (x - y) \curlyvee y, \tag{3.5.49}$$
$$x = (x \curlyvee y) - (y - x), \tag{3.5.50}$$
$$x = (x \curlywedge y) \curlyvee (x - y). \tag{3.5.51}$$

From (3.5.42) follows $x \leqslant x \curlyvee y$.

Proposition 3.5.11. *If for some $x, y \in G$ there exists $x \curlyvee y$, then for all $u \in G$, $\bar{z}, \bar{w} \in G^n$, $i = 1, \ldots, n$ there are also elements $x[\bar{z}] \curlyvee y[\bar{z}]$ and $u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y]$, and the following identities are satisfied:*

$$(x \curlyvee y)[\bar{z}] = x[\bar{z}] \curlyvee y[\bar{z}], \tag{3.5.52}$$
$$u[\bar{w}|_i(x \curlyvee y)] = u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y]. \tag{3.5.53}$$

Proof. Suppose that an element $x \curlyvee y$ exists. Then $x \leqslant a$ and $y \leqslant a$ for some $a \in G$, which, by the l-regularity of the relation ω, implies $x[\bar{z}] \leqslant a[\bar{z}]$ and $y[\bar{z}] \leqslant a[\bar{z}]$ for any $\bar{z} \in G^n$. This means that $x[\bar{z}] \curlyvee y[\bar{z}]$ exists and

$$\begin{aligned}
(x \curlyvee y)[\bar{z}] &\overset{(3.5.39)}{=} (a - ((a - x) \curlywedge (a - y)))[\bar{z}] \\
&\overset{(3.5.9)}{=} a[\bar{z}] - ((a - x) \curlywedge (a - y))[\bar{z}] \\
&\overset{(3.5.27)}{=} a[\bar{z}] - ((a - x)[\bar{z}] \curlywedge (a - y)[\bar{z}]) \\
&\overset{(3.5.9)}{=} a[\bar{z}] - ((a[\bar{z}] - x[\bar{z}]) \curlywedge (a[\bar{z}] - y[\bar{z}])) \\
&\overset{(3.5.39)}{=} x[\bar{z}] \curlyvee y[\bar{z}].
\end{aligned}$$

This proves (3.5.52).

Further, from $x \leqslant a$, $y \leqslant a$ and the i-regularity of ω, we obtain $u[\bar{w}|_i\, x] \leqslant u[\bar{w}|_i\, a]$ and $u[\bar{w}|_i\, y] \leqslant u[\bar{w}|_i\, a]$. Hence, there exists an element $u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y]$. Since $x \leqslant x \curlyvee y$ and $y \leqslant x \curlyvee y$, we also have $u[\bar{w}|_i\, x] \leqslant u[\bar{w}|_i(x \curlyvee y)]$ and $u[\bar{w}|_i\, y] \leqslant u[\bar{w}|_i(x \curlyvee y)]$, which, according to (3.5.48), gives

$$u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y] \leqslant u[\bar{w}|_i(x \curlyvee y)]. \tag{3.5.54}$$

On the other side, the existence of $u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y]$ implies,

$$u[\bar{w}|_i\, x] \leqslant u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y] \quad \text{and} \quad u[\bar{w}|_i\, y] \leqslant u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y].$$

Moreover, by (3.5.38) and (3.5.50), we have

$$u[\bar{w}|_i(x \curlyvee y)] - u[\bar{w}|_i(y - x)] \leqslant u[\bar{w}|_i((x \curlyvee y) - (y - x))] = u[\bar{w}|_i\, x].$$

Consequently,

$$u[\bar{w}|_i(x \curlyvee y)] - u[\bar{w}|_i(y - x)] \leqslant u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y]. \tag{3.5.55}$$

But $y - x \leqslant y$, so, $u[\bar{w}|_i(y-x)] \leqslant u[\bar{w}|_i\, y]$ and

$$u[\bar{w}|_i(y-x)] \leqslant u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y].$$

This and (3.5.55) guarantee the existence of an element

$$(u[\bar{w}|_i(x \curlyvee y)] - u[\bar{w}|_i(y-x)]) \curlyvee u[\bar{w}|_i(y-x)]$$

such that

$$(u[\bar{w}|_i(x \curlyvee y)] - u[\bar{w}|_i(y-x)]) \curlyvee u[\bar{w}|_i(y-x)] \leqslant u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y]. \quad (3.5.56)$$

Since $u[\bar{w}|_i(y-x)] \leqslant u[\bar{w}|_i\, y] \leqslant u[\bar{w}|_i(x \curlyvee y)]$, the last inequality and (3.5.49) imply

$$(u[\bar{w}|_i(x \curlyvee y)] - u[\bar{w}|_i(y-x)]) \curlyvee u[\bar{w}|_i(y-x)] = u[\bar{w}|_i(x \curlyvee y)],$$

which together with (3.5.56) gives

$$u[\bar{w}|_i(x \curlyvee y)] \leqslant u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y].$$

Comparing this inequality with (3.5.54) we obtain (3.5.53). □

Corollary 3.5.12. *If for some $x, y \in G$ an element $x \curlyvee y$ exists, then for any polynomial $t \in T_n(G)$ an element $t(x) \curlyvee t(y)$ also exists and $t(x \curlyvee y) = t(x) \curlyvee t(y)$.*

Proposition 3.5.13. *For all $x, y \in G$ and all polynomials $t_1, t_2 \in T_n(G)$ we have:*

$$t_1(x \curlywedge y) \curlywedge t_2(x-y) = 0.$$

Proof. Let $t_1(x \curlywedge y) \curlywedge t_2(x-y) = h$. Obviously $h \leqslant t_1(x \curlywedge y)$ and $h \leqslant t_2(x-y)$. Since $t_2(x-y) \leqslant t_2(x)$, we have $h \leqslant t_2(x)$. Thus, $x \curlywedge y \leqslant x$, $h \leqslant t_1(x \curlywedge y)$ and $h \leqslant t_2(x)$. This, in view of Proposition 3.5.6 and (3.5.11), gives $h \leqslant t_2(x \curlywedge y)$. Consequently,

$$h \leqslant t_2(x-y) \curlywedge t_2(x \curlywedge y). \quad (3.5.57)$$

Further, by (3.5.37) and (3.5.18), we obtain

$$\begin{aligned} t_2(x-y) - t_2(x \curlywedge y) &= (t_2(x) - t_2(x \curlywedge y)) - t_2(x \curlywedge y) \\ &= t_2(x) - t_2(x \curlywedge y) = t_2(x-y). \end{aligned}$$

Therefore, by (3.5.23), we have

$$t_2(x-y) \curlywedge t_2(x \curlywedge y) = t_2(x-y) - (t_2(x-y) - t_2(x \curlywedge y)) = t_2(x-y) - t_2(x-y) = 0,$$

which together with (3.5.57) implies $h \leqslant 0$. So, $h = 0$. This completes the proof. □

Proposition 3.5.14. *For all $x, y, z, g \in G$ and all polynomials $t_1, t_2 \in T_n(G)$ the following conditions are true:*

$$t_1(x \curlywedge y) \curlywedge t_2(y) = t_1(x \curlywedge y) \curlywedge t_2(x \curlywedge y), \tag{3.5.58}$$

$$t_1(x \curlywedge y \curlywedge z) \curlywedge t_2(y) \leqslant t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z), \tag{3.5.59}$$

$$g \leqslant t_1(x \curlywedge y) \wedge g \leqslant t_2(y \curlywedge z) \longrightarrow g \leqslant t_2(x \curlywedge y \curlywedge z). \tag{3.5.60}$$

Proof. To prove (3.5.58) observe first that for $z = t_1(x \curlywedge y) \curlywedge t_2(y)$ we have $z \leqslant t_1(x \curlywedge y)$ and $z \leqslant t_2(y)$. Since the relation ω is weakly steady and $x \curlywedge y \leqslant y$, from the above we conclude $z \leqslant t_2(x \curlywedge y)$, i.e., $t_1(x \curlywedge y) \curlywedge t_2(y) \leqslant t_2(x \curlywedge y)$. This, by (3.5.29), implies

$$t_1(x \curlywedge y) \curlywedge t_2(y) \leqslant t_1(x \curlywedge y) \curlywedge t_2(x \curlywedge y).$$

On the other side, the stability of ω and $x \curlywedge y \leqslant y$ imply $t_2(x \curlywedge y) \leqslant t_2(y)$ for every $t_2 \in T_n(G)$. Hence

$$t_1(x \curlywedge y) \curlywedge t_2(x \curlywedge y) \leqslant t_1(x \curlywedge y) \curlywedge t_2(y)$$

by (3.5.29). This completes the proof of (3.5.58).

Further:

$$\begin{aligned} t_1(x \curlywedge y \curlywedge z) \curlywedge t_2(y) &= t_1((x \curlywedge z) \curlywedge y) \curlywedge t_2(y) \\ &\overset{(3.5.58)}{=} t_1((x \curlywedge z) \curlywedge y) \curlywedge t_2((x \curlywedge z) \curlywedge y) \\ &\leqslant t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z) \end{aligned}$$

proves (3.5.59).

Finally, let $g \leqslant t_1(x \curlywedge y)$ and $g \leqslant t_2(y \curlywedge z)$. Then

$$\begin{aligned} & g \leqslant t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z) \overset{(3.5.26)}{=} t_1(x \curlywedge y) \curlywedge (t_2(y) - t_2(y - z)) \\ & \overset{(3.5.32)}{=} \Big(t_1(x \curlywedge y) \curlywedge t_2(y)\Big) - \Big(t_1(x \curlywedge y) \curlywedge t_2(y - z)\Big) \\ & \overset{(3.5.58)}{=} \Big(t_1(x \curlywedge y) \curlywedge t_2(x \curlywedge y)\Big) - \Big(t_1(x \curlywedge y) \curlywedge t_2(y - z)\Big) \\ & \overset{(3.5.32)}{=} t_1(x \curlywedge y) \curlywedge \Big(t_2(x \curlywedge y) - t_2(y - z)\Big) \leqslant t_2(x \curlywedge y) - t_2(y - z) \\ & \overset{(3.5.38)}{\leqslant} t_2\Big((x \curlywedge y) - (y - z)\Big) \overset{(3.5.34)}{=} t_2(x \curlywedge y \curlywedge z). \end{aligned}$$

This proves (3.5.60) and completes the proof of our proposition. □

Corollary 3.5.15. *For all $x, y, z \in G$ and all polynomials $t_1, t_2 \in T_n(G)$ we have:*

$$t_1(x \curlywedge y \curlywedge z) \curlywedge t_2(y) = t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z). \tag{3.5.61}$$

Proof. We have $t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z) \leqslant t_1(x \curlywedge y)$ and $t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z) \leqslant t_2(y \curlywedge z)$, so by (3.5.60) we obtain $t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z) \leqslant t_1(x \curlywedge y \curlywedge z)$. Considering now that $t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z) \leqslant t_2(y \curlywedge z) \leqslant t_2(y)$, by (3.5.28), we get $t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z) \leqslant t_1(x \curlywedge y \curlywedge z) \curlywedge t_2(y)$. Taking now into account the condition (3.5.59) we obtain (3.5.61). □

Definition 3.5.16. By a *determining pair* of a subtraction Menger algebra $(G, o, -, 0)$ of rank n we mean an ordered pair (ε^*, W), where ε is a v-regular equivalence relation defined on (G, o), $\varepsilon^* = \varepsilon \cup \{(e_1, e_1), \ldots, (e_n, e_n)\}$, $e_1, \ldots, e_n$ are selectors of a unitary extension (G^*, o^*) of (G, o) and W is the empty set or an l-ideal of (G, o) which is an ε-class.

Definition 3.5.17. A nonempty subset F of a subtraction Menger algebra $(G, o, -, 0)$ of rank n is called a *filter* if:
(1) $0 \notin F$;
(2) $x \in F \wedge x \leqslant y \longrightarrow y \in F$;
(3) $x \in F \wedge y \in F \longrightarrow x \curlywedge y \in F$
for all $x, y \in G$.

If $a, b \in G$ and $a \nleqslant b$, then $[\,a) = \{x \in G \,|\, a \leqslant x\}$ is a filter with $a \in [\,a)$ and $b \notin [\,a)$. By Zorn's Lemma, the collection of filters which contain an element a, but do not contain an element b, has a maximal element which is denoted by $F_{a,b}$. Using this filter we define the following three sets:

$$W_{a,b} = \{x \in G \,|\, (\forall t \in T_n(G))\, t(x) \notin F_{a,b}\},$$
$$\varepsilon_{a,b} = \{(x, y) \in G \times G \,|\, x \curlywedge y \notin W_{a,b} \vee x, y \in W_{a,b}\},$$
$$\varepsilon^*_{a,b} = \varepsilon_{a,b} \cup \{(e_1, e_1), \ldots, (e_n, e_n)\}.$$

Proposition 3.5.18. *For any $a, b \in G$, the pair $(\varepsilon^*_{a,b}, W_{a,b})$ is the determining pair of the algebra $(G, o, -, 0)$.*

Proof. First we show that $\varepsilon_{a,b}$ is an equivalence relation on G. It is clear that this relation is reflexive and symmetric. To prove its transitivity, let $(x, y), (y, z) \in \varepsilon_{a,b}$. We have four possibilities:

(a) $x \curlywedge y \notin W_{a,b} \wedge y \curlywedge z \notin W_{a,b}$,

(b) $x \curlywedge y \notin W_{a,b} \wedge y, z \in W_{a,b}$,

(c) $x, y \in W_{a,b} \wedge y \curlywedge z \notin W_{a,b}$,

(d) $x, y \in W_{a,b} \wedge y, z \in W_{a,b}$.

In the case (a), we have $t_1(x \curlywedge y), t_2(y \curlywedge z) \in F_{a,b}$ for some $t_1, t_2 \in T_n(G)$. Since $F_{a,b}$ is a filter, then, obviously, $t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z) \in F_{a,b}$. This, according to (3.5.61), implies $t_1(x \curlywedge y \curlywedge z) \curlywedge t_2(y) \in F_{a,b}$. But $t_1(x \curlywedge y \curlywedge z) \curlywedge t_2(y) \leqslant t_1(x \curlywedge z)$, hence also $t_1(x \curlywedge z) \in F_{a,b}$, i.e., $x \curlywedge z \notin W_{a,b}$. Thus, $(x, z) \in \varepsilon_{a,b}$.

In the case (b), from $x \curlywedge y \notin W_{a,b}$ follows $t(x \curlywedge y) \in F_{a,b}$ for some polynomial $t \in T_n(G)$. But $x \curlywedge y \leqslant y$, and consequently $t(x \curlywedge y) \leqslant t(y)$. Thus $t(y) \in F_{a,b}$, i.e., $y \notin W_{a,b}$, which is a contradiction. Hence the case (b) is impossible. Analogously we can show that the case (c) is also impossible. The case (d) is obvious, because in this case $x, z \in W_{a,b}$, which means that $(x, z) \in \varepsilon_{a,b}$. This completes the proof that $\varepsilon_{a,b}$ is transitive.

Moreover, if $x \in W_{a,b}$, then $t(x) \notin F_{a,b}$ for every $t \in T_n(G)$. In particular, for all $t(x) = t'(u[\bar{w}|_i\, x]) \in T_n(G)$ we have $t'(u[\bar{w}|_i\, x]) \notin F_{a,b}$. Thus, $u[\bar{w}|_i\, x] \in W_{a,b}$ for every $i = 1, \dots, n$. Hence, $W_{a,b}$ is an i-ideal of (G, o), and consequently, an l-ideal. It is clear that $W_{a,b}$ is an $\varepsilon_{a,b}$-class.

Next, we prove that the relation $\varepsilon_{a,b}$ is v-regular. Let $x \equiv y(\varepsilon_{a,b})$. Then $x \curlywedge y \notin W_{a,b}$ or $x, y \in W_{a,b}$. In the case $x, y \in W_{a,b}$ we obtain $u[\bar{w}|_i\, x], u[\bar{w}|_i\, y] \in W_{a,b}$ because $W_{a,b}$ is an l-ideal of (G, o). Thus, $u[\bar{w}|_i\, x] \equiv u[\bar{w}|_i\, y](\varepsilon_{a,b})$. In the case where $x \curlywedge y \notin W_{a,b}$ the elements $u[\bar{w}|_i\, x]$, $u[\bar{w}|_i\, y]$ simultaneously belong or not belong to $W_{a,b}$. Indeed, if $u[\bar{w}|_i\, x]$, $u[\bar{w}|_i\, y] \in W_{a,b}$, then obviously $u[\bar{w}|_i\, x] \equiv u[\bar{w}|_i\, y](\varepsilon_{a,b})$. Now, if $u[\bar{w}|_i\, x] \notin W_{a,b}$, then $t(u[\bar{w}|_i\, x]) \in F_{a,b}$ for some $t \in T_n(G)$. Since $x \curlywedge y \notin W_{a,b}$, then also $t_1(x \curlywedge y) \in F_{a,b}$ for some $t_1 \in T_n(G)$. Thus $t_1(x \curlywedge y) \curlywedge t(u[\bar{w}|_i\, x]) \in F_{a,b}$, which, by (3.5.58), implies $t_1(x \curlywedge y) \curlywedge t(u[\bar{w}|_i(x \curlywedge y)]) \in F_{a,b}$. But $t_1(x \curlywedge y) \curlywedge t(u[\bar{w}|_i(x \curlywedge y)]) \leqslant t(u[\bar{w}|_i\, y])$, hence $t(u[\bar{w}|_i\, y]) \in F_{a,b}$, i.e., $u[\bar{w}|_i\, y] \notin W_{a,b}$. So, we have shown that $x \curlywedge y \notin W_{a,b}$ and $u[\bar{w}|_i\, x] \notin W_{a,b}$ imply $u[\bar{w}|_i\, y] \notin W_{a,b}$. Similarly we can show that $x \curlywedge y \notin W_{a,b}$ and $u[\bar{w}|_i\, y] \notin W_{a,b}$ imply $u[\bar{w}|_i\, x] \notin W_{a,b}$. Therefore, we have proved that in the case $x \curlywedge y \notin W_{a,b}$ the elements $u[\bar{w}|_i\, x]$, $u[\bar{w}|_i\, y]$ simultaneously belong or not belong to $W_{a,b}$.

So, if for $x \curlywedge y \notin W_{a,b}$ we have $u[\bar{w}|_i\, x], u[\bar{w}|_i\, y] \in W_{a,b}$, then clearly $u[\bar{w}|_i\, x] \equiv u[\bar{w}|_i\, y](\varepsilon_{a,b})$. Therefore assume that $u[\bar{w}|_i\, x] \notin W_{a,b}$ (hence $u[\bar{w}|_i\, y] \notin W_{a,b}$). Thus, $x \curlywedge y \notin W_{a,b}$, $u[\bar{w}|_i\, x] \notin W_{a,b}$, i.e., $t(x \curlywedge y) \in F_{a,b}$, $t_1(u[\bar{w}|_i\, x]) \in F_{a,b}$ for some $t, t_1 \in T_n(G)$. Hence, $t(y \curlywedge x \curlywedge y) \curlywedge t_1(u[\bar{w}|_i\, x]) \in F_{a,b}$. From this, according to (3.5.61), we obtain $t(y \curlywedge x) \curlywedge t_1(u[\bar{w}|_i(x \curlywedge y)]) \in F_{a,b}$. This implies $t_1(u[\bar{w}|_i(x \curlywedge y)]) \in F_{a,b}$. Since $u[\bar{w}|_i(x \curlywedge y)] \leqslant u[\bar{w}|_i\, x]$ and $u[\bar{w}|_i(x \curlywedge y)] \leqslant u[\bar{w}|_i\, y]$, we have $u[\bar{w}|_i(x \curlywedge y)] \leqslant u[\bar{w}|_i\, x] \curlywedge u[\bar{w}|_i\, y]$, which, by the stability of ω gives $t_1(u[\bar{w}|_i(x \curlywedge y)]) \leqslant t_1(u[\bar{w}|_i\, x] \curlywedge u[\bar{w}|_i\, y])$. Consequently, $t_1(u[\bar{w}|_i\, x] \curlywedge u[\bar{w}|_i\, y]) \in F_{a,b}$, so $u[\bar{w}|_i\, x] \curlywedge u[\bar{w}|_i\, y] \notin W_{a,b}$, i.e., $u[\bar{w}|_i\, x] \equiv u[\bar{w}|_i\, y](\varepsilon_{a,b})$. In this way we have proved that the relation $\varepsilon_{a,b}$ is i-regular for every $i = 1, \dots, n$. Thus it is v-regular. □

Proposition 3.5.19. *All equivalence classes of $\varepsilon_{a,b}$, except of $W_{a,b}$, are filters.*

Proof. Indeed, let $H \neq W_{a,b}$ be an arbitrary class of $\varepsilon_{a,b}$. If $x \in H$ and $x \leqslant y$, then $x \curlywedge y = x \notin W_{a,b}$, consequently, $(x, y) \in \varepsilon_{a,b}$. Hence, $y \in H$. Further, let $x, y \in H$, then $(x, y) \in \varepsilon_{a,b}$. Thus $x \curlywedge y \notin W_{a,b}$, i.e., $t(x \curlywedge y) \in F_{a,b}$ for some $t \in T_n(G)$. But $x \curlywedge y = x \curlywedge (x \curlywedge y)$, hence, $t(x \curlywedge (x \curlywedge y)) \in F_{a,b}$ and $x \curlywedge (x \curlywedge y) \notin W_{a,b}$. So $x \equiv x \curlywedge y(\varepsilon_{a,b})$. This implies $x \curlywedge y \in H$. Thus, we have shown that H is a filter. □

Proposition 3.5.20. *If $x \curlyvee y$ exists for some $x, y \in W_{a,b}$, then $x \curlyvee y \in W_{a,b}$.*

Proof. Let $x \curlyvee y$ exist for some $x, y \in W_{a,b}$. If $x \curlyvee y \notin W_{a,b}$, then $t(x \curlyvee y) \in F_{a,b}$ for some $t \in T_n(G)$, and, according to Corollary 3.5.12, $t(x \curlyvee y) = t(x) \curlyvee t(y)$. If $t(x) \notin F_{a,b}$, then $F_{a,b}$ is a proper subset of the set

$$U = \{u \in G \mid (\exists z \in F_{a,b})\, z \curlywedge t(x) \leqslant u\}$$

because $t(x) \in U$.

We show that U is a filter. $0 \notin U$ because, by (3.5.13), we have $0 \leqslant z \curlywedge t(x)$ for any $z \in F_{a,b}$. Let $s \in U$ and $s \leqslant r$. Then $z \curlywedge t(x) \leqslant s$ for some $z \in F_{a,b}$. Consequently, $z \curlywedge t(x) \leqslant r$, so $r \in U$. Now let $s \in U$ and $r \in U$, i.e., $z_1 \curlywedge t(x) \leqslant s$ and $z_2 \curlywedge t(x) \leqslant r$ for some $z_1, z_2 \in F_{a,b}$. Since $F_{a,b}$ is a filter, we have $z_1 \curlywedge z_2 \in F_{a,b}$. Hence, $(z_1 \curlywedge z_2) \curlywedge t(x) \leqslant s \curlywedge r$, which implies $s \curlywedge r \in U$. Thus U is a filter. But by assumption $F_{a,b} \subset U$ is a maximal filter, which does not contain b, so $b \in U$. Consequently, $z_1 \curlywedge t(x) \leqslant b$ for some $z_1 \in F_{a,b}$. Similarly, if $t(y) \notin F_{a,b}$, then $z_2 \curlywedge t(y) \leqslant b$ for some $z_2 \in F_{a,b}$. This implies $z \curlywedge t(x) \leqslant b$ and $z \curlywedge t(y) \leqslant b$ for $z = z_1 \curlywedge z_2$. Hence $(z \curlywedge t(x)) \curlyvee (z \curlywedge t(y))$ exists and

$$(z \curlywedge t(x)) \curlyvee (z \curlywedge t(y)) = z \curlywedge (t(x) \curlyvee t(y)) = z \curlywedge t(x \curlyvee y) \in F_{a,b}$$

by (3.5.45). But $(z \curlywedge t(x)) \curlyvee (z \curlywedge t(y)) \leqslant b$ by (3.5.48), so $z \curlywedge t(x \curlyvee y) \leqslant b$. Since $z \curlywedge t(x \curlyvee y) \in F_{a,b}$, then, obviously, $b \in F_{a,b}$, which is impossible. So, $t(x) \in F_{a,b}$ or $t(y) \in F_{a,b}$, hence $x \notin W_{a,b}$ or $y \notin W_{a,b}$, which is contrary to the assumption that $x, y \in W_{a,b}$. Thus, the assumption that $x \curlyvee y \notin W_{a,b}$ is incorrect. Therefore $x \curlyvee y \in W_{a,b}$. □

It is clear that a representation $P : G \to \mathcal{F}(A^n, A)$ of a Menger algebra (G, o) is a representation of a subtraction Menger algebra $(G, o, -, 0)$ if

$$P(x - y) = P(x) \setminus P(y) \quad \text{and} \quad P(0) = \varnothing$$

for all $x, y \in G$.

With each determining pair (ε^*, W) is associated a simplest representation $P_{(\varepsilon^*,W)}$ of (G, o), which assigns to each element $g \in G$ an n-place function $P_{(\varepsilon^*,W)}(g)$ defined on $\mathcal{H} = \mathcal{H}_0 \cup \{\{e_1\}, \ldots, \{e_n\}\}$, where $\mathcal{H}_0$ is the set of all ε-classes of G different from W such that

$$(H_1, \ldots, H_n, H) \in P_{(\varepsilon,W)}(g) \longleftrightarrow g[H_1 \cdots H_n] \subset H,$$

for $(H_1, \ldots, H_n) \in \mathcal{H}_0^n \cup \{(\{e_1\}, \ldots, \{e_n\})\}$ and $H \in \mathcal{H}$.

Theorem 3.5.21. *Each subtraction Menger algebra of rank n is isomorphic to some difference Menger algebra of n-place functions.*

Proof. Let $(G, o, -, 0)$ be a subtraction Menger algebra of rank n. Then the sum

$$P = \sum_{a,b \in G,\, a \not\leqslant b} P_{(\varepsilon^*_{a,b}, W_{a,b})}$$

of the family $\big(P_{(\varepsilon^*_{a,b}, W_{a,b})}\big)_{a,b \in G,\, a \not\leqslant b}$ of simplest representations of (G, o) is a representation of (G, o).

Now we show that P is a representation of $(G, o, -, 0)$. Let $\mathcal{H}_0$ be the set of all $\varepsilon_{a,b}$-classes of G different from $W_{a,b}$. Consider the collection of $H_1, \ldots, H_n$, $H \in \mathcal{H}$, where $\mathcal{H} = \mathcal{H}_0 \cup \{\{e_1\}, \ldots, \{e_n\}\}$, such that $(H_1, \ldots, H_n, H) \in P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1 - g_2)$ for some $g_1, g_2 \in G$. Then, obviously, $(g_1 - g_2)[H_1 \cdots H_n] \subset H \neq W_{a,b}$. Thus $(g_1 - g_2)[\bar{x}] \in H$ for each $\bar{x} \in H_1 \times \cdots \times H_n$, which, by (3.5.9), gives $g_1[\bar{x}] - g_2[\bar{x}] \in H$. But $g_1[\bar{x}] - g_2[\bar{x}] \leqslant g_1[\bar{x}]$ and H is a filter (Proposition 3.5.19), hence $g_1[\bar{x}] \in H$. Thus $(g_1[\bar{x}] - g_2[\bar{x}]) \curlywedge g_2[\bar{x}] = 0$, by (3.5.31). Consequently, $(g_1[\bar{x}] - g_2[\bar{x}]) \curlywedge g_2[\bar{x}] \in W_{a,b}$, because the other $\varepsilon_{a,b}$-classes, since they are filters, do not contain 0. This means that $g_1[\bar{x}] - g_2[\bar{x}] \not\equiv g_2[\bar{x}](\varepsilon_{a,b})$. Hence, $g_2[\bar{x}] \notin H$. Therefore $g_1[H_1 \cdots H_n] \subset H$ and $g_2[h_1 \cdots H_n] \cap H = \varnothing$, which implies

$$(H_1, \ldots, H_n, H) \in P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1) \setminus P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_2).$$

In this way, we have proved the inclusion

$$P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1 - g_2) \subset P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1) \setminus P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_2). \tag{3.5.62}$$

To show the reverse inclusion let

$$(H_1, \ldots, H_n, H) \in P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1) \setminus P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_2).$$

Then $(H_1, \ldots, H_n, H)$ is in $P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1)$ but not in $P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_2)$, i.e., $g_1[H_1 \cdots H_n] \subset H$ and $g_2[H_1 \cdots H_n] \cap H = \varnothing$. So, $g_1[\bar{x}] \in H$ and $g_2[\bar{x}] \notin H$ for $\bar{x} \in H_1 \times \cdots \times H_n$. Since $g_1[\bar{x}] \curlywedge g_2[\bar{x}] \notin W_{a,b}$ implies $g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon_{a,b})$ and $g_2[\bar{x}] \in H$, we get a contradiction and conclude that $g_1[\bar{x}] \curlywedge g_2[\bar{x}] \in W_{a,b}$.

If $g_1[\bar{x}] - g_2[\bar{x}] \in W_{a,b}$, then, by (3.5.51) and Proposition 3.5.20, we obtain $g_1[\bar{x}] = (g_1[\bar{x}] \curlywedge g_2[\bar{x}]) \curlyvee (g_1[\bar{x}] - g_2[\bar{x}]) \in W_{a,b}$. Consequently, $g_1[\bar{x}] \in W_{a,b}$, which is impossible because $g_1[\bar{x}] \in H$. Thus, $(g_1[\bar{x}] - g_2[\bar{x}]) \curlywedge g_1[\bar{x}] = g_1[\bar{x}] - g_2[\bar{x}] \notin W_{a,b}$. Hence, $g_1[\bar{x}] - g_2[\bar{x}] \equiv g_1[\bar{x}](\varepsilon_{a,b})$. This implies that $(g_1 - g_2)[\bar{x}] = g_1[\bar{x}] - g_2[\bar{x}] \in H$. Therefore, $(g_1 - g_2)[H_1 \cdots H_n] \subset H$, i.e., $(H_1, \ldots, H_n, H) \in P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1 - g_2)$. So, we have proved

$$P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1) \setminus P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_2) \subset P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1 - g_2).$$

This together with (3.5.62) proves

$$P_{(\varepsilon^*_{a,b},W_{a,b})}(g_1 - g_2) = P_{(\varepsilon^*_{a,b},W_{a,b})}(g_1) \setminus P_{(\varepsilon^*_{a,b},W_{a,b})}(g_2),$$

which means that $P(g_1 - g_2) = P(g_1) \setminus P(g_2)$ for $g_1, g_2 \in G$. Further on, $P(0) = P(0-0) = P(0) \setminus P(0) = \varnothing$, so P is a representation of $(G, o, -, 0)$ by n-place functions.

We show that this representation is faithful. Let $P(g_1) = P(g_2)$ for some $g_1, g_2 \in G$. If $g_1 \neq g_2$, then both inequalities $g_1 \leqslant g_2$ and $g_2 \leqslant g_1$ can not occur the same time. Suppose that $g_1 \nleqslant g_2$. Then $g_1 \in F_{g_1,g_2}$ and, consequently,

$$(\{e_1\}, \ldots, \{e_n\}, F_{g_1,g_2}) \in P_{(\varepsilon^*_{g_1,g_2},W_{g_1,g_2})}(g_2).$$

Since $P_{(\varepsilon^*_{g_1,g_2},W_{g_1,g_2})}(g_1) = P_{(\varepsilon^*_{g_1,g_2},W_{g_1,g_2})}(g_2)$, then, obviously,

$$(\{e_1\}, \ldots, \{e_n\}, F_{g_1,g_2}) \in P_{(\varepsilon^*_{g_1,g_2},W_{g_1,g_2})}(g_2).$$

Thus $\{g_2\} = g_2[\{e_1\} \cdots \{e_n\}] \subset F_{g_1,g_2}$, hence $g_2 \in F_{g_1,g_2}$. This is a contradiction because F_{g_1,g_2} is a filter containing g_1 but not containing g_2. The case $g_2 \nleqslant g_1$ is analogous. So, the supposition $g_1 \neq g_2$ is not true. Hence $g_1 = g_2$ and P is a faithful representation. The theorem is proved. □

3.6 Restrictive Menger algebras

The algebra $(\Phi, O, \rhd)$, where Φ is the set of some n-place functions closed with respect to the Menger composition O and the restrictive product $\rhd$ of functions, will be called a *restrictive Menger algebra of n-place functions.*

Theorem 3.6.1. *An algebra $(G, o, \blacktriangleright)$ of type $(n+1, 2)$ is isomorphic to a restrictive Menger algebra of n-place functions if and only if (G, o) is a Menger algebra of rank n, $(G, \blacktriangleright)$ is an idempotent semigroup and the following three identities hold:*

$$x[(y_1 \blacktriangleright z_1) \cdots (y_n \blacktriangleright z_n)] = y_1 \blacktriangleright \cdots \blacktriangleright y_n \blacktriangleright x[z_1 \cdots z_n], \tag{3.6.1}$$

$$(x \blacktriangleright y)[z_1 \cdots z_n] = x[z_1 \cdots z_n] \blacktriangleright y[z_1 \cdots z_n], \tag{3.6.2}$$

$$x \blacktriangleright y \blacktriangleright z = y \blacktriangleright x \blacktriangleright z. \tag{3.6.3}$$

Proof. Necessity. Let $(\Phi, O, \rhd)$ be a restrictive Menger algebra of n-place functions. Of course, $\rhd$ is an idempotent operation. Moreover,

$$(\alpha \rhd \beta) \rhd \gamma = \gamma \circ \triangle_{\mathrm{pr}_1(\alpha \rhd \beta)}$$

for all $\alpha, \beta, \gamma \in \Phi$. Indeed, if $(\bar{a}, \bar{a}) \in \triangle_{\mathrm{pr}_1(\alpha \rhd \beta)}$, then $\bar{a} \in \mathrm{pr}_1(\alpha \rhd \beta)$, i.e., $(\bar{a}, b) \in \alpha \rhd \beta$ for some b. Thus $(\bar{a}, b) \in \alpha \rhd \beta = \beta \circ \triangle_{\mathrm{pr}_1 \alpha}$. So, there is $\bar{c}$

for which $(\bar{a},\bar{c}) \in \triangle_{\mathrm{pr}_1\alpha}$ and $(\bar{c},b) \in \beta$. Hence $\bar{a} = \bar{c} \in \mathrm{pr}_1\alpha$, and consequently, $(\bar{a},b) \in \beta$. Therefore $\bar{a} \in \mathrm{pr}_1\alpha$ and $\bar{a} \in \mathrm{pr}_1\beta$, which implies $(\bar{a},\bar{a}) \in \triangle_{\mathrm{pr}_1\alpha}$ and $(\bar{a},\bar{a}) \in \triangle_{\mathrm{pr}_1\beta}$. Consequently, $(\bar{a},\bar{a}) \in \triangle_{\mathrm{pr}_1\beta} \circ \triangle_{\mathrm{pr}_1\alpha}$. This proves the inclusion $\triangle_{\mathrm{pr}_1(\alpha\rhd\beta)} \subset \triangle_{\mathrm{pr}_1\beta} \circ \triangle_{\mathrm{pr}_1\alpha}$. The proof of the converse inclusion is similar.

Using this identity, we obtain

$$
\begin{aligned}
(\alpha \rhd \beta) \rhd \gamma = \gamma \circ \triangle_{\mathrm{pr}_1(\alpha\rhd\beta)} &= \gamma \circ \triangle_{\mathrm{pr}_1\beta} \circ \triangle_{\mathrm{pr}_1\alpha} \\
&= (\gamma \circ \triangle_{\mathrm{pr}_1\beta}) \circ \triangle_{\mathrm{pr}_1\alpha} = (\beta \rhd \gamma) \circ \triangle_{\mathrm{pr}_1\alpha} = \alpha \rhd (\beta \rhd \gamma),
\end{aligned}
$$

which proves that $(G,\rhd)$ is a semigroup.

For all $\alpha,\beta_i,\gamma_i \in \mathbf{\Phi}$, $i = 1,\ldots,n$ we have also

$$
\begin{aligned}
\alpha[(\beta_1 \rhd \gamma_1)\cdots(\beta_n \rhd \gamma_n)] &= \alpha[(\gamma_1 \circ \triangle_{\mathrm{pr}_1\beta_1})\cdots(\gamma_n \circ \triangle_{\mathrm{pr}_1\beta_n})] \\
&= \alpha[\gamma_1\cdots\gamma_n] \circ \triangle_{\mathrm{pr}_1\beta_n} \circ \cdots \circ \triangle_{\mathrm{pr}_1\beta_1} \\
&= \beta_1 \rhd \cdots \rhd \beta_n \rhd \alpha[\gamma_1\cdots\gamma_n].
\end{aligned}
$$

This proves (3.6.1).

Now let $\alpha,\beta,\gamma_i \in \mathbf{\Phi}$, $i = 1,\ldots,n$, where $\mathbf{\Phi} \subset \mathcal{F}(A^n,A)$. Suppose that $(\bar{a},c) \in (\alpha \rhd \beta)[\gamma_1\cdots\gamma_n]$ for some $\bar{a} \in A^n$, $c \in A$. Then there exists $\bar{b} \in A^n$ such that $(\bar{a},b_i) \in \gamma_i$, $i = 1,\ldots,n$, and $(\bar{b},c) \in \alpha \rhd \beta$. But $\bar{b} \in \mathrm{pr}_1\alpha$ implies $(\bar{b},d) \in \alpha$ for some $d \in A$. Thus $(\bar{a},d) \in \alpha[\gamma_1\cdots\gamma_n]$ and $\bar{a} \in \mathrm{pr}_1\alpha[\gamma_1\cdots\gamma_n]$. This, together with $(\bar{a},c) \in \beta[\gamma_1\cdots\gamma_n]$, proves $(\bar{a},c) \in \alpha[\gamma_1\cdots\gamma_n] \rhd \beta[\gamma_1\cdots\gamma_n]$. Therefore

$$(\alpha \rhd \beta)[\gamma_1\cdots\gamma_n] \subset \alpha[\gamma_1\cdots\gamma_n] \rhd \beta[\gamma_1\cdots\gamma_n].$$

To prove the converse inclusion let $(\bar{a},c) \in \alpha[\gamma_1\cdots\gamma_n] \rhd \beta[\gamma_1\cdots\gamma_n]$. Then $\bar{a} \in \mathrm{pr}_1\alpha[\gamma_1\cdots\gamma_n]$ and $(\bar{a},c) \in \beta[\gamma_1\cdots\gamma_n]$. From $\bar{a} \in \mathrm{pr}_1\alpha[\gamma_1\cdots\gamma_n]$ we deduce the existence of a d such that $(\bar{a},d) \in \alpha[\gamma_1\cdots\gamma_n]$. We can therefore select $\bar{e} = (e_1,\ldots,e_n) \in A^n$ for which $(\bar{a},e_i) \in \gamma_i$, $i = 1,\ldots,n$, and $(\bar{e},d) \in \alpha$. Obviously $(\bar{a},c) \in \beta[\gamma_1\cdots\gamma_n]$ implies $(\bar{a},b_i) \in \gamma_i$, $i = 1,\ldots,n$, and $(\bar{b},c) \in \beta$ for some $\bar{b} \in A^n$. Since all γ_i are functions, from $(\bar{a},e_i) \in \gamma_i$ and $(\bar{a},b_i) \in \gamma_i$ we obtain $b_i = e_i$, $i = 1,\ldots,n$, i.e., $\bar{b} = \bar{e}$. So, $(\bar{b},d) \in \alpha$. Hence $\bar{b} \in \mathrm{pr}_1\alpha$ and $(\bar{b},c) \in \alpha \rhd \beta$. Thus $(\bar{a},c) \in (\alpha \rhd \beta)[\gamma_1\cdots\gamma_n]$, which proves the inclusion

$$\alpha[\gamma_1\cdots\gamma_n] \rhd \beta[\gamma_1\cdots\gamma_n] \subset (\alpha \rhd \beta)[\gamma_1\cdots\gamma_n].$$

This completes the proof of (3.6.2).

For all $\alpha,\beta,\gamma \in \mathbf{\Phi}$ we have also

$$
\begin{aligned}
(\alpha \rhd \beta) \rhd \gamma &= \gamma \circ \triangle_{\mathrm{pr}_1(\alpha\rhd\beta)} = \gamma \circ \triangle_{\mathrm{pr}_1\beta} \circ \triangle_{\mathrm{pr}_1\alpha} = \gamma \circ \triangle_{\mathrm{pr}_1\beta\cap\mathrm{pr}_1\alpha} \\
&= \gamma \circ \triangle_{\mathrm{pr}_1\alpha\cap\mathrm{pr}_1\beta} = \gamma \circ \triangle_{\mathrm{pr}_1\alpha} \circ \triangle_{\mathrm{pr}_1\beta} = (\alpha \rhd \gamma) \circ \triangle_{\mathrm{pr}_1\beta} = \beta \rhd (\alpha \rhd \gamma).
\end{aligned}
$$

This proves (3.6.3). The proof of the necessity part is complete. □

The proof of the sufficiency will be based on some auxiliary results. In order to introduce them note that every algebra $(G, o, \blacktriangleright)$ of type $(n+1, 2)$ satisfying all the conditions of Theorem 3.6.1 will be called a *restrictive Menger algebra of rank n*. In this algebra, we define two binary relations

$$\alpha_d = \{(x,y) \,|\, x \blacktriangleright y = x\} \qquad \text{and} \qquad \alpha_s = \{(x,y) \,|\, y \blacktriangleright x = x\}.$$

As one can verify without any difficulty, these relations are, respectively, a stable order and quasi-order in the semigroup $(G, \blacktriangleright)$.

Proposition 3.6.2. *For any $x, y \in G$ we have*

$$\alpha_d^{-1} \circ \alpha_d \cap \alpha_s = \alpha_d, \tag{3.6.4}$$

$$(x \blacktriangleright y, y) \in \alpha_d, \quad (x \blacktriangleright y, x) \in \alpha_s. \tag{3.6.5}$$

Proof. If $(x,z) \in \alpha_d^{-1} \circ \alpha_d$ and $(x,z) \in \alpha_s$, then there is y such that $(x,y) \in \alpha_d$ and $(y,z) \in \alpha_d^{-1}$. Hence $(z,y) \in \alpha_d$ and $x \blacktriangleright y = x$, $z \blacktriangleright y = z$, $z \blacktriangleright x = x$, which together with (3.6.3) give

$$x \blacktriangleright z = x \blacktriangleright (z \blacktriangleright y) = (z \blacktriangleright x) \blacktriangleright y = x \blacktriangleright y = x.$$

Thus $(x,z) \in \alpha_d$, i.e., $\alpha_d^{-1} \circ \alpha_d \cap \alpha_s \subset \alpha_d$. To prove the converse inclusion, observe that if $(x,y) \in \alpha_d$, then $x \blacktriangleright y = x$. Hence, by (3.6.3), we have $x = x \blacktriangleright x = (x \blacktriangleright y) \blacktriangleright x = (y \blacktriangleright x) \blacktriangleright x = y \blacktriangleright x$. Thus $(x,y) \in \alpha_s$, i.e., $\alpha_d \subset \alpha_s$. Since α_d is reflexive, we have $\triangle_G \subset \alpha_d$ and $\triangle_G \subset \alpha_d^{-1}$, which means that $\alpha_d = \triangle_G \circ \alpha_d \subset \alpha_d^{-1} \circ \alpha_d$. These two inclusions imply $\alpha_d \subset \alpha_d^{-1} \circ \alpha_d \cap \alpha_s$. This completes the proof of (3.6.4).

The proof of (3.6.5) is trivial. □

Proposition 3.6.3. *If $(G, o, \blacktriangleright)$ is a restrictive Menger algebra of rank n, then in the algebra (G, o) the relation α_d is stable, the relation α_s is l-regular and v-negative and the condition* (3.2.2)*, where $x \leqslant y \longleftrightarrow (x,y) \in \alpha_d$ and $x \sqsubset y \longleftrightarrow (x,y) \in \alpha_s$, is satisfied.*

Proof. Let $x \leqslant y$ and $x_i \leqslant y_i$ for all $i = 1, \ldots, n$. Then $x = x \blacktriangleright y$ and $x_i = x_i \blacktriangleright y_i$, $i = 1, \ldots, n$. From $x = x \blacktriangleright y$, by (3.6.2), follows $x[x_1 \cdots x_n] = (x \blacktriangleright y)[x_1 \cdots x_n] = x[x_1 \cdots x_n] \blacktriangleright y[x_1 \cdots x_n]$, which implies $x[x_1 \cdots x_n] \leqslant y[x_1 \cdots x_n]$. From the identities $x_i = x_i \blacktriangleright y_i$ and (3.6.1) we obtain $y[x_1 \cdots x_n] = y[x_1 \blacktriangleright y_1 \cdots x_n \blacktriangleright y_n] = x_1 \blacktriangleright \cdots \blacktriangleright x_n \blacktriangleright y[y_1 \cdots y_n]$, whence, by (3.6.5), we obtain $x_1 \blacktriangleright \cdots \blacktriangleright x_n \blacktriangleright y[y_1 \cdots y_n] \leqslant y[y_1 \cdots y_n]$, and consequently, $y[x_1 \cdots x_n] \leqslant y[y_1 \cdots y_n]$. Therefore $x[x_1 \cdots x_n] \leqslant y[y_1 \cdots y_n]$. This proves that α_d is stable.

Now let $x \sqsubset y$, i.e., $y \blacktriangleright x = x$. Then $(y \blacktriangleright x)[z_1 \cdots z_n] = x[z_1 \cdots z_n]$, whence, by (3.6.2), we deduce $y[z_1 \cdots z_n] \blacktriangleright x[z_1 \cdots z_n] = x[z_1 \cdots z_n]$. Thus

$x[z_1 \cdots z_n] \sqsubset y[z_1 \cdots z_n]$. So, α_s is l-regular. Further, using the idempotency of $\blacktriangleright$, (3.6.1) and (3.6.3) we obtain

$$\begin{aligned} & x[y_1 \cdots y_n] \\ & \quad = x[(y_1 \blacktriangleright y_1) \cdots (y_{i-1} \blacktriangleright y_{i-1})\,((y_i \blacktriangleright y_i) \blacktriangleright y_i)\,(y_{i+1} \blacktriangleright y_{i+1}) \cdots (y_n \blacktriangleright y_n)] \\ & \quad = y_1 \blacktriangleright \cdots \blacktriangleright y_{i-1} \blacktriangleright (y_i \blacktriangleright y_i) \blacktriangleright y_{i+1} \blacktriangleright \cdots \blacktriangleright y_n \blacktriangleright x[y_1 \cdots y_n] \\ & \quad = y_i \blacktriangleright y_1 \blacktriangleright \cdots \blacktriangleright y_i \blacktriangleright \cdots \blacktriangleright y_n \blacktriangleright x[y_1 \cdots y_n] \\ & \quad = y_i \blacktriangleright x[(y_1 \blacktriangleright y_1) \cdots (y_{i-1} \blacktriangleright y_{i-1})\,(y_i \blacktriangleright y_i)\,(y_{i+1} \blacktriangleright y_{i+1}) \cdots (y_n \blacktriangleright y_n)] \\ & \quad = y_i \blacktriangleright x[y_1 \cdots y_n], \end{aligned}$$

which implies $x[y_1 \cdots y_n] \sqsubset y_i$ for every $i = 1, \ldots, n$. This shows that α_s is v-negative.

To prove (3.2.2) assume that $x \leqslant y$, $z \sqsubset x$ and $z \sqsubset u[\bar{w}\,|_i y]$. Then $x = x \blacktriangleright y$, $x \blacktriangleright z = z$ and $u[\bar{w}\,|_i y] \blacktriangleright z = z$. The last identity implies $x \blacktriangleright u[\bar{w}\,|_i y] \blacktriangleright z = x \blacktriangleright z$, whence we obtain $u[\bar{w}\,|_i (x \blacktriangleright y)] \blacktriangleright z = z$. So, $u[\bar{w}\,|_i x] \blacktriangleright z = z$, i.e., $z \sqsubset u[\bar{w}\,|_i x]$. This completes the proof of (3.2.2) and the proof of our Proposition. □

Proposition 3.6.4. *Let (ε^*, W) be the determining pair of a Menger algebra (G, o) of rank n, where $\varepsilon^* = \varepsilon \cup \{(e_1, e_1), \ldots, (e_n, e_n)\}$, $e_1, \ldots, e_n$ — are the selectors in (G^*, o^*). Then*

$$P_{(\varepsilon^*, W)}(g_1 \blacktriangleright g_2) = P_{(\varepsilon^*, W)}(g_1) \rhd P_{(\varepsilon^*, W)}(g_2) \tag{3.6.6}$$

holds for all $g_1, g_2 \in G$ if and only if for all $g_1, g_2 \in G$ the implications

$$g_1 \blacktriangleright g_2 \notin W \longrightarrow g_1 \notin W \wedge\, g_1 \blacktriangleright g_2 \equiv g_2(\varepsilon), \tag{3.6.7}$$

$$g_1, g_2 \notin W \longrightarrow g_1 \blacktriangleright g_2 \equiv g_2(\varepsilon) \tag{3.6.8}$$

are satisfied.

Proof. It is obvious that the conditions (3.6.7) and (3.6.8) are respectively equivalent to

$$\begin{aligned} & y \notin W \wedge\, g_1 \blacktriangleright g_2 \equiv y(\varepsilon) \longrightarrow g_1 \notin W \wedge\, g_2 \equiv y(\varepsilon), \\ & y \notin W \wedge\, g_1 \notin W \wedge\, g_2 \equiv y(\varepsilon) \longrightarrow g_1 \blacktriangleright g_2 \equiv y(\varepsilon). \end{aligned}$$

The above system of implications is equivalent to the statement

$$(\forall y \notin W)\ (g_1 \blacktriangleright g_2 \equiv y(\varepsilon) \longleftrightarrow g_1 \notin W \wedge\, g_2 \equiv y(\varepsilon)).$$

Applying (3.6.2) to this equivalence, we conclude that it is equivalent to

$$(\forall \bar{x})(\forall y \notin W)\left((g_1 \blacktriangleright g_2)[\bar{x}] \equiv y(\varepsilon) \longleftrightarrow g_1[\bar{x}] \notin W \wedge g_2[\bar{x}] \equiv y(\varepsilon)\right),$$

where $\bar{x} \in (G \setminus W)^n$. This is equivalent to (3.6.6). □

Proof of Theorem 3.6.1. *Sufficiency.* Let $(G, o, \blacktriangleright)$ be the restrictive Menger algebra of rank n. It is evident that the system $(G, o, \alpha_d, \alpha_s)$ is a f.o.p. Menger algebra. So, we can use the constructions presented in the proof of Theorem 3.2.1. In particular, we consider the following representation of a Menger algebra (G, o):

$$P = \sum_{g \in G} P_{(\varepsilon_g^*, W_v(\alpha_s\langle g\rangle))},$$

where $\varepsilon_g = \varepsilon_v(\alpha_d\langle g\rangle) \cap \varepsilon_v(\alpha_s\langle g\rangle)$. Theorem 3.2.1 shows that P is a faithful representation of (G, o) by n-place functions. For this representation we have also $\zeta_P = \alpha_d$ and $\chi_P = \alpha_s$.

The proof of the equality $P(g_1 \blacktriangleright g_2) = P(g_1) \rhd P(g_2)$ can be reduced to the proof that all determining pairs of the type $(\varepsilon_g^*, W_v(\alpha_s\langle g\rangle))$ satisfy condition (3.6.6). In view of Proposition 3.6.4, this fact will be proved if we prove that these determining pairs satisfy the conditions (3.6.7) and (3.6.8). To prove (3.6.7), observe first that $W_v(\alpha_s\langle g\rangle) = G \setminus \alpha_s\langle g\rangle$ for every $g \in G$, because α_s is v-negative. From (3.6.1) we conclude $x \blacktriangleright t(y) = t(x \blacktriangleright y)$ for all $x, y \in G$, $t \in T_n(G)$. So, if $g_1 \blacktriangleright g_2 \notin W_v(\alpha_s\langle g\rangle)$, then $(g, g_1 \blacktriangleright g_2) \in \alpha_s$, i.e., $g_1 \blacktriangleright g_2 \blacktriangleright g = g$. As $(g_1 \blacktriangleright g_2, g_1) \in \alpha_s$, then $(g, g_1) \in \alpha_s$. Therefore $g_1 \notin W_v(\alpha_s\langle g\rangle)$. Now, let us show that the condition $g_2 \equiv g_1 \blacktriangleright g_2(\varepsilon_g)$, i.e., $g_2 \equiv g_1 \blacktriangleright g_2(\varepsilon_{\alpha_i\langle g\rangle})$ for every $i = d, s$, is equivalent to the condition

$$(\forall t \in T_n(G))\left(t(g_2) \in \alpha_i\langle g\rangle \longleftrightarrow t(g_1 \blacktriangleright g_2) \in \alpha_i\langle g\rangle\right) \qquad \text{for every } i = d, s.$$

Let $t(g_2) \in \alpha_d\langle g\rangle$, i.e., $g \blacktriangleright t(g_2) = g$. Then $t(g \blacktriangleright g_2) = g$. But $g_1 \notin W_v(\alpha_s\langle g\rangle)$, i.e., $g_1 \blacktriangleright g = g$. Hence $t(g_1 \blacktriangleright g \blacktriangleright g_2) = g$. Consequently, $t(g \blacktriangleright g_1 \blacktriangleright g_2) = g$. Therefore $g \blacktriangleright t(g_1 \blacktriangleright g_2) = g$, which means that $t(g_1 \blacktriangleright g_2) \in \alpha_d\langle g\rangle$. So, $t(g_2) \in \alpha_d\langle g\rangle$ implies $t(g_1 \blacktriangleright g_2) \in \alpha_d\langle g\rangle$. On the other hand, if $t(g_1 \blacktriangleright g_2) \in \alpha_d\langle g\rangle$, then $g \blacktriangleright t(g_1 \blacktriangleright g_2) = g$, and, consequently, $g \blacktriangleright g_1 \blacktriangleright t(g_2) = g$. This, by (3.6.3), is equivalent to $g_1 \blacktriangleright g \blacktriangleright t(g_2) = g$. Because $g_1 \blacktriangleright g = g$, the above implies $g \blacktriangleright t(g_2) = g$, i.e., $t(g_2) \in \alpha_d\langle g\rangle$. Hence $g_2 \equiv g_1 \blacktriangleright g_2(\varepsilon_{\alpha_d\langle g\rangle})$. Now let $t(g_2) \in \alpha_s\langle g\rangle$, i.e., $t(g_2) \blacktriangleright g = g$, which is equivalent to $g \blacktriangleright t(g_2) \blacktriangleright g = g$. From this, we have $t(g \blacktriangleright g_2) \blacktriangleright g = g$. But $g = g_1 \blacktriangleright g$, so, using condition (3.6.3) we get $g = t(g_1 \blacktriangleright g \blacktriangleright g_2) \blacktriangleright g = t(g \blacktriangleright g_1 \blacktriangleright g_2) \blacktriangleright g = g \blacktriangleright t(g_1 \blacktriangleright g_2) \blacktriangleright g = t(g_1 \blacktriangleright g_2) \blacktriangleright g = g$. Hence $t(g_1 \blacktriangleright g_2) \in \alpha_s\langle g\rangle$. Thus $t(g_2) \in \alpha_s\langle g\rangle$ implies $t(g_1 \blacktriangleright g_2) \in \alpha_s\langle g\rangle$. In a similar way, we can verify the converse implication. These two implications prove $g_2 \equiv g_1 \blacktriangleright g_2(\varepsilon_{\alpha_s\langle g\rangle})$. In this way, we have proved that $g_2 \equiv g_1 \blacktriangleright g_2(\varepsilon_g)$. This completes the proof of (3.6.7).

The proof of (3.6.8) is analogous. □

A restrictive Menger algebra $(G, o, \blacktriangleright)$ of rank $n \geqslant 2$ is called *special*, if for all $i = 1, \ldots, n,\ u, x, y \in G,\ \bar{w} \in G^n$ the following condition

$$u[\bar{w}|_i x] = u[\bar{w}|_i y] \longrightarrow u[\bar{w}|_i x] \blacktriangleright x = u[\bar{w}|_i y] \blacktriangleright y \qquad (3.6.9)$$

is satisfied.

Theorem 3.6.5. *A Menger algebra $(G, o, \blacktriangleright)$ of rank $n \geqslant 2$ is isomorphic to a restrictive Menger algebra of reversive n-place functions if and only if it is special.*

Proof. Let $(\Phi, O, \rhd)$ be a restrictive Menger algebra of reversive n-place functions. Assume that $\varphi[\bar{\psi}|_i\alpha] = \varphi[\bar{\psi}|_i\beta]$ for some $\varphi, \alpha, \beta \in \Phi$, $\bar{\psi} \in \Phi^n$, $i \in \{1, \ldots, n\}$. Then $\varphi[\bar{\psi}|_i\alpha] \circ \triangle_H = \varphi[\bar{\psi}|_i\beta] \circ \triangle_H$, where $H = \mathrm{pr}_1\varphi[\bar{\psi}|_i\alpha] = \mathrm{pr}_1\varphi[\bar{\psi}|_i\beta]$. Hence $\varphi[\bar{\psi}|_i\alpha \circ \triangle_H] = \varphi[\bar{\psi}|_i\beta \circ \triangle_H]$. Since all functions in Φ are reversive, the last equality implies $\alpha \circ \triangle_H = \beta \circ \triangle_H$, i.e., $\varphi[\bar{\psi}|_i\alpha] \rhd \alpha = \varphi[\bar{\psi}|_i\beta] \rhd \beta$, which proves the necessity.

To prove the sufficiency, consider a special restrictive Menger algebra $(G, o, \blacktriangleright)$ of rank $n \geqslant 2$ and the relation $\varepsilon_m\langle\alpha_d^{-1} \circ \alpha_d, \alpha_s\langle g\rangle\rangle$ defined on (G, o) for every fixed $m \in \mathbb{N}$ and $g \in G$. We prove by induction that this relation satisfies the implication

$$(g_1, g_2) \in \varepsilon_m\langle\alpha_d^{-1} \circ \alpha_d, \alpha_s\langle g\rangle\rangle \longrightarrow g \blacktriangleright g_1 = g \blacktriangleright g_2. \qquad (3.6.10)$$

First assume that $(g_1, g_2) \in \varepsilon_0\langle\alpha_d^{-1} \circ \alpha_d, \alpha_s\langle g\rangle\rangle$. Then $(g_1, g_2) \in \alpha_d^{-1} \circ \alpha_d$ and $g_1, g_2 \in \alpha_s\langle g\rangle$, whence $g_1 \blacktriangleright g_2 = g_2 \blacktriangleright g_1$ and $g_1 \blacktriangleright g = g,\ \ g_2 \blacktriangleright g = g$. Therefore $g \blacktriangleright g_1 \blacktriangleright g_2 = g \blacktriangleright g_2 \blacktriangleright g_1$ and $g_1 \blacktriangleright g \blacktriangleright g_2 = g_2 \blacktriangleright g \blacktriangleright g_1$. So, $g \blacktriangleright g_2 = g \blacktriangleright g_1$, which means that for $m = 0$ the implication (3.6.10) is true.

Let it be valid for some $m = k$. Then for $(g_1, g_2) \in \varepsilon_{k+1}\langle\alpha_d^{-1} \circ \alpha_d, \alpha_s\langle g\rangle\rangle$ there are $q, r, u, s, c, d, z, \bar{w}, \bar{v}$ for which we have $g_1 = u[\bar{w}|_q c],\ g_2 = u[\bar{w}|_q d]$, $g_1, g_2 \in \alpha_s\langle g\rangle$ and $(s[\bar{v}|_r c], z),\ (z, s[\bar{v}|_r d]) \in \varepsilon_k\langle\alpha_d^{-1} \circ \alpha_d, \alpha_s\langle g\rangle\rangle$. By hypothesis, the last condition implies

$$g \blacktriangleright s[\bar{v}|_r c] = g \blacktriangleright s[\bar{v}|_r d].$$

Hence $s[\bar{v}|_r(g \blacktriangleright c)] = s[\bar{v}|_r(g \blacktriangleright d)]$, which according to (3.6.9), gives $s[\bar{v}|_r(g \blacktriangleright c)] \blacktriangleright g \blacktriangleright c = s[\bar{v}|_r(g \blacktriangleright d)] \blacktriangleright g \blacktriangleright d$. Further on, using the idempotency of the operation $\blacktriangleright$ and the conditions (3.6.1), (3.6.3), we obtain

$$s[\bar{v}|_r c] \blacktriangleright g \blacktriangleright c = s[\bar{v}|_r d] \blacktriangleright g \blacktriangleright d. \qquad (3.6.11)$$

As $s[\bar{v}|_r c],\ s[\bar{v}|_r d] \in \alpha_s\langle g\rangle$, then $s[\bar{v}|_r c] \blacktriangleright g = g$ and $s[\bar{v}|_r d] \blacktriangleright g = g$. This, together with (3.6.11), gives $g \blacktriangleright c = g \blacktriangleright d$. Therefore, $u[\bar{w}|_q(g \blacktriangleright c)] = u[\bar{w}|_q(g \blacktriangleright d)]$, which is equivalent to $g \blacktriangleright u[\bar{w}|_q c] = g \blacktriangleright u[\bar{w}|_q d]$. Thus

$g \blacktriangleright g_1 = g \blacktriangleright g_2$. So, the implication (3.6.10) is valid for $m = k+1$. Hence, by induction, it is valid for every $m \in \mathbb{N}$.

Using the implication (3.6.10) it is not difficult to prove that

$$(g_1, g_2) \in \varepsilon_m \langle \alpha_d^{-1} \circ \alpha_d\,, \alpha_s\langle g_1\rangle\rangle \longrightarrow (g_1, g_2) \in \alpha_d, \tag{3.6.12}$$

$$\left.\begin{array}{r}(g_1, g_2) \in \varepsilon_m \langle \alpha_d^{-1} \circ \alpha_d, \alpha_s\langle g\rangle\rangle, \\ x[\bar{y}|_i g_2] \in \alpha_s\langle g\rangle\end{array}\right\} \longrightarrow x[\bar{y}|_i g_1] \in \alpha_s\langle g\rangle \tag{3.6.13}$$

for all $m \in \mathbb{N}$, $i = 1, \ldots, n$, $g, g_1, g_2, x \in G$, $\bar{y} \in G^n$.

Consider now the family of determining pairs $\big((\varepsilon_g^*, W_g)\big)_{g \in G}$ of a Menger algebra (G, o), where $\varepsilon_g^* = \varepsilon\langle \alpha_d^{-1} \circ \alpha_d, \alpha_s\langle g\rangle\rangle \cup \{(e_1, e_1), \ldots, (e_n, e_n)\}$, $W_g = G \setminus \alpha_s\langle g\rangle$, $e_1, \ldots, e_n$ — the selectors of (G^*, o^*). Let P be the sum of the family of simplest representations determined by these pairs. Since the relations α_d and α_s satisfy the conditions (3.6.12) and (3.6.13), the system $(G, o, \alpha_d, \alpha_s)$ satisfies all assumptions of Theorem 3.3.1, i.e., it is a strong f.o.p. Menger algebra. So, P, as it was shown in the proof of Theorem 3.3.1, is a faithful representation of (G, o) by reversive n-place functions.

To complete this proof, we must prove that $P(g_1 \blacktriangleright g_2) = P(g_1) \rhd P(g_2)$ for all $g_1, g_2 \in G$. For this, it is sufficient to show that each determining pair (ε_g^*, W_g) satisfies (3.6.7) and (3.6.8). Let $g_1 \blacktriangleright g_2 \in \alpha_s\langle g\rangle$, i.e., let $g_1 \blacktriangleright g_2 \blacktriangleright g = g$. Then $g = g_1 \blacktriangleright g_1 \blacktriangleright g_2 \blacktriangleright g = g_1 \blacktriangleright g$. Thus $g_1 \in \alpha_s\langle g\rangle$. Similarly we show $g_2 \in \alpha_s\langle g\rangle$. From $(g_2, g_2) \in \alpha_d\langle g\rangle$ and $(g_1 \blacktriangleright g_2, g_2) \in \alpha_d$ we conclude

$$(g_2, g_1 \blacktriangleright g_2) \in (\alpha_d^{-1} \circ \alpha_d) \cap \alpha_s\langle g\rangle \times \alpha_s\langle g\rangle \subset \varepsilon\langle \alpha_d^{-1} \circ \alpha_d\,, \alpha_s\langle g\rangle\rangle,$$

which proves (3.6.7). If $g_1, g_2 \in \alpha_s\langle g\rangle$, then $g_1 \blacktriangleright g = g$, $g_2 \blacktriangleright g = g$ and $(g_1 \blacktriangleright g) \blacktriangleright (g_2 \blacktriangleright g) = g \blacktriangleright g = g$. But, on the other side,

$$(g_1 \blacktriangleright g) \blacktriangleright (g_2 \blacktriangleright g) = (g_1 \blacktriangleright g_2) \blacktriangleright g \blacktriangleright g = (g_1 \blacktriangleright g_2) \blacktriangleright g.$$

Hence $(g_1 \blacktriangleright g_2) \blacktriangleright g = g$. This gives $g_1 \blacktriangleright g_2 \in \alpha_s\langle g\rangle$. Now, applying the just proved relation $(g_1 \blacktriangleright g_2, g_2) \in \alpha_d^{-1} \circ \alpha_d$, we obtain

$$(g_1 \blacktriangleright g_2, g_2) \in \varepsilon_0\langle \alpha_d^{-1} \circ \alpha_d, \alpha_s\langle g\rangle\rangle \subset \varepsilon\langle \alpha_d^{-1} \circ \alpha_d, \alpha_s\langle g\rangle\rangle.$$

So, condition (3.6.8) also takes place. The proof is complete. □

Let Φ be the set of (reversive) n-place functions closed with respect to the Menger composition O, set-theoretic intersection $\cap$ and restrictive product $\rhd$ of functions. Algebras, isomorphic to the algebra of the form $(\Phi, O, \cap, \rhd)$, are called (*strong*) *restrictive* $\curlywedge$*-Menger algebras of rank* n. Their characterization is given by the following theorem.

Theorem 3.6.6. *An algebra* $(G, o, \curlywedge, \blacktriangleright)$ *of type* $(n+1, 2, 2)$ *is a restrictive* $\curlywedge$*-Menger algebra of rank* n *if and only if*
(a) $(G, o, \blacktriangleright)$ *is a restrictive Menger algebra of rank* n,
(b) $(G, \curlywedge)$ *is a semilattice,*
(c) *the identities*

$$x \curlywedge (y \blacktriangleright z) = y \blacktriangleright (x \curlywedge z), \tag{3.6.14}$$

$$(x \curlywedge y) \blacktriangleright y = x \curlywedge y \tag{3.6.15}$$

and (3.4.1) *are satisfied.*

Proof. The necessity of these conditions is obvious. We prove the sufficiency. For this assume that $(G, o, \curlywedge, \blacktriangleright)$ satisfies all these conditions. First observe that the relation α_d coincides with the natural order $\leqslant$ in the semilattice $(G, \curlywedge)$. Indeed, if $x \leqslant y$, i.e., $x \curlywedge y = x$, then $(x \curlywedge y) \blacktriangleright y = x \blacktriangleright y$. This, according to (3.6.15), gives $x \curlywedge y = x \blacktriangleright y$. Hence $x \blacktriangleright y = x$, i.e., $(x, y) \in \alpha_d$. Conversely, if $(x, y) \in \alpha_d$, then $y \curlywedge (x \blacktriangleright y) = y \curlywedge x$, and $x \blacktriangleright y = y \curlywedge x$, by (3.6.14). So, $y \curlywedge x = x$, i.e., $x \leqslant y$, which was to be proved. Further on, since $(G, o, \blacktriangleright)$ is a restrictive Menger algebra, $(G, o, \alpha_d, \alpha_s)$, as it was proved, is a f.o.p. Menger algebra. Thus, α_d is stable in (G, o), and α_s is l -regular and v-negative.

Now, we show that $(G, o, \curlywedge, \alpha_s)$ satisfies all assumptions of Theorem 3.6.1. For this it is sufficient to check just the conditions (3.4.13)–(3.4.15). Let $x \sqsubset x \curlywedge y$, where $\sqsubset$ is the relation α_s, then $(x \curlywedge y) \blacktriangleright x = x$. From this, according to (3.6.15), we get $x \curlywedge y = x$, i.e., $x \leqslant y$. So, (3.4.13) is satisfied. Next, from (3.6.15) we have $x_i \curlywedge y_i \leqslant y_i$ for every $i = 1, \ldots, n$, which together with the stability of α_d gives $u[(x_1 \curlywedge y_1) \cdots (x_n \curlywedge y_n)] \leqslant u[y_1 \cdots y_n]$. This proves (3.4.14). To prove (3.4.15) observe that

$$y \blacktriangleright z = z \blacktriangleright y \,\wedge\, x \sqsubset y \,\wedge\, x \sqsubset u[\bar{w}|_i z] \longrightarrow x \sqsubset u[\bar{w}|_i (y \curlywedge z)] \tag{3.6.16}$$

for all $x, y, z, u \in G$, $\bar{w} \in G^n$ and $i = 1, \ldots, n$. Indeed, if $y \blacktriangleright z = z \blacktriangleright y$, then

$$z \curlywedge (y \blacktriangleright z) = z \curlywedge (z \blacktriangleright y).$$

This, according to (3.6.14), gives $y \blacktriangleright z = z \blacktriangleright (z \curlywedge y)$. Since $z \curlywedge y \sqsubset z$, from the above formula we obtain $z \blacktriangleright (z \curlywedge y) = z \curlywedge y$, and, consequently, $y \blacktriangleright z = z \curlywedge y$. In view of the fact that the order α_s is v-negative the condition $x \sqsubset u[\bar{w}|_i z]$ implies $x \sqsubset z$, i.e., $z \blacktriangleright x = x$. Thus, according to the definition of α_s, the condition $x \sqsubset u[\bar{w}|_i z]$ is equivalent to the condition $u[\bar{w}|_i z] \blacktriangleright x = x$. Therefore

$$x = y \blacktriangleright x = y \blacktriangleright u[\bar{w}|_i z] \blacktriangleright x = u[\bar{w}|_i (y \blacktriangleright z)] \blacktriangleright x = u[\bar{w}|_i (z \curlywedge y)] \blacktriangleright x,$$

i.e., $x \sqsubset u[\bar{w}|_i (y \curlywedge z)]$. This proves (3.6.16). Now, according to (3.6.14), we have $(y \curlywedge z) \blacktriangleright (z \curlywedge v) = z \blacktriangleright (y \curlywedge z \curlywedge v)$ and $(z \curlywedge v) \blacktriangleright (y \curlywedge z) = z \blacktriangleright (y \curlywedge z \curlywedge v)$. Hence

$$(y \curlywedge z) \blacktriangleright (z \curlywedge v) = (z \curlywedge v) \blacktriangleright (y \curlywedge z).$$

So, from (3.6.16), we can conclude (3.4.15). Thus, all the constructions and arguments, used in the proof of Theorem 3.4.5 are valid for $(G, o, \curlywedge, \alpha_s)$.

Let P be the sum of the family of these simplest representations, which correspond to the determining pairs from the family $((\varepsilon_g^*, W_g))_{g\in G}$, where

$$\varepsilon_g = \{(g_1, g_2) \,|\, g \sqsubset g_1 \curlywedge g_2 \ \ or \ \ g_1, g_2 \in W_g\}, \qquad W_g = G \setminus \alpha_s\langle g\rangle .$$

In the proof of Theorem 3.4.5, it was proved that the representation P is faithful and satisfies the identity $P(g_1 \curlywedge g_2) = P(g_1) \cap P(g_2)$. It also satisfies the identity $P(g_1 \blacktriangleright g_2) = P(g_1) \rhd P(g_2)$. To verify this identity it is sufficient to prove that the conditions (3.6.7) and (3.6.8) are valid for each determining pair. Let $g_1 \blacktriangleright g_2 \notin W_g$, i.e., $g \sqsubset g_1 \blacktriangleright g_2$. Then $g \sqsubset g_1$, which gives $g_1 \notin W_g$. Since $g_1 \blacktriangleright g_2 \leqslant g_2$ implies $(g_1 \blacktriangleright g_2) \curlywedge g_2 = g_1 \blacktriangleright g_2$, we have also $(g_1 \blacktriangleright g_2) \curlywedge g_2 \notin W_g$, i.e., $g_1 \blacktriangleright g_2 \equiv g_2(\varepsilon_g)$. So, condition (3.6.7) is satisfied. Now, if $g_1 \notin W_g$ and $g_2 \notin W_g$, then $g_1 \blacktriangleright g = g$ and $g_2 \blacktriangleright g = g$. Hence $g_1 \blacktriangleright g_2 \blacktriangleright g = g_1 \blacktriangleright g = g$, i.e., $g \sqsubset (g_1 \blacktriangleright g_2)$. As it is easy to see, since $(g_1 \blacktriangleright g_2) \blacktriangleright g_2 = g_2 \blacktriangleright (g_1 \blacktriangleright g_2)$, by applying (3.6.16) in the relation above we obtain $g \sqsubset (g_1 \blacktriangleright g_2) \curlywedge g_2$, i.e., $g_1 \blacktriangleright g_2 \equiv g_2(\varepsilon_g)$. This proves (3.6.8) and completes our proof. □

Corollary 3.6.7. *A restrictive $\curlywedge$-Menger algebra of rank $n \geqslant 2$ is strong if and only if it satisfies* (3.4.3).

Proof. Indeed, let $u[\bar{w}|_i g_1] \notin W_g$ and $u[\bar{w}|_i g_1] \equiv u[\bar{w}|_i g_2](\varepsilon_g)$, where W_g and ε_g are as in the last proof. Then $g \sqsubset u[\bar{w}|_i g_1] \curlywedge u[\bar{w}|_i g_2]$. Applying (3.4.3) to this condition, we obtain $g \sqsubset u[\bar{w}|_i(g_1 \curlywedge g_2)]$. But $u[\bar{w}|_i(g_1 \curlywedge g_2)] \sqsubset g_1 \curlywedge g_2$ implies $g \sqsubset g_1 \curlywedge g_2$, i.e., $g_1 \equiv g_2(\varepsilon_g)$. So, the restriction of ε_g to $\alpha_s\langle g\rangle$ is v-cancellative. Thus, P is a representation by reversive n-place functions, which was to be proved. □

Finally, we consider *restrictive $\mathcal{D}$-Menger algebras of n-place functions*, i.e., algebras of the form $(\Phi, O, \rhd, \overset{n+1}{\triangle}_A)$, where $\Phi \subset \mathcal{F}(A^n, A)$ and $n \geqslant 2$.

Theorem 3.6.8. *An algebra $(G, o, \blacktriangleright, e)$ of type $(n+1, 2, 0)$, where $n \geqslant 2$, is isomorphic to some restrictive $\mathcal{D}$-Menger algebra of n-place functions if and only if*

(a) $(G, o, \blacktriangleright)$ *is a restrictive Menger algebra of rank $n \geqslant 2$,*

(b) $(G, \curlywedge)$, *where $x \curlywedge y = e[xy^{n-1}]$ for all $x, y \in G$, is a semilattice,*

(c) *the conditions* (3.4.16), (3.6.15) *and*

$$e \blacktriangleright g = g \longrightarrow g[e^n] = g \tag{3.6.17}$$

are satisfied.

Proof. As it is not difficult to see, the algebra $(G, o, \curlywedge, \blacktriangleright)$ satisfies all the assumptions of Theorem 3.6.6. Thus to prove the above theorem, we can use the same arguments as in the proof of Theorem 3.6.6, in this case however, instead of the representation from the proof of Theorem 3.6.6, we consider the representation from the proof of Theorem 3.4.6, i.e., the representation

$$P = \sum_{g \in G'} P_{(\varepsilon_g^*, W_g)} + \sum_{g \in G''} P_{(\varepsilon_g, W_g)},$$

where $G' = \{g \,|\, e \notin \alpha_s\langle g\rangle\}$, $G'' = \{g \,|\, e \in \alpha_s\langle g\rangle\}$ and $\varepsilon_g, \varepsilon_g^*, W_g$ are as in the proof of Theorem 3.6.6. Therefore $P(g_1 \blacktriangleright g_2) = P(g_1) \triangleright P(g_2)$ for all $g_1, g_2 \in G$. To prove that the representation P is faithful it is sufficient to prove that ζ_P coincides with the natural order of $(G, \curlywedge)$. The equality $P(e) = \overset{n+1}{\triangle}_A$, where A is some set, can be proved analogously as in the proof of Theorem 3.4.6. □

Finally note that *an algebra* $(G, o, \blacktriangleright, e)$ *is isomorphic to some restrictive* $\mathcal{D}$*-Menger algebra of reverse* n*-place functions, where* $n \geqslant 2$*, if and only if the assumptions of Theorem* 3.6.8 *are satisfied and the identity* (3.4.3) *holds.*

3.7 Functional Menger systems

On the set $\mathcal{F}(A^n, A)$ of all n-place functions defined on A we can consider n unary operations $\mathcal{R}_1, \ldots, \mathcal{R}_n$ such that for every function $\varphi \in \mathcal{F}(A^n, A)$ $\mathcal{R}_i\,\varphi$ is the restriction of n-place projectors defined on A to the domain of φ, i.e.,

$$\mathcal{R}_i\,\varphi = \varphi \triangleright I_i^n \tag{3.7.1}$$

for every $i = 1, \ldots, n$, where I_i^n is the ith n-place projector on A. In other words, $\mathcal{R}_i\,\varphi$ is an n-place function from $\mathcal{F}(A^n, A)$, which satisfies the conditions

$$\mathrm{pr}_1\mathcal{R}_i\,\varphi = \mathrm{pr}_1\varphi, \tag{3.7.2}$$

$$\bar{a} \in \mathrm{pr}_1\varphi \longrightarrow \mathcal{R}_i\,\varphi(\bar{a}) = a_i \tag{3.7.3}$$

for any $\bar{a} \in A^n$. Algebras of the form $(\Phi, O, \mathcal{R}_1, \ldots, \mathcal{R}_n)$, where Φ is a subset of $\mathcal{F}(A^n, A)$, are called *functional Menger systems of rank* n.

Such algebras (with some additional operations) were first studied in [80,192] and [193]. For example, V. Kafka considered in [80] an algebraic system of the form $M = (\Phi, O, \mathcal{R}_1, \ldots, \mathcal{R}_n, \mathcal{L}, \subset)$, where $\Phi \subset \mathcal{F}(A^n, A)$ and $\mathcal{L}\varphi = \overset{n+1}{\triangle}_{\mathrm{pr}_2\varphi}$ for every $\varphi \in \Phi$. Such an algebraic system satisfies the conditions:

(A_1) (Φ, O) is a Menger algebra,

(A_2) $\subset$ is an order,

(A_3) the following identities are satisfied:

$$\begin{cases} \varphi[\mathcal{R}_1\varphi\cdots\mathcal{R}_n\varphi]=\varphi, \\ \mathcal{R}_i(\varphi[\gamma_1\cdots\gamma_n])=\mathcal{R}_i((\mathcal{R}_j\varphi)[\gamma_1\cdots\gamma_n]), \\ (\mathcal{L}\varphi)[\varphi\cdots\varphi]=\varphi, \end{cases}$$

(A_4) $\mathcal{L}(\varphi[\gamma_1\cdots\gamma_n])\subset\mathcal{L}\varphi$ for all $\varphi,\gamma_1\cdots\gamma_n\in\mathbf{\Phi}$,

(A_5) $\mathbf{\Phi}$ has elements $I_1,\ldots,I_n$ such that $\varphi[I_1\cdots I_n]=\varphi$ and

$$\begin{cases} \gamma_i\subset I_i\longrightarrow\mathcal{R}_i\gamma_i\subset\gamma_i, \\ \varphi\subset\bigcap\limits_{j=1}^{n}I_j\longrightarrow\mathcal{L}\varphi\subset\varphi, \\ \gamma_k\subset I_k\longrightarrow\gamma_k[\varphi_1\cdots\varphi_n]\subset\varphi_k, \\ \bigwedge\limits_{k=1}^{n}(\gamma_k\subset I_k)\longrightarrow\varphi[\gamma_1\cdots\gamma_n]\subset\varphi, \\ \mathcal{R}_i\gamma_j\subset\bigcap\limits_{k=1}^{n}\mathcal{R}_i\gamma_k\longrightarrow I_j[\gamma_1\cdots\gamma_n]=\gamma_j, \\ \varphi\subset\psi\longrightarrow\varphi=\psi[\rho_1\cdots\rho_n]\ \text{ for some }\ \rho_1\subset I_1,\ldots,\rho_n\subset I_n \end{cases}$$

for all $\varphi,\varphi_1,\ldots,\varphi_n,\gamma_1,\ldots,\gamma_n\in\mathbf{\Phi}$ and $i,j,k\in\{1,\ldots,n\}$.

It is proved in [80] that if an algebraic system $(G,o,R_1,\ldots,R_n,L,\leqslant)$ satisfies the above conditions, then there exists an isomorphism of the algebra (G,o) into $(\mathcal{F}(A^n,A),O)$ which transforms the order $\leqslant$ into the set-theoretical inclusion. However, A_1-A_5 do not give a complete characterization of systems of the form $(\mathbf{\Phi},O,\mathcal{R}_1,\ldots,\mathcal{R}_n,\mathcal{L},\subset)$.

Let $(G,o,R_1,\ldots,R_n)$ be an algebra of type $(n+1,1,\ldots,1)$ which satisfies the identities:

$$x[\bar{y}][\bar{z}]=x[y_1[\bar{z}]\cdots y_n[\bar{z}]], \tag{3.7.4}$$

$$x[R_1x\cdots R_nx]=x, \tag{3.7.5}$$

$$x[\bar{u}|_iz][R_1y\cdots R_ny]=x[\bar{u}|_i\,z[R_1y\cdots R_ny]], \tag{3.7.6}$$

$$R_i(x[R_1y\cdots R_ny])=(R_ix)[R_1y\cdots R_ny], \tag{3.7.7}$$

$$x[R_1y\cdots R_ny][R_1z\cdots R_nz]=x[R_1z\cdots R_nz][R_1y\cdots R_ny], \tag{3.7.8}$$

$$R_i(x[\bar{y}])=R_i((R_kx)[\bar{y}]), \tag{3.7.9}$$

$$(R_ix)[\bar{y}]=y_i[R_1(x[\bar{y}])\cdots R_n(x[\bar{y}])] \tag{3.7.10}$$

for all $i,k=1,\ldots,n$.

Theorem 3.7.1. *Each functional Menger system* $(\mathbf{\Phi},O,\mathcal{R}_1,\ldots,\mathcal{R}_n)$ *of rank* n *satisfies the identities* (3.7.4)–(3.7.10).

Proof. The identity (3.7.4) is obvious. Furthermore, for every $\varphi \in \mathbf{\Phi}$ we have $\varphi[\mathcal{R}_1\varphi\cdots\mathcal{R}_n\varphi] = \varphi[(\varphi \rhd I_1^n)\cdots(\varphi \rhd I_n^n)] = \varphi \rhd \varphi[I_1^n\cdots I_n^n]$ and $\varphi[I_1^n\cdots I_n^n] = \varphi$. Thus $\varphi[\mathcal{R}_1\varphi\cdots\mathcal{R}_n\varphi] = \varphi \rhd \varphi = \varphi$, which proves (3.7.5). Since for any $\alpha, \beta \in \mathbf{\Phi}$ we have $\alpha[\mathcal{R}_1\beta\cdots\mathcal{R}_n\beta] = \beta \rhd \alpha$, for all $\varphi, \alpha, \beta \in \mathbf{\Phi}$, $\bar{\psi} \in \mathbf{\Phi}^n$ and $i = 1, \ldots, n$, we have also $\varphi[\bar{\psi}|_i\alpha][\mathcal{R}_1\beta\cdots\mathcal{R}_n\,\beta] = \beta \rhd \varphi[\bar{\psi}|_i\alpha] = \varphi[\bar{\psi}|_i(\beta \rhd \alpha)] = \varphi[\bar{\psi}|_i\,\alpha[\mathcal{R}_1\beta\cdots\mathcal{R}_n\beta]]$. This proves (3.7.6). Because $\alpha \rhd \beta \rhd \gamma = \beta \rhd \alpha \rhd \gamma$ holds for all $\alpha, \beta, \gamma \in \mathbf{\Phi}$, then, evidently, $\mathcal{R}_i(\varphi[\mathcal{R}_1\psi\cdots\mathcal{R}_n\psi]) = \mathcal{R}_i(\psi \rhd \varphi) = (\psi \rhd \varphi) \rhd I_i^n = \psi \rhd (\varphi \rhd I_i^n) = \psi \rhd \mathcal{R}_i\varphi = (\mathcal{R}_i\varphi)[\mathcal{R}_1\psi\cdots\mathcal{R}_n\psi]$ for $i = 1, \ldots, n$ and $\varphi, \psi \in \mathbf{\Phi}$. Similarly $\gamma[\mathcal{R}_1\beta\cdots\mathcal{R}_n\beta][\mathcal{R}_1\alpha\cdots\mathcal{R}_n\alpha] = \gamma[\mathcal{R}_1\alpha\cdots\mathcal{R}_n\alpha][\mathcal{R}_1\beta\cdots\mathcal{R}_n\beta]$. So, (3.7.7) and (3.7.8) are proved too.

Now let $\varphi, \psi_1, \ldots, \psi_n \in \mathbf{\Phi}$, then, according to (3.6.1)–(3.6.3) and (3.7.1), we obtain

$$\begin{aligned}\mathcal{R}_i((\mathcal{R}_k\varphi)[\psi_1\cdots\psi_n]) &= (\varphi \rhd I_k^n)[\psi_1\cdots\psi_n] \rhd I_i^n\\ &= \varphi[\psi_1\cdots\psi_n] \rhd I_k^n[\psi_1\cdots\psi_n] \rhd I_i^n\\ &= \varphi[\psi_1\cdots\psi_n] \rhd \psi_1 \rhd \cdots \rhd \psi_n \rhd \psi_k \rhd I_i^n\\ &= \psi_1 \rhd \cdots \rhd \psi_n \rhd \varphi[\psi_1\cdots\psi_n] \rhd I_i^n\\ &= \varphi[\psi_1\cdots\psi_n] \rhd I_i^n = \mathcal{R}_i(\varphi[\psi_1\cdots\psi_n]).\end{aligned}$$

So, we proved (3.7.9). Further we have

$$\begin{aligned}\mathcal{R}_k(\varphi[\psi_1\cdots\psi_n]) &= (\varphi \rhd I_i^n)[\psi_1\cdots\psi_n]\\ &= \varphi[\psi_1\cdots\psi_n] \rhd I_i^n[\psi_1\cdots\psi_n]\\ &= \varphi[\psi_1\cdots\psi_n] \rhd \psi_1 \rhd \cdots \rhd \psi_n \rhd \psi_i\\ &= \psi_1 \rhd \cdots \rhd \psi_n \rhd \varphi[\psi_1\cdots\psi_n] \rhd \psi_i\\ &= \varphi[\psi_1\cdots\psi_n] \rhd \psi_i\\ &= \psi_i[\mathcal{R}_1(\varphi[\psi_1\cdots\psi_n])\cdots\mathcal{R}_n(\varphi[\psi_1\cdots\psi_n])],\end{aligned}$$

which proves (3.7.10). The proof is complete. □

Assume that the algebra $\mathcal{G} = (G, o, R_1, \ldots, R_n)$ satisfies the identities (3.7.4)–(3.7.10) and consider on it the new operation $\blacktriangleright$ defined by the formula $x \blacktriangleright y = y[R_1x\cdots R_nx]$.

Proposition 3.7.2. $(G, o, \blacktriangleright)$ *is a restrictive Menger algebra of rank* n.

Proof. Indeed, by (3.7.4), (G, o) is a Menger algebra of rank n. From (3.7.5) and (3.7.8), we obtain:

$$x \blacktriangleright x = x \quad\text{and}\quad z \blacktriangleright (y \blacktriangleright x) = y \blacktriangleright (z \blacktriangleright x).$$

From (3.7.6), it follows that $y \blacktriangleright x[\bar{u}|_i\,z] = x[\bar{u}|_i(y \blacktriangleright z)]$ for every $i = 1, \ldots, n$. Applying the idempotency of $\blacktriangleright$ to this identity we obtain (3.6.1). Furthermore, using (3.7.7), we have

$$\begin{aligned}(x \blacktriangleright y) \blacktriangleright z &= z[R_1(x \blacktriangleright y) \cdots R_n(x \blacktriangleright y)] \\ &= z[R_1 y[R_1 x \cdots R_n x] \cdots R_n y[R_1 x \cdots R_n x]] \\ &= z[(R_1 y)[R_1 x \cdots R_n x] \cdots (R_n y)[R_1 x \cdots R_n x]] \\ &= z[R_1 y \cdots R_n y][R_1 x \cdots R_n x] \\ &= (y \blacktriangleright z)[R_1 x \cdots R_n x] = x \blacktriangleright (y \blacktriangleright z).\end{aligned}$$

This means that the operation $\blacktriangleright$ is associative. Now, according to (3.7.10),

$$\begin{aligned}(x \blacktriangleright y)[\bar{z}] &= y[R_1 x \cdots R_n x][\bar{z}] = y[(R_1 x)[\bar{z}] \cdots (R_n x)[\bar{z}]] \\ &= y[z_1[R_1 x[\bar{z}] \cdots R_n x[\bar{z}]] \cdots z_n[R_1 x[\bar{z}] \cdots R_n x[\bar{z}]]] \\ &= y[\bar{z}][R_1 x[\bar{z}] \cdots R_n x[\bar{z}]] = x[\bar{z}] \blacktriangleright y[\bar{z}],\end{aligned}$$

which proves (3.6.2). The proof is complete. □

Since $(G, o, \blacktriangleright)$ is a restrictive Menger algebra, we can consider on $\mathcal{G}$ the relations α_d and α_s, which, as it is known, satisfy all the conditions of Theorem 3.2.1. Thus, $(G, o, \alpha_d, \alpha_s)$ if a f.o.p. Menger algebra of rank n.

Below we describe the fundamental properties of α_d and α_s connected with the operations R_i, $i = 1, \ldots, n$. For simplicity's sake, the relation α_d will be denoted by $\leqslant$, while the relation α_s — by $\sqsubset$.

Proposition 3.7.3. *$R_i x = (R_i x)[R_1 x \cdots R_n x]$ and $R_i x = R_i R_k x$ hold for all $x \in G$ and $i, k = 1, \ldots, n$.*

Proof. Indeed, using (3.7.5) and (3.7.7), we get

$$R_i x = R_i x[R_1 x \cdots R_n x] = (R_i x)[R_1 x \cdots R_n x]$$

for every $x \in G$ and $i = 1, \ldots, n$. Next, using (3.7.5), (3.7.9) and the just proved identity, we obtain $R_i x = R_i x[R_1 x \cdots R_n x] = R_i(R_k x)[R_1 x \cdots R_n x] = R_i R_k x$. □

Proposition 3.7.4. *For all $x, y \in G$ and every $i = 1, \ldots, n$*
(a) *$x \leqslant y$ implies $R_i x \leqslant R_i y$,*
(b) *the condition $R_i x \leqslant R_i y$ is equivalent to $x \sqsubset y$.*

Proof. (a) Let $x \leqslant y$ for some $x, y \in G$. Then $x = x \blacktriangleright y$ and $R_i x = R_i(x \blacktriangleright y) = R_i y[R_1 x \cdots R_n x]$, which, by Proposition 3.7.3, gives $R_i x = R_i y[R_1 R_i x \cdots R_n R_i x]$. Applying (3.7.7) to this identity, we obtain $R_i x = (R_i y)[R_1 R_i x \cdots R_n R_i x]$, i.e., $R_i x = (R_i x) \blacktriangleright (R_i y)$. Thus $R_i x \leqslant R_i y$.

(b) If $x \sqsubset y$, then $y \blacktriangleright x = x$ and $R_ix[R_1y \cdots R_ny] = R_ix$ for every $i = 1, \ldots, n$. This, by means of (3.7.7), gives $(R_ix)[R_1y \cdots R_ny] = R_ix$. Applying to this identity condition (3.7.10), we obtain

$$\begin{aligned} R_ix &= (R_iy)[R_1x[R_1y \cdots R_ny] \cdots R_nx[R_1y \cdots R_ny]] \\ &= (R_iy)[R_1(y \blacktriangleright x) \cdots R_n(y \blacktriangleright x)] = (R_iy)[R_1x \cdots R_nx] \\ &= (R_iy)[R_1R_ix \cdots R_nR_ix] = (R_ix) \blacktriangleright (R_iy), \end{aligned}$$

i.e., $R_ix \leqslant R_iy$. Conversely, if $R_ix \leqslant R_iy$ holds for some $i = 1, \ldots, n$, then, by the previous proposition, it also holds for every $i = 1, \ldots, n$. Thus for every $i = 1, \ldots, n$ we have $R_ix \sqsubset R_iy$, because $\alpha_d \subset \alpha_s$. The last condition is equivalent to $(R_iy) \blacktriangleright (R_ix) = R_ix$, i.e., to $(R_ix)[R_1y \cdots R_ny] = R_ix$. Therefore we have $y \blacktriangleright R_ix = R_ix$, and consequently

$$\begin{aligned} y \blacktriangleright x = y \blacktriangleright x[R_1x \cdots R_nx] &= x[R_1x \cdots R_nx][R_1y \cdots R_ny] \\ &= x[(y \blacktriangleright R_1x) \cdots (y \blacktriangleright R_nx)] = x[R_1x \cdots R_nx] = x. \end{aligned}$$

So, $x \sqsubset y$ and the proof is complete. □

Proposition 3.7.5. *For all $i = 1, \ldots, n$ and $x, y, y_1, \ldots, y_n \in G$ we have:*
(a) $R_ix \sqsubset x$ *and* $x \sqsubset R_ix$,
(b) $(R_ix)[y_1 \cdots y_n] \leqslant y_i$,
(c) $x[R_1y \cdots R_ny] \leqslant x$.

Proof. (a) Since $x \leqslant x$ is valid for every $x \in G$, by Proposition 3.7.4 we also have $R_i(R_ix) \leqslant R_ix$. So, $R_ix \sqsubset x$. The proof of $x \sqsubset R_ix$ is analogous.
(b) Using (3.7.9) and (3.7.10) we get

$$\begin{aligned} (R_ix)[y_1 \cdots y_n] \blacktriangleright y_i &= y_i[R_1(R_ix)[y_1 \cdots y_n] \cdots R_n(R_ix)[y_1 \cdots y_n]] \\ &= y_i[R_1x[y_1 \cdots y_n] \cdots R_nx[y_1 \cdots y_n]] = (R_ix)[y_1 \cdots y_n]. \end{aligned}$$

Thus $(R_ix)[y_1 \cdots y_n] \leqslant y_i$.
(c) It is trivial: $x[R_1y \cdots R_ny] = y \blacktriangleright x \leqslant x$, by Proposition 3.7.2. □

For every element g of the algebra $\mathcal{G}$ we define the relation $\rho_g \subset G \times G$ letting:

$$\rho_g = \big(\alpha_d^{-1} \circ \alpha_d \cap \alpha_s\langle g\rangle \times \alpha_s\langle g\rangle\big) \cup \alpha_s'\langle g\rangle \times \alpha_s'\langle g\rangle,$$

where $\alpha_s'\langle g\rangle = G \setminus \alpha_s\langle g\rangle$. It is clear that ρ_g is a quasi-equivalence. On (G, o) it is also i-regular for every $i = 1, \ldots, n$. Indeed, if $(g_1, g_2) \in \rho_g$ and $g_1, g_2 \in \alpha_s'\langle g\rangle$, then $u[\bar{w}|_i\, g_1]$, $u[\bar{w}|_i\, g_2] \in \alpha_s'\langle g\rangle$ because $\alpha_s'\langle g\rangle$ is an l-ideal. So, $(u[\bar{w}|_i\, g_1],\, u[\bar{w}|_i\, g_2]) \in \rho_g$. If $g_1, g_2 \in \alpha_s\langle g\rangle$, then, obviously, $(g_1, g_2) \in \alpha_d^{-1} \circ \alpha_d$, i.e., $g_1 \leqslant z$ and $g_2 \leqslant z$ for some $z \in G$. These inequalities together with the stability of α_d imply $u[\bar{w}|_i\, g_1] \leqslant u[\bar{w}|_i\, z]$ and $u[\bar{w}|_i\, g_2] \leqslant u[\bar{w}|_i\, z]$. Hence $(u[\bar{w}|_i\, g_1], u[\bar{w}|_i\, g_2]) \in \alpha_d^{-1} \circ \alpha_d$. Suppose now that $u[\bar{w}|_i\, g_1]$ lies in $\alpha_s\langle g\rangle$. Then also $u[\bar{w}|_i\, z]$ lies in $\alpha_s\langle g\rangle$. Therefore $g_2 \leqslant z$, $g \sqsubset g_2$ and $g \sqsubset u[\bar{w}|_i\, z]$, which, according to (3.2.2), gives $g \sqsubset u[\bar{w}|_i\, g_2]$. In the same manner, we prove that $g \sqsubset u[\bar{w}|_i\, g_2]$ implies $g \sqsubset u[\bar{w}|_i\, g_1]$. So the elements $u[\bar{w}|_i\, g_1]$ and $u[\bar{w}|_i\, g_2]$ simultaneously belong or do not belong to $\alpha_s\langle g\rangle$. If they do not belong to $\alpha_s\langle g\rangle$, then obviously $(u[\bar{w}|_i\, g_1], u[\bar{w}|_i\, g_2]) \in \rho_g$. If they belong to $\alpha_s\langle g\rangle$, then

$$(u[\bar{w}|_i\, g_1], u[\bar{w}|_i\, g_2]) \in \alpha_d^{-1} \circ \alpha_d \cap \alpha_s\langle g\rangle \times \alpha_s\langle g\rangle \subset \rho_g.$$

This completes the proof of the i-regularity of ρ_g, where $i = 1, \ldots, n$.

Let ε_g by the transitive closure of ρ_g. It is clear that ε_g is a v-congruence on (G, o) and $\alpha_s'\langle g\rangle$ is its equivalence class. For each $g_1 \in G$ on the quotient set G/ε_g we define an n-place function $P_g(g_1)$ putting

$$P_g(g_1)(\varepsilon_g\langle x_1\rangle, \ldots, \varepsilon_g\langle x_n\rangle) = \varepsilon_g\langle g_1[x_1 \cdots x_n]\,\rangle \tag{3.7.11}$$

for all $x_1, \ldots, x_n \in G$ such that $g_1[x_1 \cdots x_n] \in \alpha_s\langle g\rangle$. Otherwise this function is not defined. It is not difficult to verify that P_g is a representation of a Menger algebra (G, o). Using this function we can prove the main result of this section.

Theorem 3.7.6. *An algebra $\mathcal{G} = (G, o, R_1, \ldots, R_n)$ of type $(n+1, 1, \ldots, 1)$ is isomorphic to a functional Menger algebra of rank n if and only if it satisfies the identities* (3.7.4)–(3.7.10).

Proof. The necessity of the above identities follows from Theorem 3.7.1. To prove the sufficiency of the identities (3.7.4)–(3.7.10) assume that the algebra $\mathcal{G}$ satisfies them and P is the sum of the family of representations $(P_g)_{g\in G}$ of (G, o) by n-place functions. We prove that the representation P is faithful and $P(R_i x) = \mathcal{R}_i P(x)$ for each $x \in G$ and $i = 1, \ldots, n$.

First we prove $\alpha_d = \zeta_P$ and $\alpha_s = \chi_P$. Let $g_1 \leqslant g_2$, then $g_1[\bar{x}] \leqslant g_2[\bar{x}]$ for every $\bar{x} \in G^n$. Since $g_1[\bar{x}] \in \alpha_s\langle g\rangle$ implies $g_2[\bar{x}] \in \alpha_s\langle g\rangle$, the inclusion $\alpha_d \subset \alpha_d^{-1} \circ \alpha_d$ gives $(g_1[\bar{x}], g_2[\bar{x}]) \in \alpha_d^{-1} \circ \alpha_d \cap \alpha_s\langle g\rangle \times \alpha_s\langle g\rangle \subset \varepsilon_g$. So, $(g_1, g_2) \in \zeta_{P_g}$ for each $g \in G$, i.e., $\alpha_d \subset \zeta_P$. Conversely, if $(g_1, g_2) \in \zeta_P$, then, for any $g \in G$, $\bar{x} \in G^n$, from $g_1[\bar{x}] \in \alpha_s\langle g\rangle$ we conclude $(g_1[\bar{x}], g_2[\bar{x}]) \in \varepsilon_g$, which, according to (3.7.5), for $\bar{x} = (R_1 g_1, \ldots, R_n g_1)$ and $g = g_1$ gives $(g_1, g_2[R_1 g_1 \cdots R_n g_1]) \in \varepsilon_g$. But ε_g is a transitive closure of ρ_d, thus in $\alpha_s\langle g_1\rangle$ one can find elements $x_1, \ldots, x_{m-1}$ for which $(g_1, x_1) \in \alpha_d^{-1} \circ \alpha_d$, $(x_1, x_2) \in \alpha_d^{-1} \circ \alpha_d, \ldots, (x_{m-1}, g_2[R_1 g_1 \cdots R_n g_1]) \in \alpha_d^{-1} \circ \alpha_d$. This means that for some elements $z_1, \ldots, z_m \in G$ we have $g_1 \leqslant z_1$, $x_1 \leqslant z_1$, $x_1 \leqslant z_2$, $x_2 \leqslant z_2$, $\ldots$, $x_{m-1} \leqslant z_m$, $g_2[R_1 g_1 \cdots R_n g_1] \leqslant z_m$. Because $g_1 \leqslant z_1$ and $x_1 \leqslant z_1$ together with $g_1 \sqsubset x_1$ imply $g_1 \leqslant x_1$, we have $g_1 \leqslant z_2$. Similarly from $g_1 \leqslant z_2$, $x_2 \leqslant z_2$ and $g_1 \sqsubset x_2$ we conclude $g_1 \leqslant x_2$ and so on. Proceeding this way, after a finite number of steps, we get

$g_1 \leqslant g_2[R_1g_1 \cdots R_ng_1]$, which, by Proposition 3.7.5 (c), gives $g_1 \leqslant g_2$. Hence $\zeta_P \subset \alpha_d$, and consequently $\alpha_d = \zeta_P$. The proof of $\alpha_s = \chi_P$ is similar.

From the above it follows, in particular, that the representation P is faithful.

The proof of the identity $P(R_ix) = \mathcal{R}_iP(x)$, $x \in G$, $i = 1, \dots, n$, is based on the relation ε_g. Since P is the sum of representations P_g, $g \in G$, and, by Proposition 3.7.5, the equality $P(R_ix)(a_1, \dots, a_n) = a_i$ is true for every $(a_1, \dots, a_n) \in \mathrm{pr}_1 P(R_ix)$, it is necessary to show that for each $g \in G$ $(a_1, \dots, a_n) \in \mathrm{pr}_1 P_g(R_ix)$ implies $P_g(a_1, \dots, a_n) = a_i$. According to the definition of P_g, this last inclusion means that for elements $y_1, \dots, y_n \in G$, from $(R_ix)[y_1 \cdots y_n] \in \alpha_s\langle g\rangle$ we can deduce $(R_ix)[y_1 \cdots y_n] \equiv y_i(\varepsilon_g)$. So, let $g \sqsubset (R_ix)[y_1 \cdots y_n]$. Since $(R_ix)[y_1 \cdots y_n] \leqslant y_i$, then obviously we have $((R_ix)[y_1 \cdots y_n], y_i) \in \rho_g \subset \varepsilon_g$, which was to be proved. □

Algebras of the form $(\Phi, O, \cap, \mathcal{R}_1, \dots, \mathcal{R}_n)$, where $\Phi \subset \mathcal{F}(A^n, A)$ and $\cap$ is a set-theoretic intersection, are called *functional Menger $\cap$-algebras of n-place functions.*

Theorem 3.7.7. *An algebra $(G, o, \curlywedge, R_1, \dots, R_n)$ of type $(n+1, 2, 1, \dots, 1)$ is isomorphic to some functional Menger $\cap$-algebra of n-place functions if and only if*
(a) $(G, o, R_1, \dots, R_n)$ *is a functional Menger system of rank n,*
(b) $(G, \curlywedge)$ *is a semilattice,*
(c) *the identities*

$$x \curlywedge y[R_1z \cdots R_nz] = (x \curlywedge y)[R_1z \cdots R_nz], \tag{3.7.12}$$

$$x[R_1(x \curlywedge y) \cdots R_n(x \curlywedge y)] = x \curlywedge y. \tag{3.7.13}$$

and (3.4.1) *are satisfied.*

Proof. The necessity of these conditions is obvious. To prove their sufficiency assume that the algebra $(G, o, \curlywedge, R_1, \dots, R_n)$ of type $(n+1, 2, 1, \dots, 1)$ satisfies all these conditions. By Proposition 3.7.2 the algebra $(G, o, \blacktriangleright)$ with the operation $x \blacktriangleright y = y[R_1x \cdots R_nx]$ is a restrictive Menger algebra of rank n. In this algebra the conditions (3.7.12) and (3.7.13) have the form

$$x \curlywedge (z \blacktriangleright y) = z \blacktriangleright (x \curlywedge y) \quad \text{and} \quad (x \curlywedge y) \blacktriangleright x = x \curlywedge y.$$

Since $(G, \curlywedge)$ is a semilattice, the last identity is equivalent to the identity (3.6.15). This means that $(G, o, \curlywedge, \blacktriangleright)$ satisfies the conditions of Theorem 3.6.6. So, it is a restrictive $\curlywedge$-algebra of rank n, i.e., it is isomorphic to some algebra of the form $(\Phi, O, \cap, \triangleright)$, where $\Phi \subset \mathcal{F}(A^n, A)$ and $\triangleright$ is the restrictive product of n-place functions.

Let $P = \sum_{g\in G} P_{(\varepsilon_g, W_g)}$ be the sum of simplest representations considered in the proof of Theorem 3.6.6. Then P is a faithful representation of $(G, o, \curlywedge, \blacktriangleright)$ by n-place functions. Let us show that P also satisfies

$$P(R_i g) = \mathcal{R}_i P(g)$$

for all $g \in G$ and $i = 1, \ldots, n$. First observe that

$$g_1 \leqslant g_2 \longleftrightarrow P(g_1) \subset P(g_2),$$

$$g_1 \sqsubset g_2 \longleftrightarrow \mathrm{pr}_1 P(g_1) \subset \mathrm{pr}_1 P(g_2).$$

Indeed, for $g_1, g_2 \in G$ we have $g_1 \leqslant g_2 \longleftrightarrow g_1 \curlywedge g_2 = g_1 \longleftrightarrow P(g_1 \curlywedge g_2) = P(g_1) \longleftrightarrow P(g_1) \cap P(g_2) = P(g_1) \longleftrightarrow P(g_1) \subset P(g_2)$, which proves the first equivalence. Analogously $g_1 \sqsubset g_2 \longleftrightarrow g_1 = g_2 \blacktriangleright g_1 \longleftrightarrow P(g_1) = P(g_2 \blacktriangleright g_1) \longleftrightarrow P(g_1) = P(g_2) \rhd P(g_1) \longleftrightarrow \mathrm{pr}_1 P(g_1) \subset \mathrm{pr}_1 P(g_2)$.

The above two equivalences together with Proposition 3.7.5 (a) imply $\mathrm{pr}_1 P(R_i g) \subset \mathrm{pr}_1 P(g)$ and $\mathrm{pr}_1 P(g) \subset \mathrm{pr}_1 P(R_i g)$, whence we obtain the identity $\mathrm{pr}_1 P(R_i g) = \mathrm{pr}_1 P(g) = \mathrm{pr}_1 \mathcal{R}_i P(g)$. Thus,

$$\mathrm{pr}_1 P(R_i g) = \mathrm{pr}_1 \mathcal{R}_1 P(g). \tag{3.7.14}$$

This means that for every $(\varepsilon_g\langle x_1\rangle, \ldots, \varepsilon_g\langle x_n\rangle) \in \mathrm{pr}_1 P_{(\varepsilon_g, W_g)}(R_i x)$ we have $g \sqsubset (R_i x)[x_1 \cdots x_n]$, which by Proposition 3.7.5 (b) implies $g \sqsubset x_i$ for every $i = 1, \ldots, n$. Consequently, $g \sqsubset x_i \curlywedge (R_i x)[x_1 \cdots x_n]$. The last relation, according to the definition of ε_g, gives $(R_i x)[x_1 \cdots x_n] \equiv x_i(\varepsilon_g)$, i.e.,

$$P_{(\varepsilon_g, W_g)}(R_i x)(\varepsilon_g\langle x_1\rangle, \ldots, \varepsilon_g\langle x_n\rangle) = \varepsilon_g\langle x_i\rangle$$

for all $g \in G$ and $i = 1, \ldots, n$. From this, in view of (3.7.14), we deduce $P(R_i g) = \mathcal{R}_i P(g)$, which proves that P is a faithful representation of $(G, o, \curlywedge, R_1, \ldots, R_n)$ by n-place functions.[5] □

3.8 Notes on Chapter 3

The task of finding an axiom system for a class of algebras of n-place functions (for a fixed n) closed under superposition was formulated by K. Menger in his article [114]. A similar task for functional Menger systems ordered by inclusion was considered by V. Kafka [80]. But in [80] this task was not fulfilled completely because the proof of the main theorem of this work is not complete. Investigations of functional semigroups were initiated by B. Schweizer and A. Sklar in [190, 192, 193].

[5] Note that every algebra $\mathcal{G}_\curlywedge = (G, o, \curlywedge, R_1, \ldots, R_n)$ of type $(n+1, 2, 1, \ldots, 1)$ satisfying all the conditions of Theorem 3.7.7 will be called a *functional Menger* $\curlywedge$*-algebra of rank* n.

The first results on restrictive Menger algebras were obtained by V. S. Trokhimenko in [216] and [230]. The first result on Menger algebras of reversive multiplace functions can be found in [215]. Research on $\mathcal{P}$-algebras of multiplace functions was started in [222]. Further studies were conducted in [224, 239] and [32]. Theorem 3.7.7 is proved in the last paper.

The restrictive multiplication of binary relations and functions was introduced by V. V. Vagner [246] in connection with some problems on differential geometry. This operation was further studied in [10, 122, 175].

The study of restrictive semigroups was initiated by V. V. Vagner [250], research on restrictive bisemigroups of binary relations — by B. M. Schein [170] (see also [173, 174]).

The first abstract characterization of semigroups of binary relations ordered by the set-theoretic inclusion was given by K. A. Zareckii [263]. In this paper he also gives an abstract characterization of ordered semigroups of transitive and reflexive binary relations. Later investigations were conducted by B. M. Schein in [164,166,167]. In [166] B. M. Schein proved that a semigroup of transitive and reflexive relations, where set-theoretic inclusion of relations is not considered, can be characterized by an infinite system of elementary axioms which can be written in the language of the first order predicates containing the sign "=". This system is not equivalent to any finite system of elementary axioms. Abstract characterizations of ordered semigroups of binary relations ordered by set-theoretic inclusion of relations, inclusion of first projections and inclusion of second projections were given in [167]. These three inclusions were considered in various combinations: all three, pairwise and each of them separately.

Stable orders on homomorphic images of symmetrical semigroups of partial functions were described by M. G. Mogilevskii [121].

The systematical study of representations of abstract semigroups by partial one-to-one transformations was started by B. M. Schein in his two papers [160] and [161]. In those two papers he proved that each semigroup satisfying some infinite system of elementary axioms can be isomorphically embedded into inverse semigroup of partial transformations.

An axiomatization of $\cap$-semigroups and inverse $\cap$-semigroups of one-to-one transformations was given by V. S. Garvackii [46], an abstract characterization of $\cup$-semigroups and $(\cap, \cup)$-semigroups of transformations — by B. M. Schein in [181, 182].

Chapter 4

Relations between functions

During the investigation of Menger algebras of multiplace functions, we find various properties of functions, which may be interesting for us, for example, the existence of a set of fixed points, the definiteness on the same set, the compatibility on some sets, and so on. It is known that an isomorphism of Menger algebras do not preserves such information. In order to preserve it, we must insert into the signature of Menger algebras some additional relations, which may be used for the abstract characterization of these properties. In the previous chapter, we have described only two such relations – the inclusion of functions ζ_Φ and the inclusion of domains χ_Φ. In this chapter, we will consider Menger algebras of functions containing, in addition to these two relations, other naturally defined relations such as, for example, the relation of semi-compatibility, connectivity, and co-definability.

4.1 Stabilizers of Menger algebras

A nonempty subset H of a (f.o.p., f.o., p.q-o.) Menger algebra (G, o) of rank n is called a (*reversive*) *stabilizer* if there exists an isomorphism P of this algebra onto some (respectively, f.o.p., f.o., p.q-o.) Menger algebra of (reversive) n-place functions such that H is the preimage of the stabilizer of some point. So, H contains all those elements of (G, o) that are mapped by P onto functions having the same common fixed point.

Lemma 4.1.1. *For a nonempty subset H of G to be a stabilizer of a f.o.p. Menger algebra (G, o, ζ, χ) of rank n it is necessary and sufficient that it is a quasi-stable, l-unitary, ζ-saturated normal v-complex and that there exists such χ-saturated subset $A \subset G$ containing H, that*

$$g_1 \in H \wedge g_2 \in H \wedge t(g_1) \in A \longrightarrow t(g_2) \in A, \tag{4.1.1}$$

$$g_1 \leqslant g_2 \wedge g_1 \in A \wedge u[\bar{w}|_i\, g_2] \in H \longrightarrow u[\bar{w}|_i\, g_1] \in H, \tag{4.1.2}$$

$$g_1 \leqslant g_2 \wedge g_1 \in A \wedge u[\bar{w}|_i\, g_2] \in A \longrightarrow u[\bar{w}|_i\, g_1] \in A \tag{4.1.3}$$

for all $t \in T_n(G)$, $i = 1, \ldots, n$, $g_1, g_2 \in G$, $u \in G \cup \{e_i\}$, $\bar{w} \in G^n$, where $x \leqslant y$ means $(x, y) \in \zeta$ and $e_1, \ldots, e_n$ are the selectors of (G^*, o^*).

Proof. Necessity. Let $(\mathbf{\Phi}, O, \zeta_{\mathbf{\Phi}}, \chi_{\mathbf{\Phi}})$ be some f.o.p. Menger algebra of n-place functions and $H^a_{\mathbf{\Phi}}$ be the stabilizer of a point a. For $\varphi \in H^a_{\mathbf{\Phi}}$, we have $\varphi(a,\ldots,a) = a$ and

$$\varphi[\varphi\cdots\varphi](a,\ldots,a) = \varphi(\varphi(a,\ldots,a),\ldots,\varphi(a,\ldots,a)) = \varphi(a,\ldots,a) = a,$$

i.e., $\varphi[\varphi\cdots\varphi] \in H^a_{\mathbf{\Phi}}$. So, $H^a_{\mathbf{\Phi}}$ is quasi-stable. Now let $\varphi[\psi\cdots\psi] \in H^a_{\mathbf{\Phi}}$ and $\psi \in H^a_{\mathbf{\Phi}}$, then

$$a = \varphi[\psi\cdots\psi](a,\ldots,a) = \varphi(\psi(a,\ldots,a),\ldots,\psi(a,\ldots,a)) = \varphi(a,\ldots,a).$$

Hence $\varphi \in H^a_{\mathbf{\Phi}}$. Thus, $H^a_{\mathbf{\Phi}}$ is l-unitary. Analogously we can show that $H^a_{\mathbf{\Phi}}$ is $\zeta_{\mathbf{\Phi}}$-saturated. Furthermore, if $\varphi(\bar{a}) = \psi(\bar{a})$ holds for some $\varphi, \psi \in \mathbf{\Phi}$ and $\bar{a} \in \mathrm{pr}_1\,\varphi \cap \mathrm{pr}_1\,\psi$, then it is not difficult to see that for every $t \in T_n(\mathbf{\Phi})$ we have $t(\varphi)\langle\bar{a}\rangle = t(\psi)\langle\bar{a}\rangle$. So, if $\varphi, \psi, t(\varphi) \in H^a_{\mathbf{\Phi}}$, then $t(\psi) \in H^a_{\mathbf{\Phi}}$. Thus, $H^a_{\mathbf{\Phi}}$ is a normal v-complex.

Consider the set

$$A_a = \{\varphi \in \mathbf{\Phi} \mid (a,\ldots,a) \in \mathrm{pr}_1\varphi\}.$$

It is clear that A_a is $\chi_{\mathbf{\Phi}}$-saturated and $H^a_{\mathbf{\Phi}} \subseteq A_a$. If $t(\varphi) \in A_a$ for some $\varphi, \psi \in H^a_{\mathbf{\Phi}}$ and $t \in T_n(\mathbf{\Phi})$, then $(a,\ldots,a) \in \mathrm{pr}_1 t(\varphi)$. Because $\varphi(a,\ldots,a) = a = \psi(a,\ldots,a)$, thus $t(\varphi)(a,\ldots,a) = t(\psi)(a,\ldots,a)$. So, $t(\psi) \in A_a$. This proves (4.1.1). Now let $\varphi \subset \psi$ and $\alpha[\bar{\chi}|_i\,\psi] \in H^a_{\mathbf{\Psi}}$ for some $\alpha, \psi \in \mathbf{\Phi}$, $\bar{\chi} \in \mathbf{\Phi}^n$, $i = 1,\ldots,n$ and $\varphi \in A_a$. Since $(a,\ldots,a) \in \mathrm{pr}_1\varphi$ together with $\varphi \subset \psi$ imply $(a,\ldots,a) \in \mathrm{pr}_1\psi$ and $\psi(a,\ldots,a) = \varphi(a,\ldots,a)$, we obtain

$$\begin{aligned} a = \alpha[\bar{\chi}|_i\,\psi](a,\ldots,a) &= \alpha(\bar{\chi}(a,\ldots,a)|_i\,\psi(a,\ldots,a)) \\ &= \alpha(\bar{\chi}(a,\ldots,a)|_i\,\varphi(a,\ldots,a)) = \alpha[\bar{\chi}|_i\,\varphi](a,\ldots,a), \end{aligned}$$

where $\bar{\chi}(a,\ldots,a)$ denotes the vector $(\chi_1(a,\ldots,a),\ldots,\chi_n(a,\ldots,a))$. The above means that $\alpha[\bar{\chi}|_i\,\varphi] \in H^a_{\mathbf{\Phi}}$. In this way we have shown (4.1.2). The proof of (4.1.3) is similar.

Sufficiency. Assume that all the conditions of the lemma are fulfilled. Then, as it is not difficult to see, for any $g_1, g_2 \in G$, $t \in T_n(G)$

$$g_1 \leqslant g_2 \,\wedge\, g_1 \in A \,\wedge\, t(g_2) \in A \longrightarrow t(g_1) \in A, \tag{4.1.4}$$

$$g_1 \leqslant g_2 \,\wedge\, g_1 \in A \,\wedge\, t(g_2) \in H \longrightarrow t(g_1) \in H. \tag{4.1.5}$$

Let us consider the representation $P = P_1 + P_{(\varepsilon^*,\,A')}$ of a Menger algebra (G, o) by n-place functions, where P_1 is some faithful representation of a f.o.p. Menger algebra (G, o, ζ, χ), and $P_{(\varepsilon^*,\,A')}$ is this simplest representation that corresponds to the determining pair (ε^*, A'), where $\varepsilon = \varepsilon_v(H) \cap \varepsilon_v(A)$,

$\varepsilon^* = \varepsilon \cup \{(e_1, e_1), \ldots, (e_n, e_n)\}$, $A' = G \setminus A$ and $e_1, \ldots, e_n$ are selectors in (G^*, o^*). Since the representation P_1 is faithful, we have $\zeta = \zeta_{P_1}$ and $\chi = \chi_{P_1}$. So, the representation P is also faithful.

The proof that P is an isomorphism of (G, o, ζ, χ) is based on the following two inclusions: $\zeta \subset \zeta_{(\varepsilon^*, A')}$ and $\chi \subset \chi_{(\varepsilon^*, A')}$. To prove the first inclusion, let $g_1 \leqslant g_2$ and $g_1[\bar{x}] \in A$ for some $\bar{x} \in B = G^n \cup \{(e_1, \ldots, e_n)\}$. Then, from $g_1 \leqslant g_2$ and the stability of ζ, we conclude $t(g_1[\bar{x}]) \leqslant t(g_2[\bar{x}])$ for all $t \in T_n(G)$, $\bar{x} \in B$. If $t(g_1[\bar{x}]) \in H$, then also $t(g_2[\bar{x}]) \in H$, because H is ζ-saturated by the assumption. Similarly, $t(g_2[\bar{x}]) \in H$, together with $g_1[\bar{x}] \leqslant g_2[\bar{x}]$, $g_1[\bar{x}] \in A$, and according to (4.1.5), gives $t(g_1[\bar{x}]) \in H$. Therefore, $g_1[\bar{x}] \equiv g_2[\bar{x}]\,(\varepsilon_v(H))$. In a similar way, using (4.1.4), we obtain $g_1[\bar{x}] \equiv g_2[\bar{x}]\,(\varepsilon_v(A))$. From the above, we conclude $g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon)$, i.e., $(g_1, g_2) \in \zeta_{(\varepsilon^*, A')}$. Thus $\zeta \subset \zeta_{(\varepsilon^*, A')}$. To prove the second inclusion, let $g_1 \sqsubset g_2$ and $g_1[\bar{x}] \in A$, where $\bar{x} \in B$ and $\sqsubset$ denotes the relation χ. Since this relation is l-regular, we have also $g_1[\bar{x}] \sqsubset g_2[\bar{x}]$, and, consequently, $g_2[\bar{x}] \in A$. Thus $(g_1, g_2) \in \chi_{(\varepsilon^*, A')}$, which proves $\chi \subset \chi_{(\varepsilon^*, A')}$. So, $\zeta_P = \zeta_{P_1} \cap \zeta_{(\varepsilon^*, A')} = \zeta \cap \zeta_{(\varepsilon^*, A')} = \zeta$. Using the same argument, we can prove the equality $\chi_P = \chi$. These two equalities show that P is an isomorphism.

Now let $g_1, g_2 \in H$. If $t(g_1) \in A$ for some $t \in T_n(G)$, then $t(g_2) \in A$, by (4.1.1). Similarly $t(g_2) \in A$ implies $t(g_1) \in A$. So, $g_1 \equiv g_2(\varepsilon_v(A))$. This means that H is contained in some $\varepsilon_v(A)$-class. But H, as a normal v-complex, is a $\varepsilon_v(H)$-class disjoint with $W_v(H)$. Because A' is a subset of $W_v(H)$, H also is a ε-class disjoint with A'. If $g \in H$, then $g[g \cdots g] \in H$, by the quasi-stability of H, and as a consequence $g[H \cdots H] \subset H$, by the v-regularity of the relation ε. So, $P_{(\varepsilon^*, A')}(g)(c, \ldots, c) = c$, where c is the element used to index the ε-class H. On the other hand, if $g[H \cdots H] \subset H$, then for every $h \in H$ we have $g[h \cdots h] \in H$, which, by the l-unitarity, gives $g \in H$. Likewise, we have shown that $g \in H$ if and only if $P_{(\varepsilon^*, A')}(g)(c, \ldots, c) = c$. In view of the fact that P is the sum of representations, we come to the conclusion that all the functions from $P(H)$, and only them, have c as a common fixed point. □

Let (G, o, ζ, χ) be a f.o.p. Menger algebra of rank n, $H \subset G$ and $\rho = \zeta \cup H \times H$. Denoting by $C[H]$ the $(\rho, \chi)^*$-closure of H, we can formulate the following theorem.

Theorem 4.1.2. *A nonempty subset H of G is a stabilizer of a f.o.p. Menger algebra (G, o, ζ, χ) of rank n if and only if it is a quasi-stable, l-unitary and ζ-saturated normal v-complex such that*

$$g_1 \leqslant g_2 \wedge g_1 \in C[H] \wedge u[\bar{w}|_i\, g_2] \in H \longrightarrow u[\bar{w}|_i\, g_1] \in H \tag{4.1.6}$$

for all $g_1, g_2 \in G$, $u \in G \cup \{e_i\}$, $\bar{w} \in G^n$, $i = 1, \ldots, n$.

Proof. Necessity. Let H^a_Φ be the stabilizer of a in a f.o.p. Menger algebra $(\boldsymbol{\Phi}, O, \zeta_\Phi, \chi_\Phi)$ of n-place functions. We have, of course, $C[H^a_\Phi] = \bigcup\limits_{n=0}^{\infty} \overset{n}{C}(H^a_\Phi)$, where $\varphi \in C(H^a_\Phi)$ if and only if $(\alpha,\beta) \in \zeta_\Phi \cup H^a_\Phi \times H^a_\Phi$, $(t(\alpha),\varphi) \in \chi_\Phi$ and $\alpha, t(\beta) \in H^a_\Phi$ for some $\alpha,\beta \in \mathbf{\Phi}$, $t \in T_n(\mathbf{\Phi})$. Using induction it is not difficult to show that for all $\varphi \in \mathbf{\Phi}$ and $m \in \mathbb{N}$ the following implication

$$\varphi \in \overset{m}{C}(H^a_\Phi) \longrightarrow (a,\ldots,a) \in \mathrm{pr}_1\varphi \tag{4.1.7}$$

takes place. Suppose that $\alpha \subset \beta$, $\alpha \in C[H^a_\Phi]$ and $\varphi[\bar\psi|_i\,\beta] \in H^a_\Phi$ for some $\alpha,\beta,\varphi \in \mathbf{\Phi}$, $\bar\psi \in \mathbf{\Phi}^n$, $i \in \{1,\ldots,n\}$. Since $\alpha \in \overset{m}{C}(H^a_\Phi)$ for some $m \in \mathbb{N}$, then, as a consequence of (4.1.7), we obtain $(a,\ldots,a) \in \mathrm{pr}_1\alpha$. But $\alpha \subset \beta$, so, $\alpha(a,\ldots,a) = \beta(a,\ldots,a)$ and

$$\begin{aligned} a &= \varphi[\bar\psi|_i\,\beta](a,\ldots,a) = \varphi(\bar\psi(a,\ldots,a)|_i\,\beta(a,\ldots,a)) \\ &= \varphi(\bar\psi(a,\ldots,a)|_i\,\alpha(a,\ldots,a)) = \varphi[\bar\psi|_i\,\alpha](a,\ldots,a). \end{aligned}$$

Hence $\varphi[\bar\psi|_i\,\alpha] \in H^a_\Phi$, which proves (4.1.6).

Sufficiency. Let the subset H satisfy all the conditions of the theorem. Then (4.1.6) corresponds to condition (4.1.2) from the last lemma, where $A = C[H]$. If $g_1 \leqslant g_2$ and $g_1,\, u[\bar w|_i\,g_2] \in C[H]$, then $(g_1,g_2) \in \rho$, $u[\bar w|_i\,g_1] \sqsubset u[\bar w|_i\,g_2]$ and $g_1,\, u[\bar w|_i\,g_2] \in C[H]$. Since $C[H]$ is $(\rho,\chi)^*$-closed, $u[\bar w|_i\,g_1] \in C[H]$. This proves (4.1.3). In a similar way we can prove that $C[H]$ satisfies (4.1.1). Obviously $C[H]$ is χ-saturated and $H \subset C[H]$. This means that all the conditions of Lemma 4.1.1 are satisfied. Hence H is a stabilizer of a f.o.p. Menger algebra (G,o,ζ,χ). The theorem is proved. □

In view of $C[H] = \bigcup\limits_{n=0}^{\infty} \overset{n}{C}(H)$, the implication (4.1.6) is equivalent to the system of conditions $(A'_m)_{m\in\mathbb{N}}$, where

$$A'_m\,:\; g_1 \leqslant g_2 \,\wedge\, u[\bar w|_i\,g_2] \in H \,\wedge\, g_1 \in \overset{m}{C}(H) \longrightarrow u[\bar w|_i\,g_1] \in H$$

for any $g_1,g_2 \in G$, $u \in G \cup \{e_i\}$, $\bar w \in G^n$, $i = 1,\ldots,n$. Besides, from $\overset{m}{C}(H) \subset \overset{m+1}{C}(H)$, follows that A'_{m+1} implies A'_m. Moreover, condition A'_m can be rewritten in the form $A_m : A''_m \longrightarrow u[\bar w|_i\,g_1] \in H$, where

$$A''_m : \begin{cases} \quad g_1 \leqslant g_2 \,\wedge\, u[\bar w|_i\,g_2] \in H, \\ \quad (a_1 \leqslant b_1 \,\vee\, a_1,b_1 \in H) \,\wedge\, t_1(a_1) \sqsubset g, \\ \bigwedge\limits_{k=1}^{2^{m-1}-1} \begin{pmatrix} (a_{2k} \leqslant b_{2k} \,\vee\, a_{2k},b_{2k} \in H) \,\wedge\, t_{2k}(a_{2k}) \sqsubset a_k, \\ (a_{2k+1} \leqslant b_{2k+1} \,\vee\, a_{2k+1},b_{2k+1} \in H), \\ t_{2k+1}(a_{2k+1}) \sqsubset t_k(b_k) \end{pmatrix}, \\ \bigwedge\limits_{k=2^{m-1}}^{2^m-1} (a_k \in H \wedge t_k(b_k) \in H) \end{cases}$$

for all $g_1, g_2, a_k, b_k \in G$, $u \in G \cup \{e_i\}$, $\bar{w} \in G^n$, $t_k \in T_n(G)$, $i = 1, \ldots, n$, $k = 1, \ldots, 2^m - 1$. This means that *the stabilizer of a f.o.p. Menger algebra (G, o, ζ, χ) of rank n is in particular its quasi-stable, l-unitary and ζ-saturated normal v-complex H, which satisfies the system of implications $(A_m)_{m\in\mathbb{N}}$.*

In the next two theorems, we present the characterization of stabilizers in f.o. and p.q-o. Menger algebras of rank n.

Theorem 4.1.3. *A nonempty subset H of a f.o. Menger algebra (G, o, ζ) of rank n is its stabilizer if and only if it is a quasi-stable, l-unitary and ζ-saturated normal v-complex, which for all $g_1, g_2 \in G$, $t_1, t_2 \in T_n(G)$ satisfies the implication*

$$g_1 \leqslant g_2 \wedge t_1(g_1) \in H \wedge t_2(g_2) \in H \longrightarrow t_2(g_1) \in H. \tag{4.1.8}$$

Proof. We omit the proof of the necessity of (4.1.8) because it is not difficult. To prove the sufficiency assume that all the conditions of the theorem are satisfied and consider the representation $P = P_1 + P_{(\varepsilon_v^*(H),\, W_v(H))}$ of a Menger algebra (G, o) by n-place functions, where P_1 is an isomorphism of a f.o. Menger algebra (G, o, ζ) onto some f.o. Menger algebra of n-place functions and $P_{(\varepsilon_v^*(H),\, W_v(H))}$ is the simplest representation of (G, o). First we show that

$$\zeta \subset \zeta_{(\varepsilon_v^*(H),\, W_v(H))}.$$

Let $g_1 \leqslant g_2$ and $g_1[\bar{x}] \notin W_v(H)$ for some $\bar{x} \in B$. Then there exists $t_1 \in T_n(G)$ such that $t_1(g_1[\bar{x}]) \in H$. In view of the stability of ζ from $g_1 \leqslant g_2$ we conclude $t(g_1[\bar{x}]) \leqslant t(g_2[\bar{x}])$. But by the assumption, H is also ζ-saturated, so $t(g_2[\bar{x}]) \in H$. Conversely, if $t(g_2[\bar{x}]) \in H$, then, according to $g_1[\bar{x}] \leqslant g_2[\bar{x}]$ and $t_1(g_1[\bar{x}]) \in H$, by (4.1.8), we obtain $t(g_1[\bar{x}]) \in H$. Thus $g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon_v(H))$, i.e., $(g_1, g_2) \in \zeta_{(\varepsilon_v^*(H),\, W_v(H))}$. Hence we have shown $\zeta \subset \zeta_{(\varepsilon_v^*(H),\, W_v(H))}$. Further, in a similar way as in the proof of Lemma 4.1.1, we show that P is an isomorphism of (G, o, ζ), and H is the $\varepsilon_v(H)$-class disjoint with $W_v(H)$, which corresponds to the common fixed point of all functions $P(g)$, where $g \in H$, and only these functions. □

Theorem 4.1.4. *For a nonempty subset H of a p.q-o. Menger algebra (G, o, χ) of rank n to be its stabilizer it is necessary and sufficient to be a quasi-stable l-unitary normal v-complex.*

Proof. Let all the conditions of the theorem are fulfilled. Consider the representation $P = P_1 + P_{(\varepsilon_v^*(H),\, \varnothing)}$ of a Menger algebra (G, o) by n-place functions, where P_1 is an isomorphism of (G, o, χ). Then $\chi = \chi_{P_1}$ and P is one-to-one. Since $P_{(\varepsilon_v^*(H),\, \varnothing)}$ is a representation of (G, o) by full n-place functions, we have $\chi \subset \chi_{(\varepsilon_v^*(H),\, \varnothing)} = G \times G$, whence $\chi_P = \chi_{P_1} \cap \chi_{(\varepsilon_v^*(H),\varnothing)} = \chi \cap G \times G = \chi$. So, P is a faithful representation of (G, o, χ). But H is a normal v-complex, therefore, obviously, H is a $\varepsilon_v(H)$-class, which (as in the previous proofs) implies that $P(H)$ is the stabilizer of some point. □

Corollary 4.1.5. *For a nonempty subset H of a Menger algebra (G, o) of rank n the following two statements are equivalent:*
(a) *H is a stabilizer of a Menger algebra (G, o),*
(b) *H is a quasi-stable l-unitary normal v-complex of (G, o).*

Proof. The implication (a)⟶(b) is obvious. To prove the converse implication assume that (b) is valid and consider the relation $\chi = G\times G$. Then $(G, o, G\times G)$ is a p.q-o. Menger algebra, which, according to Theorem 4.1.4, means that the subset H is its stabilizer. So, i.e., (b) implies (a). □

Lemma 4.1.6. *A nonempty subset H of a strong f.o.p. Menger algebra (G, o, ζ, χ) of rank $n \geqslant 2$ is its reversive stabilizer if and only if it is quasi-stable, l-unitary and there exists a χ-saturated subset $A \subset G$ containing H, such that*

$$(x, y) \in \varepsilon\langle\rho, A\rangle \wedge x \in H \longrightarrow y \in H, \tag{4.1.9}$$

$$(x, y) \in \varepsilon\langle\rho, A\rangle \wedge u[\bar{w}|_i\, x] \in A \longrightarrow u[\bar{w}|_i\, y] \in A \tag{4.1.10}$$

for all $x, y \in G$, $u \in G \cup \{e_i\}$, $\bar{w} \in G^n$, $i = 1, \ldots, n$, where ρ means $\zeta^{-1} \circ \zeta \cup H \times H$.

Proof. Necessity. Let $(\Phi, O, \zeta_\Phi, \chi_\Phi)$ be a f.o.p. Menger algebra of reversive n-place functions, H^a_Φ — the stabilizer of a, A_a — the set of all $\varphi \in \Phi$ for which $(a, \ldots, a) \in \mathrm{pr}_1\varphi$. First we show, using induction, that for all $m \in \mathbb{N}$ and $\alpha, \beta \in \Phi$ the following implication

$$(\alpha, \beta) \in \varepsilon_m\langle\rho_\Phi, A_a\rangle \longrightarrow \alpha(a, \ldots, a) = \beta(a, \ldots, a), \tag{4.1.11}$$

where $\rho_\Phi = \zeta_\Phi^{-1} \circ \zeta_\Phi \cup H^a_\Phi \times H^a_\Phi$, is fulfilled. Indeed, $(\alpha, \beta) \in \varepsilon_0\langle\rho_\Phi, A_a\rangle$ means that $\alpha \subset \gamma$, $\beta \subset \gamma$ for some $\gamma \in \Phi$, and $(a, \ldots, a) \in \mathrm{pr}_1\,\alpha \cap \mathrm{pr}_1\,\beta$ or $\alpha(a, \ldots, a) = a = \beta(a, \ldots, a)$. So, in both cases $\alpha(a, \ldots, a) = \beta(a, \ldots, a)$, which proves (4.1.11) for $m = 0$. Assume now, that it is fulfilled for some fixed $m \in \mathbb{N}$. If $(\alpha, \beta) \in \varepsilon_{m+1}\langle\rho_\Phi, A_a\rangle$, then $\alpha = \varphi[\bar{\psi}|_q\gamma] \in A_a$, $\beta = \varphi[\bar{\psi}|_q\chi] \in A_a$, $(f[\bar{g}|_r\gamma], \theta) \in \varepsilon_m\langle\rho_\Phi, A_a\rangle$, $(\theta, f[\bar{g}|_r\chi]) \in \varepsilon_m\langle\rho_\Phi, A_a\rangle$ for some $\varphi, \chi, \gamma, f, \theta \in \Phi$, $\bar{\psi}, \bar{g} \in \Phi^n$, $q, r \in \{1, \ldots, n\}$. Whence

$$f[\bar{g}|_r\gamma](a, \ldots, a) = \theta(a, \ldots, a) = f[\bar{g}|_r\chi](a, \ldots, a),$$

which proves

$$\gamma(a, \ldots, a) = \chi(a, \ldots, a),$$

because all functions in $\mathbf{\Phi}$ are reversive. Therefore

$$\alpha(a,\ldots,a) = \varphi[\bar{\psi}|_q\gamma](a,\ldots,a) = \varphi(\bar{\psi}(a,\ldots,a)|_q\gamma(a,\ldots,a))$$
$$= \varphi(\bar{\psi}(a,\ldots,a)|_q\chi(a,\ldots,a)) = \varphi[\bar{\psi}|_q\chi](a,\ldots,a) = \beta(a,\ldots,a),$$

where $\bar{\psi}(a,\ldots,a) = (\psi_1(a,\ldots,a),\ldots,\psi_n(a,\ldots,a))$. Thus, (4.1.11) is true for $m+1$, and consequently, for every $m \in \mathbb{N}$.

Now let $(\alpha,\beta) \in \varepsilon\langle\rho_{\mathbf{\Phi}}, A_a\rangle$ and $\alpha \in H^a_{\mathbf{\Phi}}$. Then $(\alpha,\beta) \in \varepsilon_m\langle\rho_{\mathbf{\Phi}}, A_a\rangle$ for some $m \in \mathbb{N}$, whence, by (4.1.11), we obtain $\alpha(a,\ldots,a) = \beta(a,\ldots,a)$. But $\alpha(a,\ldots,a) = a$, so, $\beta(a,\ldots,a) = a$, i.e., $\beta \in H^a_{\mathbf{\Phi}}$. This proves (4.1.9). The proof of (4.1.10) is analogous.

Sufficiency. Let all the conditions of the lemma be fulfilled. The subset A is χ-saturated, so, the subset $A' = G \setminus A$ is an l-ideal of a Menger algebra (G,o). Indeed, if $u[\bar{w}|_i x] \in A$, then $u[\bar{w}|_i x] \sqsubset x$ together with the v-negativeness of χ imply $x \in A$. Now let $(g_1,g_2) \in \varepsilon_m\langle\rho, A\rangle$, $x[\bar{y}|_i g_1] \sqsubset g$ and $x[\bar{y}|_i g_2] \in A$. But $\varepsilon_m\langle\rho, A\rangle \subset \varepsilon\langle\rho, A\rangle$, so, $(g_1,g_2) \in \varepsilon\langle\rho, A\rangle$ and $x[\bar{y}|_i g_2] \in A$. This, by (4.1.10), gives $x[\bar{y}|_i g_1] \in A$. Hence $g \in A$ because A is χ-saturated. So, the subset A is $\langle\rho,\chi\rangle$-closed. Thus, the ordered pair (ε^*, A'), where $\varepsilon^* = \varepsilon\langle\rho, A\rangle \cup \{(e_1,e_1),\ldots,(e_n,e_n)\}$, $e_1,\ldots,e_n$ — the selectors in (G^*, o^*), is a determining pair, and $P_{(\varepsilon^*,A')}$ is the simplest representation of a Menger algebra (G,o) by reversive n-place functions. As $H \times H \subset \varepsilon\langle\rho, A\rangle$, then H is contained in some $\varepsilon\langle\rho, A\rangle$-class. Further, if $(x,y) \in \varepsilon\langle\rho, A\rangle$ and $x \in H$, then, according to (4.1.9), we have $y \in H$. Thus, H is an $\varepsilon\langle\rho, A\rangle$-class disjoint from A'.

Let $g_1 \leqslant g_2$ and $g_1[\bar{x}] \in A$ for some $\bar{x} \in G^n \cup \{(e_1,\ldots,e_n)\}$. Since ζ is l-regular, then $g_1[\bar{x}] \leqslant g_2[\bar{x}]$, which together with $\zeta \subset \chi$ and the χ-saturation of A implies $g_2[\bar{x}] \in A$, therefore $(g_1[\bar{x}], g_2[\bar{x}]) \in \varepsilon\langle\rho, A\rangle$. Hence $(g_1,g_2) \in \zeta_{(\varepsilon^*,\, A')}$, and consequently, $\zeta \subset \zeta_{(\varepsilon^*,\, A')}$. The proof of $\chi \subset \chi_{(\varepsilon^*,\, A')}$ is similar.

Now let P_1 be an arbitrary faithful representation of a strong f.o.p. Menger algebra (G,o,ζ,χ) by reversive n-place functions. Then the representation

$$P = P_1 + P_{(\varepsilon^*,\, A')}$$

is an isomorphism of (G,o,ζ,χ) such that H is the preimage of the stabilizer of some point. The proof of this fact is identical to the corresponding part of the proof of Lemma 4.1.1. □

The $\langle\rho,\chi\rangle$-closure of H in a strong f.o.p. Menger algebra (G,o,ζ,χ) of rank n, where $\rho = \zeta^{-1} \circ \zeta \cup H \times H$, will be denoted by $F[H]$.

Theorem 4.1.7. *A nonempty subset H of a strong f.o.p. Menger algebra (G,o,ζ,χ) of rank $n \geqslant 2$ is its reversive stabilizer if and only if it is quasi-stable, l-unitary and satisfies the condition*

$$(x,y) \in \varepsilon\langle\rho, F[H]\rangle \wedge x \in H \longrightarrow y \in H \tag{4.1.12}$$

for all $x,y \in G$.

Proof. *Necessity*. Let H^a_Φ be the stabilizer of the point a of a f.o.p. Menger algebra $(G, o, \zeta_\Phi, \chi_\Phi)$ of reversive n-place functions. First of all, let us show that for any $m, k \in \mathbb{N}$ and $\alpha, \beta \in \mathbf{\Phi}$ the following conditions are fulfilled:

$$\alpha \in F_k(H^a_\Phi) \longrightarrow \alpha\langle a, \dots, a\rangle \neq \varnothing, \tag{4.1.13}$$

$$(\alpha, \beta) \in \varepsilon_m\langle \rho_\Phi, F_k[H^a_\Phi]\rangle \longrightarrow \alpha\langle a, \dots, a\rangle = \beta\langle a, \dots, a\rangle, \tag{4.1.14}$$

where $\rho_\Phi = \zeta_\Phi^{-1} \circ \zeta_\Phi \cup H^a_\Phi \times H^a_\Phi$ and $F_k(H^a_\Phi)$ are as in Section 2.1 (p. 35) with $\rho = \rho_\Phi$, $\sigma = \chi_\Phi$. Assume that $P(k)$ is the one-place predicate over the set of natural numbers, which is the conjunction of the conditions (4.1.13) and (4.1.14), where α, β, m are connected by the universal quantifier. For $k = 0$ and $\alpha \in H^a_\Phi = F_0(H^a_\Phi)$, we have $\alpha(a, \dots, a) = a$, which implies $\alpha\langle a, \dots, a\rangle \neq \varnothing$. If $(\alpha, \beta) \in \varepsilon_m\langle \rho_\Phi, F_0(H^a_\Phi)\rangle = H^a_\Phi \times H^a_\Phi$, then $\alpha\langle a, \dots, a\rangle = \{a\} = \beta\langle a, \dots, a\rangle$. So, $P(0)$ is fulfilled.

Assume now that $P(k)$ is fulfilled for some fixed $k \in \mathbb{N}$ and consider $\alpha \in F_{k+1}(H^a_\Phi)$. Then $(\varphi_1, \varphi_2) \in \varepsilon_m\langle \rho_\Phi, F_0(H^a_\Phi)\rangle$, $\mathrm{pr}_1\psi[\bar{\chi}|_i\varphi_1] \subset \mathrm{pr}_1\alpha$ and $\psi[\bar{\chi}|_i\varphi_2] \in F_k(H^a_\Phi)$ for some $\varphi_1, \varphi_2, \psi \in \mathbf{\Phi}$, $\bar{\chi} \in \mathbf{\Phi}^n$, $i \in \{1, \dots, n\}$. According to the assumption, this last condition means that $\psi[\bar{\chi}|_i\,\varphi_2]\langle a, \dots, a\rangle \neq \varnothing$, whence $\varphi_2\langle a, \dots, a\rangle \neq \varnothing$. Therefore from the first condition we obtain $\varphi_1\langle a, \dots, a\rangle = \varphi_2\langle a, \dots, a\rangle \neq \varnothing$. Thus

$$\varphi_1(a, \dots, a) = \varphi_2(a, \dots, a)$$

and

$$\begin{aligned}\varnothing \neq \psi[\bar{\chi}|_i\,\varphi_2]\langle a, \dots, a\rangle &= \{\psi(\bar{\chi}(a, \dots, a)|_i\,\varphi_2(a, \dots, a))\}\\ &= \{\psi(\bar{\chi}(a, \dots, a)|_i\,\varphi_1(a, \dots, a))\} = \psi[\bar{\chi}|_i\,\varphi_1]\langle a, \dots, a\rangle,\end{aligned}$$

whence $(a, \dots, a) \in \mathrm{pr}_1\psi[\bar{\chi}|_i\,\varphi_1] \subset \mathrm{pr}_1\alpha$. So, (4.1.13) is valid for $k+1$. Now let $(\alpha, \beta) \in \varepsilon_0\langle \rho_\Phi, F_{k+1}(H^a_\Phi)\rangle$, i.e., $\alpha, \beta \in H^a_\Phi$ or $\alpha \subset \gamma$, $\beta \subset \gamma$ for some $\gamma \in \mathbf{\Phi}$ and $\alpha, \beta \in F_{k+1}(H^a_\Phi)$. Since (4.1.13) is true for $k+1$, $(a, \dots, a) \in \mathrm{pr}_1\alpha \cap \mathrm{pr}_1\beta$. Hence from $\alpha \subset \gamma$, $\beta \subset \gamma$ we have $\alpha(a, \dots, a) = \gamma(a, \dots, a) = \beta(a, \dots, a)$, and consequently, $\alpha\langle a, \dots, a\rangle = \beta\langle a, \dots, a\rangle$. For $\alpha, \beta \in H^a_\Phi$, the last equality is evident. So, condition (4.1.14) is true for $k+1$ and $m = 0$. Suppose that it is valid for $k+1$ and some $m \in \mathbb{N}$. If

$$(\alpha, \beta) \in \varepsilon_{m+1}\langle \rho_\Phi, F_{k+1}(H^a_\Phi)\rangle,$$

then we have $\alpha = \varphi[\bar{\psi}|_q\,\gamma] \in F_{k+1}(H^a_\Phi)$, $\beta = \varphi[\bar{\psi}|_q\,\chi] \in F_{k+1}(H^a_\Phi)$ and $\{(f[\bar{g}|_r\,\gamma], \theta), (\theta, f[\bar{g}|_r\,\chi])\} \subset \varepsilon_m\langle \rho_\Phi, F_{k+1}(H^a_\Phi)\rangle$ for some $\varphi, \chi, \gamma, f, \theta \in \mathbf{\Phi}$, $\bar{\psi}, \bar{g} \in \mathbf{\Phi}^n$, $q, r \in \{1, \dots, n\}$. The last inclusion together with the above supposition implies

$$f[\bar{g}|_r\,\gamma]\langle a, \dots, a\rangle = \theta\langle a, \dots, a\rangle = f[\bar{g}|_r\,\chi]\langle a, \dots, a\rangle.$$

But our functions are reversive, so

$$\gamma\langle a, \dots, a\rangle = \chi\langle a, \dots, a\rangle \neq \varnothing.$$

Consequently

$$\begin{aligned}\alpha\langle a,\ldots,a\rangle &= \{\varphi(\bar{\psi}(a,\ldots,a)|_q\,\gamma(a,\ldots,a))\}\\ &= \{\varphi(\bar{\psi}(a,\ldots,a)|_q\,\chi(a,\ldots,a))\} = \beta\langle a,\ldots,a\rangle.\end{aligned}$$

This proves that (4.1.14) is true for $k+1$ and $m+1$. Hence, by induction, it is valid for $k+1$ and all $m \in \mathbb{N}$. So $P(k+1)$ is fulfilled. From the above we conclude that $P(k)$ is fulfilled for all $k \in \mathbb{N}$.

Now let $(\alpha,\beta) \in \varepsilon\langle\rho_\Phi, F[H^a_\Phi]\rangle$ and $\alpha \in H^a_\Phi$. Then one can find such $m,k \in \mathbb{N}$ for which $(\alpha,\beta) \in \varepsilon_m\langle\rho_\Phi, F_k(H^a_\Phi)\rangle$, which, according to (4.1.13) and (4.1.14), gives $\alpha\langle a,\ldots,a\rangle = \beta\langle a,\ldots,a\rangle \neq \varnothing$. So, $\alpha \in H^a_\Phi$ implies $\beta\langle a,\ldots,a\rangle = \{a\}$, i.e., $\beta \in H^a_\Phi$. The condition (4.1.12) is proved.

Sufficiency. Observe that the set $F[H]$ plays the same role in our theorem as A in Lemma 4.1.6. Indeed, H is contained in $F[H]$ and $F[H]$ is χ-saturated, because it is $\langle\rho,\chi\rangle$-closed, where $\rho = \zeta^{-1}\circ\zeta \cup H\times H$. The condition (4.1.12) corresponds to the implication (4.1.9). Moreover, if $(x,y) \in \varepsilon\langle\rho_\Phi, F[H]\rangle$ and $u[\bar{w}|_i\,x] \in F[H]$, then, according to (2.1.9), there exists $m \in \mathbb{N}$ such that $(x,y) \in \varepsilon_m\langle\rho_\Phi, F[H]\rangle$. But $u[\bar{w}|_i\,y] \sqsubset u[\bar{w}|_i\,y]$, $u[\bar{w}|_i\,x] \in F[H]$ and $F[H]$ is $\langle\rho,\chi\rangle$-closed, so $u[\bar{w}|_i\,y] \in F[H]$. This means that the analog of (4.1.10) is also satisfied. Therefore all the conditions of Lemma 4.1.6 are fulfilled and H is a reversive stabilizer of a strong p.q-o. Menger algebra. □

As it is not difficult to see that condition (4.1.12) is equivalent to the system $(B_m)_{m\in\mathbb{N}}$, where

$$B_m:\quad (x,y) \in \varepsilon_m\langle\rho_\Phi, F_m(H)\rangle \wedge x \in H \longrightarrow y \in H,$$

which for every fixed $m \in \mathbb{N}$ can be written in the elementary form by a finite number of steps.

Using the same method as in the proof of Theorem 4.1.3, we can prove that *a nonempty subset H of a strong f.o. Menger algebra (G,o,ζ) of rank $n \geqslant 2$ is a reversive stabilizer if and only if it is a quasi-stable, strong, l-unitary and ζ-saturated subset of G.* But in this case, we must consider the isomorphic representation $P = P_1 + P_{(\varepsilon^*_v(H),\,W_v(H))}$, where P_1 is some faithful representation of (G,o,ζ), and $P_{(\varepsilon^*_v(H),\,W_v(H))}$ is the simplest representation of (G,o) by reversive n-place functions. The last assertion is a consequence of the fact that H is a strong subset of (G,o).

From the above argument we obtain the following corollary.

Corollary 4.1.8. *For a nonempty subset H of a strong Menger algebra (G,o) of rank $n \geqslant 2$ the following two conditions are equivalent:*
(a) *H is a reversive stabilizer of (G,o),*
(b) *H is a quasi-stable strong and l-regular algebra subset of (G,o).*

Proof. The implication (a)$\longrightarrow$(b) is obvious. To prove the converse implication assume that (b) is valid and consider a strong f.o. Menger algebra $(G, o, \overset{\wedge}{\zeta})$, where $\overset{\wedge}{\zeta}$ is defined by (3.3.6). Let $x \in H$ and $(x, y) \in \overset{\wedge}{\zeta}$, i.e., $x \in H$ and $\hat{y} \subset \hat{x}$. Since $x \in H$, we have $\hat{x} \subset H$, whence $\hat{y} \subset H$ and as a consequence $y \in H$. So, H is $\overset{\wedge}{\zeta}$-saturated. Thus, H is a reversive stabilizer of $(G, o, \overset{\wedge}{\zeta})$. This proves (a). □

Let $F^0[H]$ be the $\langle\rho^0, \chi\rangle$-closure of $H \subset G$ in a strong p.q-o. Menger algebra (G, o, χ) of rank $n \geqslant 2$, where $\rho^0 = \triangle_G \cup H \times H$.

Theorem 4.1.9. *For a nonempty subset H of G to be a reversive stabilizer of a strong p.q-o. Menger algebra (G, o, χ) of rank $n \geqslant 2$ it is necessary and sufficient to be quasi-stable, l-unitary and the following condition*

$$(x, y) \in \varepsilon\langle\rho^0, F^0[H]\rangle \wedge x \in H \longrightarrow y \in H \tag{4.1.15}$$

to be satisfied for all $x, y \in G$.

Proof. Let H^a_Φ be the stabilizer of a in a p.q-o. Menger algebra (Φ, O, χ_Φ) of reversive n-place functions. Since $\triangle_\Phi \subset \zeta_\Phi$, we have $F^0[H^a_\Phi] \subset F[H^a_\Phi]$, whence $\varepsilon\langle\rho^0_\Phi, F^0[H^a_\Phi]\rangle \subset \varepsilon\langle\rho_\Phi, F[H^a_\Phi]\rangle$, where ρ_Φ, $F[H^a_\Phi]$ are as in the proof of Theorem 4.1.7 and $\rho^0_\Phi = \triangle_\Phi \cup H^a_\Phi \times H^a_\Phi$. So, if $(\alpha, \beta) \in \varepsilon\langle\rho^0_\Phi, F^0[H^a_\Phi]\rangle$ and $\alpha \in H^a_\Phi$, then $(\alpha, \beta) \in \varepsilon\langle\rho_\Phi, F[H^a_\Phi]\rangle$ and $\alpha \in H^a_\Phi$, which, by Theorem 4.1.7, implies that $\beta \in H^a_\Phi$. This proves the necessity of condition (4.1.15).

To prove the sufficiency consider the representation $P = P_1 + P_{(\varepsilon^*, W)}$ of a Menger algebra (G, o) by reversive n-place functions, where P_1 is an isomorphism of (G, o, χ) onto some p.q-o. Menger algebra of reversive n-place functions and (ε^*, W) is the determining pair of (G, o), where

$$\varepsilon^* = \varepsilon\langle\rho^0, F^0[H]\rangle \cup \{(e_1, e_1), \ldots, (e_n, e_n)\}, \quad W = G \setminus F^0[H],$$

$e_1, \ldots, e_n$ — the selectors of (G^*, o^*). Analogously as in the proof of Lemma 4.1.6 we prove that H is an ε-class disjoint from W and P is an isomorphism of (G, o, χ, H) onto some system $(\Phi, O, \chi_\Phi, H^a_\Phi)$ of reversive n-place functions. This completes our proof. □

Comparing the conditions (4.1.12) and (4.1.15), we come to the conclusion that (4.1.15) is equivalent to the system of conditions $\left(B^0_m\right)$, where B^0_m is obtained from B_m as a result of the change of ρ into ρ^0. Similarly as B_m, every B^0_m can be written in elementary form by a finite number of steps.

Now we consider Menger algebras of the form $(\Phi, O, \cap, \rhd)$. Such algebras are called *restrictive $\cap$-Menger algebras of n-place functions.* The abstract analog of such algebras is called a *restrictive $\curlywedge$-Menger algebra of rank n* and is denoted by $(G, o, \curlywedge, \blacktriangleright)$. Stabilizers in such algebras are described by the following theorem.

Theorem 4.1.10. *For a nonempty subset H to be a stabilizer of a restrictive $\curlywedge$-Menger algebra $(G, o, \curlywedge, \blacktriangleright)$ of rank n it is necessary and sufficient to be a α_d-saturated,*[1] *l-unitary, quasi-stable and $\curlywedge$-stable*[2] *subset and there exists such α_s-saturated, $\blacktriangleright$-stable subset A of a set G, containing H, that*

$$g_1 \in A \wedge g_2 \in H \longrightarrow g_1 \blacktriangleright g_2 \in H \tag{4.1.16}$$

for all $g_1, g_2 \in G$.

Proof. Necessity. Let $(\Phi, O, \cap, \rhd)$ be some restrictive $\cap$-Menger algebra of n-place functions and H_Φ^a be the stabilizer of a point a. In the same way as in the proof of Lemma 4.1.1 we can prove that H_Φ^a is l-unitary and quasi-stable. If $\varphi \in H_\Phi^a$ and $(\varphi, \psi) \in \alpha_d$, where $\varphi, \psi \in \Phi$, then $\varphi(a, \ldots, a) = a$ and $\varphi \subset \psi$, whence $\psi(a, \ldots, a) = a$, hence $\psi \in H_\Phi^a$. So, H_Φ^a is α_d-saturated. Obviously $H_\Phi^a \subset A_a = \{\varphi \in \Phi \mid (a, \ldots, a) \in \mathrm{pr}_1\varphi\}$.

Now let $(\varphi, \psi) \in \alpha_s$, where $\varphi, \psi \in \Phi$, and $\varphi \in A_a$. Then $\mathrm{pr}_1\varphi \subset \mathrm{pr}_1\psi$ and $(a, \ldots, a) \in \mathrm{pr}_1\varphi$. Therefore $(a, \ldots, a) \in \mathrm{pr}_1\psi$, whence $\psi \in A_a$. So, A_a is α_s-saturated. It is obvious that A_a is $\rhd$-stable. Finally, let $\varphi \in A_a$ and $\psi \in H_\Phi^a$, then clearly $(a, \ldots, a) \in \mathrm{pr}_1\varphi$ and $\psi(a, \ldots, a) = a$, whence $(\varphi \rhd \psi)(a, \ldots, a) = a$, i.e., $(\varphi \rhd \psi) \in H_\Phi^a$. This proves (4.1.16).

Sufficiency. Let all the conditions of the theorem be fulfilled. Since $(G, o, \curlywedge, \blacktriangleright)$ is a restrictive $\curlywedge$-Menger algebra of rank n, it is isomorphic to some restrictive $\cap$-Menger algebra of n-place functions. Let P_1 be this isomorphism.

Consider the binary relation

$$\varepsilon[A] = \{(g_1, g_2) \mid g_1 \curlywedge g_2 \in A \ \vee \ g_1, g_2 \in A'\}$$

on the given algebra, where $A' = G \setminus A$. It is a v-regular equivalence. Indeed, the reflexivity and the symmetry of $\varepsilon[A]$ are obvious. To prove the transitivity let $(g_1, g_2) \in \varepsilon[A]$ and $(g_2, g_3) \in \varepsilon[A]$. If $g_1, g_2, g_3 \in A'$, then, evidently, $(g_1, g_3) \in \varepsilon[A]$. Therefore, suppose that $g_1 \curlywedge g_2 \in A$ and $g_2 \curlywedge g_3 \in A$. Then $(g_1 \curlywedge g_2) \blacktriangleright (g_2 \curlywedge g_3) \in A$ because A is $\blacktriangleright$-stable. Hence, by (3.6.14), we have $g_3 \curlywedge ((g_1 \curlywedge g_2) \blacktriangleright g_2) \in A$. According to (3.6.15), the last condition may be written as $g_3 \curlywedge (g_1 \curlywedge g_2) \in A$. Since $g_3 \curlywedge (g_1 \curlywedge g_2) \leqslant g_1 \curlywedge g_3$ and $\alpha_d \subset \alpha_s$ then from the α_s-saturatability of A we obtain $g_1 \curlywedge g_3 \in A$. Therefore $(g_1, g_3) \in \varepsilon[A]$. So, the transitivity of $\varepsilon[A]$ is proved. Hence, $\varepsilon[A]$ is an equivalence relation.

It is also v-regular. Indeed, let $(g_1, g_2) \in \varepsilon[A]$. If $u[\bar{w}|_i g_1], u[\bar{w}|_i g_2] \in A'$, where $u \in G$, $\bar{w} \in G^n$, then, evidently, $u[\bar{w}|_i g_1] \equiv u[\bar{w}|_i g_2](\varepsilon[A])$. But from $g_1 \curlywedge g_2 \in A$ and $u[\bar{w}|_i g_1] \in A$, according to the $\blacktriangleright$-stability of A, we obtain $(g_1 \curlywedge g_2) \blacktriangleright u[\bar{w}|_i g_1] \in A$, whence $u[\bar{w}|_i (g_1 \curlywedge g_2) \blacktriangleright g_1] \in A$, by (3.6.1). Hence

[1] The definitions of relations α_d and α_s see p. 137.

[2] A subset H of G is called $*$-*stable*, where $* \in \{\curlywedge, \blacktriangleright\}$, if for every $x, y \in G$ the condition $x, y \in H \longrightarrow x * y \in H$ is fulfilled.

$u[\bar{w}|_i g_1 \curlywedge g_2] \in A$. Since $u[\bar{w}|_i g_1 \curlywedge g_2] \leqslant u[\bar{w}|_i g_2]$ and A is α_d-saturated, then $u[\bar{w}|_i g_2] \in A$. Similarly we show that $u[\bar{w}|_i g_1] \in A$ follows from $g_1 \curlywedge g_2 \in A$ and $u[\bar{w}|_i g_2] \in A$. Thus $u[\bar{w}|_i g_1]$ and $u[\bar{w}|_i g_2]$ belong or not belong to A simultaneously. In the case when $u[\bar{w}|_i g_1], u[\bar{w}|_i g_2] \in A$ it is not difficult to show that $u[\bar{w}|_i g_1 \curlywedge g_2] \leqslant u[\bar{w}|_i g_1]$ and $u[\bar{w}|_i g_1 \curlywedge g_2] \leqslant u[\bar{w}|_i g_2]$. Therefore $u[\bar{w}|_i g_1 \curlywedge g_2] \leqslant u[\bar{w}|_i g_1] \curlywedge u[\bar{w}|_i g_2]$. Since $u[\bar{w}|_i g_1 \curlywedge g_2] \in A$, we have $u[\bar{w}|_i g_1] \curlywedge u[\bar{w}|_i g_2] \in A$. Hence $u[\bar{w}|_i g_1] \equiv u[\bar{w}|_i g_2](\varepsilon[A])$. So, $\varepsilon[A]$ is i-regular for every $i = 1, \ldots, n$, therefore it is v-regular.

By assumption the set A is α_s-saturated, hence it is an l-ideal of the restrictive $\curlywedge$-Menger algebra $(G, o, \curlywedge, \blacktriangleright)$.

Now we are able to prove that the mapping $P_{(\varepsilon[A],A')} : g \mapsto P_{(\varepsilon[A],A')}(g)$ is a representation of a restrictive $\curlywedge$-Menger algebra $(G, o, \curlywedge, \blacktriangleright)$ by n-place functions. To prove this fact it is sufficient to verify that the pair $(\varepsilon[A], A')$ satisfies the conditions (3.4.8), (3.4.9), (3.4.10) of Proposition 3.4.2 and the conditions (3.6.7), (3.6.8) of Proposition 3.6.4. Let $g_1 \curlywedge g_2 \in A$. Given that $g_1 \curlywedge g_2 \leqslant g_1$ and A is α_d-saturated, we obtain $g_1 \in A$. This proves (3.4.8). Besides, from $g_1 \curlywedge g_2 \in A$, it follows that $(g_1, g_2) \in \varepsilon[A]$, which verifies (3.4.9). Now let $g_1 \in A$ and $(g_1, g_2) \in \varepsilon[A]$, then $g_1 \curlywedge g_2 \in A$, whence $(g_1 \curlywedge g_2) \curlywedge g_1 \in A$, hence $g_1 \curlywedge g_2 \equiv g_1(\varepsilon[A])$. So, condition (3.4.10) is fulfilled. Next, let $g_1 \blacktriangleright g_2 \in A$. Since $(g_1 \blacktriangleright g_2, g_1) \in \alpha_s$, then $g_1 \in A$. It is known that $(g_1 \blacktriangleright g_2, g_1) \in \alpha_s$, whence $(g_1 \blacktriangleright g_2) \curlywedge g_2 = g_1 \blacktriangleright g_2$, therefore $(g_1 \blacktriangleright g_2) \curlywedge g_2 \in A$. This gives $g_1 \blacktriangleright g_2 \equiv g_2(\varepsilon[A])$. Thus (3.6.7) is also valid. Now let $g_1, g_2 \in A$. By the $\blacktriangleright$-stability of A we obtain $g_1 \blacktriangleright g_2 \in A$, from which follows $g_1 \blacktriangleright g_2 \equiv g_2(\varepsilon[A])$. This verifies (3.6.8). So, $P_{(\varepsilon[A],A')}$ is a representation of $(G, o, \curlywedge, \blacktriangleright)$ by n-place functions.

Consider now the representation $P = P_1 + P_{(\varepsilon[A],A')}$ of a restrictive $\curlywedge$-Menger algebra $(G, o, \curlywedge, \blacktriangleright)$ by n-place. It is clear that this representation is faithful, because P_1 is an isomorphism. Note that a subset H is an $\varepsilon[A]$-class, which does not coincides with A'. Indeed, $H \subset A$, hence, evidently, $H \neq A'$. Moreover, if $g_1, g_2 \in H$, then, by the $\curlywedge$-stability of H, we obtain $g_1 \curlywedge g_2 \in H$, whence $g_1 \curlywedge g_2 \in A$. Therefore $g_1 \equiv g_2(\varepsilon[A])$. If $g_1 \in H$ and $g_1 \equiv g_2(\varepsilon[A])$, then $g_1 \in A$, hence $g_2 \in A$. But $g_1 \equiv g_2(\varepsilon[A])$ and $g_1, g_2 \in A$, according to the definition of $\varepsilon[A]$, implies $g_1 \curlywedge g_2 \in A$. So, $g_1 \curlywedge g_2 \in A$ and $g_1 \in H$, therefore, by (4.1.16), we obtain $(g_1 \curlywedge g_2) \blacktriangleright g_1 \in H$, whence, by (3.6.15), we have $g_1 \curlywedge g_2 \in H$. This together with $g_1 \curlywedge g_2 \leqslant g_2$ implies $g_2 \in H$. So, H is an $\varepsilon[A]$-class disjoint from A'.

Let $(X_i)_{i \in I}$ be the family of $\varepsilon[A]$-classes disjoint from A' and $H = X_k$, where $k \in I$. We show that

$$g \in H \longleftrightarrow P(g)(k, \ldots, k) = k. \tag{4.1.17}$$

Indeed, if $g \in H$, then the quasi-stability of H implies $g[g \cdots g] \in H$. Whence, by the v-regularity of $\varepsilon[A]$, we get $g[H \cdots H] \subset H$. Therefore

$P_{(\varepsilon[A],\,A')}(g)(k,\ldots,k) = k$. Hence $P(g)(k,\ldots,k) = k$. Conversely, if the last equality is valid, then $g[H\cdots H] \subset H$. Thus $g[h\cdots h] \in H$ for some $h \in H$, which by l-unitarity of H, proves $g \in H$. So, condition (4.1.17) is true. We have shown in this way that H is a stabilizer of a restrictive $\curlywedge$-Menger algebra $(G, o, \curlywedge, \blacktriangleright)$. □

Let $(G, o, \curlywedge, \blacktriangleright)$ be a restrictive $\curlywedge$-Menger algebra of rank n. A subset X of this algebra is called *C-closed*, if it satisfies the condition

$$a \blacktriangleright b = g \blacktriangleright a \blacktriangleright b \,\wedge\, a, b \in X \longrightarrow g \in X \tag{4.1.18}$$

for all $a, b, g \in G$. Note that every C-closed subset is α_s-saturated and $\blacktriangleright$-stable. Indeed, putting $a = b = x$ and $g = y$ in (4.1.18), we obtain

$$x = y \blacktriangleright x \,\wedge\, x \in X \longrightarrow y \in X$$

for all $x, y \in G$. The last formula means that X is α_s-saturated. Furthermore, if $x, y \in X$, then, according to (4.1.18), from $x \blacktriangleright y = (x \blacktriangleright y) \blacktriangleright x \blacktriangleright y$, follows $x \blacktriangleright y \in X$. So, X is $\blacktriangleright$-stable.

For each subset X we define the subset $E(X)$ in the following way:

$$E(X) = \{g \in G \,|\, a \blacktriangleright b = g \blacktriangleright a \blacktriangleright b \;\text{ for some }\; a, b \in X\}, \tag{4.1.19}$$

where $g, a, b \in G$. It is not difficult to show that the following conditions

(i) $X \subset E(X)$ for all $X \subset G$,

(ii) $X \subset Y \longrightarrow E(X) \subset E(Y)$ for all $X, Y \subset G$,

(iii) $E(X) = X$ for each C-closed subset $X \subset G$

are true. We denote the least C-closed subset, containing X, by $C(X)$.

Proposition 4.1.11. *For each subset X of a restrictive $\curlywedge$-Menger algebra $(G, o, \curlywedge, \blacktriangleright)$ of rank n the equality*

$$C(X) = \bigcup_{m=0}^{\infty} E_m(X)$$

is true, where $E_0(X) = X$ and $E_m(X) = E(E_{m-1}(X))$, $m \in \mathbb{N}$.

Proof. Since $X \subset C(X)$, we have $E(X) \subset E(C(X)) = C(X)$. So, $E(X) \subset C(X)$, hence $E_2(X) = E(E(X)) \subset E(C(X)) = C(X)$. Therefore $E_2(X) \subset C(X)$. Similarly we can prove that $E_m(X) \subset C(X)$ for every $m \in \mathbb{N}$. Hence $\bigcup_{m=0}^{\infty} E_m(X) \subset C(X)$. Let us show that the subset $\bigcup_{m=0}^{\infty} E_m(X)$ is C-closed. Indeed, if $a \blacktriangleright b = g \blacktriangleright a \blacktriangleright b$ and $a, b \in \bigcup_{m=0}^{\infty} E_m(X)$, then there exist $m_1, m_2 \in \mathbb{N}$

such that $a \in E_{m_1}(X)$ and $b \in E_{m_2}(X)$. Let $m = \max\{m_1, m_2\}$, then $E_{m_1}(X) \subset E_m(X)$ and $E_{m_2}(X) \subset E_m(X)$. Thus, we have $a \blacktriangleright b = g \blacktriangleright a \blacktriangleright b$ and $a, b \in E_m(X)$. Hence by (4.1.19) we obtain $g \in E_{m+1}(X)$, therefore $g \in \bigcup\limits_{m=0}^{\infty} E_m(X)$. So, the subset $\bigcup\limits_{m=0}^{\infty} E_m(X)$ is C-closed. Since $C(X)$ is the least C-closed subset, containing X, then $C(X) \subset \bigcup\limits_{m=0}^{\infty} E_m(X)$, which completes the proof. □

Theorem 4.1.12. *For a nonempty subset H to be a stabilizer of a restrictive $\curlywedge$-Menger algebra $(G, o, \curlywedge, \blacktriangleright)$ of rank n it is necessary and sufficient that it is an α_d-saturated, l-unitary, quasi-stable and $\curlywedge$-stable subset satisfying for all $g_1, g_2 \in G$ the condition*

$$g_1 \in C(H) \wedge g_2 \in H \longrightarrow g_1 \blacktriangleright g_2 \in H. \tag{4.1.20}$$

Proof. Let $(\Phi, O, \cap, \rhd)$ be a restrictive $\cap$-Menger algebra of n-place functions and H_Φ^a be its stabilizer of a point a. At first we show, using induction, that for each $m \in \mathbb{N}$ the implication

$$f \in E_m(H_\Phi^a) \longrightarrow (a, \ldots, a) \in \mathrm{pr}_1 f, \tag{4.1.21}$$

where $f \in F$, is valid. For $m = 0$, it is obviously satisfied. Suppose that it is satisfied for some $k \in \mathbb{N}$ and consider $f \in E_{k+1}(H_\Phi^a)$. Then $\alpha \rhd \beta = f \rhd \alpha \rhd \beta$ and $\alpha, \beta \in E_k(H_\Phi^a)$ for some $\alpha, \beta \in F$. The last equality means that $\mathrm{pr}_1(\alpha \rhd \beta) \subset \mathrm{pr}_1 f$. By inductive assumption, from $\alpha, \beta \in E_k(H_\Phi^a)$ we obtain $(a, \ldots, a) \in \mathrm{pr}_1\alpha$ and $(a, \ldots, a) \in \mathrm{pr}_1\beta$. Therefore $(a, \ldots, a) \in \mathrm{pr}_1(\alpha \rhd \beta)$, hence $(a, \ldots, a) \in \mathrm{pr}_1 f$. So, this implication is true for $k+1$, and consequently, it valid for every $m \in \mathbb{N}$.

Now let $\alpha \in C(H_\Phi^a)$ and $\beta \in H_\Phi^a$, then $\beta(a, \ldots, a) = a$ and $\alpha \in E_m(H_\Phi^a)$ for some $m \in \mathbb{N}$. Hence, by (4.1.21), we obtain $(a, \ldots, a) \in \mathrm{pr}_1\alpha$. Therefore $(\alpha \rhd \beta)(a, \ldots, a) = a$ and so $\alpha \rhd \beta \in H_\Phi^a$. The condition (4.1.21) is proved.

The other conditions of the theorem were verified in the proof of Theorem 4.1.10. In this way, we have proved the necessity of all the conditions of Theorem 4.1.12.

Conversely, let all the conditions of Theorem 4.1.12 be satisfied. As it has been shown above, the subset $C(H)$ is α_d-saturated, $\blacktriangleright$-stable and $H \subset C(H)$. Thus, a restrictive $\curlywedge$-Menger algebra $(G, o, \curlywedge, \blacktriangleright)$ satisfies all the conditions formulated in Theorem 4.1.10. Therefore H is a stabilizer of this algebra. So, the sufficiency of the conditions of Theorem 4.1.12 is proved. □

Note that using induction it is not difficult to prove the following proposition:

Proposition 4.1.13. *For any subset X of a restrictive $\curlywedge$-Menger algebra $(G, o, \curlywedge, \blacktriangleright)$ of rank n and for all $g \in G$, $m \in \mathbb{N}$ the condition $g \in E_m(X)$ is fulfilled if and only if there are elements $a_i, b_i \in G$, $i = 1, 2, \ldots, 2^m - 1$ such that the formula*

$$A_m : \quad \begin{cases} a_1 \blacktriangleright b_1 = g \blacktriangleright a_1 \blacktriangleright b_1, \\ \displaystyle\bigwedge_{i=1}^{2^{m-1}-1} \begin{pmatrix} a_{2i} \blacktriangleright b_{2i} = a_i \blacktriangleright a_{2i} \blacktriangleright b_{2i}, \\ a_{2i+1} \blacktriangleright b_{2i+1} = b_i \blacktriangleright a_{2i+1} \blacktriangleright b_{2i+1} \end{pmatrix}, \\ \displaystyle\bigwedge_{i=2^{m-1}}^{2^m-1} (a_i \in X \wedge b_i \in X) \end{cases}$$

holds.

In accordance with Proposition 4.1.11, condition (4.1.20) is equivalent to the system $(A'_m)_{m\in\mathbb{N}}$, where

$$A'_m \; : \; g_1 \in E_m(H) \; \wedge \; g_2 \in H \longrightarrow g_1 \blacktriangleright g_2 \in H.$$

Next, by Proposition 4.1.13 we show that A'_m is equivalent to the condition

$$A''_m \; : \; g_2 \in H \wedge A_m \longrightarrow g_1 \blacktriangleright g_2 \in H$$

for all $g_1, g_2, a_i, b_i \in G$, $i = 1, 2, \ldots, 2^m - 1$.

Thus we have proved the following theorem:

Theorem 4.1.14. *For a nonempty subset H to be a stabilizer of a restrictive $\curlywedge$-Menger algebra $(G, o, \curlywedge, \blacktriangleright)$ of rank n it is necessary and sufficient that it is an α_d-saturated, l-unitary, quasi-stable and $\curlywedge$-stable subset satisfying the conditions $(A''_m)_{m\in\mathbb{N}}$.*

4.2 Stabilizers of functional Menger systems

In a similar way as in the case of Menger algebras, we will say that a nonempty subset H of G is a *stabilizer* of a functional Menger system $\mathcal{G} = (G, o, R_1, \ldots, R_n)$ if there exists a representation P of $\mathcal{G}$ by n-place functions on some set A such that $H = H_P^a$ for some point $a \in A$, where

$$H_P^a = \{g \in G \,|\, P(g)(a, \ldots, a) = a\}.$$

The description of stabilizers of functional Menger systems is based on the properties of two binary relations $\leqslant$ and $\sqsubset$ defined in the following way:

$$x \leqslant y \longleftrightarrow x = y[R_1 x \cdots R_n x], \qquad x \sqsubset y \longleftrightarrow R_1 x \leqslant R_1 y.$$

It is not difficult to see that the first relation coincides with the relation α_d defined in § 3.6, while the second one with α_s. From Proposition 3.6.3 and condition (3.6.4) we see that the first relation is a stable order, the second one – an l-regular v-negative quasi-order containing $\leqslant$, which satisfy the condition (3.2.2). Thus, the system $(G, o, \alpha_d, \alpha_s)$ is a f.o.p. Menger algebra of rank n, i.e., it satisfies all conditions of Theorem 3.2.1.

From the results obtained in Section 3.7, it follows that

$$x \leqslant y \longrightarrow R_i x \leqslant R_i y, \qquad x \sqsubset y \longleftrightarrow R_i x \leqslant R_i y,$$
$$x \sqsubset y \longleftrightarrow x[R_1 y \cdots R_n y] = x, \qquad (R_i x)[y_1 \cdots y_n] \leqslant y_i,$$
$$x[R_1 y_1 \cdots R_n y_n] \leqslant x, \qquad R_i x = R_i R_k x,$$

for all $x, y, x_i, y_i \in G$ and $i, k = 1, \ldots, n$.

Remember that every pair $(\mathcal{E}, W)$, where $\mathcal{E}$ is a v-regular equivalence defined on (G, o) and W is the empty set or an l-ideal of (G, o) which is an $\mathcal{E}$-class, determines the family $(H_a)_{a \in A_{\mathcal{E}}}$ of all $\mathcal{E}$-classes (uniquely indexed by elements of some set $A_{\mathcal{E}}$) such that $H_a \neq W$. Using this family, we can define for every $g \in G$ an n-place function $P_{(\mathcal{E},W)}(g)$ on $A_{\mathcal{E}}$ putting

$$P_{(\mathcal{E},W)}(g)(a_1, \ldots, a_n) = b \longleftrightarrow g[H_{a_1} \cdots H_{a_n}] \subset H_b, \tag{4.2.1}$$

where $(a_1, \ldots, a_n) \in \operatorname{pr}_1 P_{(\mathcal{E},W)}(g) \longleftrightarrow g[H_{a_1} \cdots H_{a_n}] \cap W = \varnothing$, and H_b is the $\mathcal{E}$-class which contains all elements of the form $g[h_1 \cdots h_n]$, $h_i \in H_{a_i}$, $i = 1, \ldots, n$. It is not difficult to see that the map $P_{(\mathcal{E},W)}\colon g \mapsto P_{(\mathcal{E},W)}(g)$ satisfies the identity

$$P_{(\mathcal{E},W)}(g[g_1 \cdots g_n]) = P_{(\mathcal{E},W)}(g)[P_{(\mathcal{E},W)}(g_1) \cdots P_{(\mathcal{E},W)}(g_n)], \tag{4.2.2}$$

i.e., $P_{(\mathcal{E},W)}$ is a representation of (G, o) by n-place functions.

Furthermore, let $T_n^\star(G)$ be the set of transformations $t\colon x \mapsto t(x)$ on G, where x is an individual variable, such that:

(a) $x \in T_n^\star(G)$,

(b) if $t(x) \in T_n^\star(G)$, then $a[\bar{b}\,|_i t(x)] \in T_n^\star(G)$ and $R_i t(x) \in T_n^\star(G)$ for all $a \in G$, $\bar{b} \in G^n$ and $i = 1, \ldots, n$,

(c) $T_n^\star(G)$ contains only elements determined in (a) and (b).

It is clear that $T_n(G) \subset T_n^\star(G)$ (see p. 26). We shall say that a subset H of a functional Menger system $\mathcal{G}$ is *a normal $v^\star$-complex*, if the condition

$$g_1, g_2, t(g_1) \in H \longrightarrow t(g_2) \in H$$

is true for all $g_1, g_2 \in G$, $t \in T_n^\star(G)$.

Using the set $T_n^\star(G)$ we can give the first characterization of stabilizers of a functional Menger system.

Theorem 4.2.15. *A nonempty subset H of G is a stabilizer of a functional Menger system $\mathcal{G} = (G, o, R_1, \ldots, R_n)$ of rank n if and only if it is a quasi-stable l-unitary normal $v^\star$-complex contained in some subset A of G such that $R_i A \subset H$, $R_i(G \setminus A) \subset G \setminus A$ and the conditions* (4.1.1), (4.1.2), *and* (4.1.3) *are satisfied for all $x, y \in G$, $\bar{w} \in G^n$, $t \in T_n^\star(G)$ and $i = 1, \ldots, n$, where the symbol $u[\bar{w}\,|_i\ ]$ may be empty.*[3]

Proof. The proof of the necessity of the conditions of the theorem is similar to the proof of the necessity of the conditions of Lemma 4.1.1. So, we prove only the sufficiency of these conditions.

Let H and A be two subsets of G satisfying all the conditions formulated in the theorem. First we shall prove that H is α_d-saturated and A is an α_s-saturated subset, i.e., that the following implications

$$x \leqslant y \wedge x \in H \longrightarrow y \in H, \tag{4.2.3}$$

$$x \sqsubset y \wedge x \in A \longrightarrow y \in A \tag{4.2.4}$$

are true.

Indeed, $x \leqslant y$ means $x = y[R_1 x \cdots R_n x]$. Since $x \in H$, $H \subset A$ and $R_i A \subset H$, we have $R_i x \in H$ for every $i = 1, \ldots, n$. This, together with the fact that H is an l-unitary normal $v^\star$-complex, implies that H is v-unitary. So, $y[R_1 x \cdots R_n x] \in H$ and $R_i x \in H$ for every $i = 1, \ldots, n$, which by the v-unitarity of H gives $y \in H$. This proves (4.2.3).

Now, if $x \sqsubset y$ and $x \in A$, then also $R_1 x \leqslant R_1 y$, i.e., $R_1 x = (R_1 y)[R_1 x \cdots R_n x]$. From $R_i A \subset H$, it follows that $R_i x \in H$, so, applying the v-unitarity of H to $(R_1 y)[R_1 x \cdots R_n x] \in H$ we obtain $R_1 y \in H$, whence we get $R_1 y \in A$. Since $R_i(G \setminus A) \subset G \setminus A$ means that

$$R_i x \in A \longrightarrow x \in A \tag{4.2.5}$$

for every $x \in G$ and $i = 1, \ldots, n$, from $R_1 y \in A$ follows $y \in A$. This completes the proof of (4.2.4).

Thus in a f.o.p. Menger algebra $(G, o, \alpha_d, \alpha_s)$ the subsets H and A satisfy all conditions of Lemma 4.1.1. Now consider the representation $P = P_1 + P_{(\varepsilon, A')}$ of a Menger algebra (G, o) by n-place functions, where P_1 is some faithful representation of a functional Menger system $\mathcal{G}$ of rank n, and $P_{(\varepsilon, A')}$ is the representation, which corresponds to the pair (ε, A'), where $\varepsilon = \varepsilon_v(H) \cap \varepsilon_v(A)$, $A' = G \setminus A$. Since the representation P_1 is faithful,[4] then the representation P is also faithful. As in Lemma 4.1.1 we can show that $g \in H$ if and only if $P_{(\varepsilon, A')}(g)(c, \ldots, c) = c$, where c is the element used to the indexation of the ε-class H.

[3] If $u[\bar{w}\,|_i\ ]$ is the empty symbol, then $u[\bar{w}\,|_i x]$ is equal to x.

[4] Note that by Theorem 3.7.6 P_1 is also a representation of $(G, o, \alpha_d, \alpha_s)$, because $\alpha_d = \zeta_P$ and $\alpha_s = \chi_P$.

To finish the proof of the theorem we must show that $P_{(\varepsilon,\,A')}$ satisfies the identity

$$P_{(\varepsilon,A')}(R_ig) = \mathcal{R}_i P_{(\varepsilon,A')}(g). \tag{4.2.6}$$

Indeed, for every

$$\bar{a} = (a_1, \ldots, a_n) \in \mathrm{pr}_1\, P_{(\varepsilon,A')}(R_ig),$$

where $H_{a_i} = \varepsilon\langle x_i\rangle$, $x_i \in A$, $i = 1, \ldots, n$, we have $(R_ig)[x_1 \cdots x_n] \in A$, whence, applying (3.7.10), we obtain $x_i[R_1g[x_1 \cdots x_n] \cdots R_ng[x_1 \cdots x_n]] \in A$. As A' is an l-ideal, the last condition implies $R_ig[x_1 \cdots x_n] \in A$ for $i = 1, \ldots, n$. Thus $g[x_1 \cdots x_n] \in A$, because in the case $g[x_1 \cdots x_n] \notin a$ we have $x_i \in G\backslash A$ and therefore $\bar{a} \in \mathrm{pr}_1\, P_{(\varepsilon,A')}(g)$. This proves the inclusion $\mathrm{pr}_1\, P_{(\varepsilon,A')}(R_ig) \subset \mathrm{pr}_1\, P_{(\varepsilon,A')}(g)$.

To prove the converse inclusion, let $\bar{a} \in \mathrm{pr}_1\, P_{(\varepsilon,A')}(g)$, where $H_{a_i} = \varepsilon\langle x_i\rangle$ with $x_i \in A$ for $i = 1, \ldots, n$. Then $g[x_1 \cdots x_n] \in A$, which, by $R_kA \subset H \subset A$, gives $R_kg[x_1 \cdots x_n] \in A$. Whence, in view of (3.7.9), we deduce $R_k(R_i)[x_1 \cdots x_n] \in A$ for $k, i = 1, \ldots, n$. From this, by (4.2.5), we get $(R_ig)[x_1 \cdots x_n] \in A$. Thus $\bar{a} \in \mathrm{pr}_1\, P_{(\varepsilon,A')}(R_ig)$ for every $i = 1, \ldots, n$.

In this way we have proved

$$\mathrm{pr}_1\, P_{(\varepsilon,A')}(R_ig) = \mathrm{pr}_1\, P_{(\varepsilon,A')}(g) = \mathrm{pr}_1\, \mathcal{R}_i P_{(\varepsilon,A')}(g) \tag{4.2.7}$$

for every $g \in G$.

Let $\bar{a} \in \mathrm{pr}_1\, P_{(\varepsilon,A')}(R_ig)$, i.e., $(R_ig)[x_1 \cdots x_n] \in A$, where $H_{a_i} = \varepsilon\langle x_i\rangle$, $x_i \in A$, $i = 1, \ldots, n$. Applying the stability of $\leqslant$ to $(R_ig)[x_1 \cdots x_n] \leqslant x_i$ we obtain $t((R_ig)[x_1 \cdots x_n]) \leqslant t(x_i)$ for every $t \in T_n(G)$ and each $i = 1, \ldots, n$. If $t((R_ig)[x_1 \cdots x_n]) \in H$, then, according (4.2.3), we get $t(x_i) \in H$. Conversely if $t(x_i) \in H$, from $(R_ig)[x_1 \cdots x_n] \leqslant x_i$ and $(R_ig)[x_1 \cdots x_n] \in A$ we can deduce, using (4.1.5), that $t((R_ig)[x_1 \cdots x_n]) \in H$. Consequently, $(R_ig)[x_1 \cdots x_n] \equiv x_i(\varepsilon_v(H))$. Similarly we can prove that $(R_ig)[x_1 \cdots x_n] \equiv x_i(\varepsilon_v(A))$. Thus $(R_ig)[x_1 \cdots x_n] \equiv x_i(\varepsilon)$ for every $i = 1, \ldots, n$. This means that $P_{(\varepsilon,A')}(R_ig)(a_1, \ldots, a_n) = a_i$. So,

$$P_{(\varepsilon,A')}(R_ig)(a_1, \ldots, a_n) = (\mathcal{R}_i P_{(\varepsilon,A')}(g))(a_1, \ldots, a_n),$$

which, together with (4.2.7), proves (4.2.6).

From (4.2.2) and (4.2.6), it follows that $P_{(\varepsilon,A')}$ is a representation of $\mathcal{G}$ by n-place functions. Thus $P = P_1 + P_{(\varepsilon,W)}$ is a faithful representation of $\mathcal{G}$ for which $H = H_P^a$. □

Let $\mathcal{G}$ be a functional Menger system of rank n and H be some subset of G. We say that a subset X of G is *C_H-closed*, if for all $a, b, c \in G$, $t \in T_n^{\star}(G)$ the implication:

$$\left.\begin{array}{l} a = b[R_1a \cdots R_na] \vee a, b \in H, \\ t(a)[R_1c \cdots R_nc] = t(a), \\ a, t(b) \in X \end{array}\right\} \longrightarrow c \in X$$

is true. In an abbreviated form this implication can be written as

$$(a \leqslant b \vee a, b \in H) \wedge t(a) \sqsubset c \wedge a, t(b) \in X \longrightarrow c \in X. \tag{4.2.8}$$

Let $C_H(X)$ denote the set of all $c \in G$ for which there exist $a, b \in G$ and $t \in T_n^\star(G)$ such that the premise of (4.2.8) is satisfied. Furthermore, let

$$C_H[X] = \bigcup_{m=0}^{\infty} \overset{m}{C}_H(X),$$

where $\overset{0}{C}_H(X) = X$ and $\overset{m+1}{C}_H(X) = C_H(\overset{m}{C}_H(X))$ for every $m = 0, 1, 2, \ldots$.

By induction we can prove that $g \in \overset{m}{C}_H(X)$ if and only if the following system of conditions is fulfilled:

$$\begin{cases} (a_1 = b_1[R_1 a \cdots R_n a] \vee a_1, b_1 \in H) \wedge t_1(a_1)[R_1 g \cdots R_n g] = t_1(a_1) \\ \bigwedge\limits_{i=1}^{2^{m-1}-1} \begin{pmatrix} a_{2i} = b_{2i}[R_1 a_{2i} \cdots R_n a_{2i}] \vee a_{2i}, b_{2i} \in H, \\ t_{2i}(a_{2i})[R_1 a_i \cdots R_n a_i] = t_{2i}(a_{2i}), \\ a_{2i+1} = b_{2i+1}[R_1 a_{2i+1} \cdots R_n a_{2i+1}] \vee a_{2i+1}, b_{2i+1} \in H, \\ t_{2i+1}(a_{2i+1})[R_1 t_i(b_i) \cdots R_n t_i(b_i)] = t_{2i+1}(a_{2i+1}) \end{pmatrix} \\ \bigwedge\limits_{i=2^{m-1}}^{2^m-1} (a_i \in X \wedge t_i(b_i) \in X) \end{cases}$$

where $a_k, b_k \in G$, $t_k \in T_n^\star(G)$.

The above system of conditions will be denoted by $\mathfrak{M}_H(X, m, g)$.

Theorem 4.2.16. *A nonempty subset H of G is a stabilizer of a functional Menger system $\mathcal{G}$ of rank n if and only if it is a quasi-stable l-unitary normal $v^\star$-complex such that $R_i H \subset H$ for every $i = 1, \ldots, n$ and*

$$x = y[R_1 x \cdots R_n x] \in C_H[H] \wedge u[\bar{w}\,|_i y] \in H \longrightarrow u[\bar{w}\,|_i x] \in H \tag{4.2.9}$$

for all $x, y, u \in G$, $\bar{w} \in G^n$, $i = 1, \ldots, n$, where the symbol $u[\bar{w}\,|_i\ ]$ may be empty.

Proof. Necessity. Let H_Φ^a be the stabilizer of a point a in a functional Menger system $(\mathbf{\Phi}, O, \mathcal{R}_1, \ldots, \mathcal{R}_n)$ of n-place functions. From the first part of the proof of Lemma 4.1.1 follows that H_Φ^a is a quasi-stable l-unitary normal $v^\star$-complex of $(\mathbf{\Phi}, O, \mathcal{R}_1, \ldots, \mathcal{R}_n)$. If $\varphi \in H_\Phi^a$, then $\varphi(a, \ldots, a) = a$, whence $\mathcal{R}_i\varphi(a, \ldots, a) = a$, i.e., $\mathcal{R}_i\varphi \in H_\Phi^a$. So, $\mathcal{R}_i H_\Phi^a \subset H_\Phi^a$ for every $i = 1, \ldots, n$.

To prove (4.2.9), we shall consider $\varphi = \psi[\mathcal{R}_1\varphi \cdots \mathcal{R}_n\varphi] \in C_{H_\Phi^a}[H_\Phi^a]$ and $\alpha[\bar{\omega}\,|_i\psi] \in H_\Phi^a$ for some $\alpha, \varphi, \psi \in \mathbf{\Phi}$, $\bar{\omega} \in \mathbf{\Phi}^n$, $i = 1, \ldots, n$, where $C_{H_\Phi^a}[H_\Phi^a] = \bigcup_{m=0}^{\infty} \overset{m}{C}_{H_\Phi^a}(H_\Phi^a)$. Then $\varphi = \psi[\mathcal{R}_1\varphi \cdots \mathcal{R}_n\varphi] \in \overset{m}{C}_{H_\Phi^a}(H_\Phi^a)$ for some $m \in \mathbb{N}$. But, as it is not difficult to see by induction, $\varphi \in \overset{m}{C}_{H^a_\Phi}(H^a_\Phi)$ implies $(a,\ldots,a) \in \operatorname{pr}_1\varphi$. The above relation means that $(a,\ldots,a) \in \operatorname{pr}_1\varphi$ and $(a,\ldots,a) \in \operatorname{pr}_1\psi[\mathcal{R}_1\varphi\cdots\mathcal{R}_n\varphi]$. Therefore

$$\begin{aligned}\varphi(a,\ldots,a) &= \psi[\mathcal{R}_1\varphi\cdots\mathcal{R}_n\varphi](a,\ldots,a)\\ &= \psi(\mathcal{R}_1\varphi(a,\ldots,a),\ldots,\mathcal{R}_n\varphi(a,\ldots,a)) = \psi(a,\ldots,a).\end{aligned}$$

Moreover, for $\alpha[\bar{\omega}\,|_i\psi] \in H^a_\Phi$ we have

$$\begin{aligned}a = \alpha[\bar{\omega}\,|_i\psi](a,\ldots,a) &= \alpha(\bar{\omega}(a,\ldots,a)\,|_i\psi(a,\ldots,a))\\ &= \alpha(\bar{\omega}(a,\ldots,a)\,|_i\varphi(a,\ldots,a)) = \alpha[\bar{\omega}\,|_i\varphi](a,\ldots,a),\end{aligned}$$

where $\bar{\omega}(a,\ldots,a)$ denotes $\omega_1(a,\ldots,a),\ldots,\omega_n(a,\ldots,a)$. So, $\alpha[\bar{\omega}\,|_i\varphi] \in H^a_\Phi$, which completes the proof of (4.2.9).

Sufficiency. Let H satisfy all the conditions of the theorem. We prove that it satisfies also all the conditions of Theorem 4.2.15. Since, by assumption, H is a quasi-stable l-unitary normal $v^\star$-complex such that $R_iH \subset H$ for every $i = 1,\ldots,n$, we must prove that it satisfies the conditions (4.1.1), (4.1.2), (4.1.3) and that $R_iA \subset H$, $R_i(G\backslash A) \subset G\backslash A$ for some $A \subset G$ containing H and all $i = 1,\ldots,n$.

First let us prove that H satisfies the condition (4.2.3). Indeed, if $x \leqslant y$ and $x \in H$, then $y[R_1x\cdots R_nx] = x \in H$ and $R_ix \in H$, $i = 1,\ldots,n$. Since H is an l-unitary normal $v^\star$-complex, H is also a v-unitary subset. Therefore $y \in H$, which completes the proof of (4.2.3).

Now let $A = C_H[H]$. Clearly we have $H \subset A$. Since $C_H[H] = \bigcup\limits_{m=0}^{\infty} \overset{m}{C}_H(H)$, to prove that $R_iU \subset H$ for every $i = 1,\ldots,n$, it is sufficient to show that

$$R_i(\overset{m}{C}_H(H)) \subset H \tag{4.2.10}$$

for every $m \in \mathbb{N}$. For $m = 0$ it is obvious because $\overset{0}{C}_H(H) = H$ and $R_iH \subset H$ for all $i = 1,\ldots,n$. Suppose that (4.2.10) is true for some $k \in \mathbb{N}$. We prove that it is valid for $k+1$. Let $g \in \overset{k+1}{C}_H(H)$. Then

$$(a \leqslant b \vee a,b \in H) \wedge\ t(a) \sqsubset g \wedge\ a, t(b) \in \overset{k}{C}_H(H)$$

for some $a,b \in G$, $t \in T^\star_n(G)$. From $a, t(b) \in \overset{k}{C}(H)$, according to our supposition, we get $R_ia, R_it(b) \in H$. If $a \leqslant b$ and $a \in H$, then $b \in H$ by (4.2.3). Thus $a, b, R_it(b) \in H$, whence $R_it(a) \in H$, because H is a normal $v^\star$-complex. The condition $t(a) \sqsubset g$ implies $R_it(a) \leqslant R_ig$, which, by (4.2.3), gives $R_ig \in H$. In a similar way $a,b \in H$ and $t(a) \sqsubset g$ prove $R_ig \in H$. We have thus shown that $R_i(\overset{k+1}{C_H}(H)) \subset H$. So, the inclusion (4.2.10) is valid for $m = k+1$ and consequently for every $m \in \mathbb{N}$. Therefore $R_iA \subset H$ for every $i = 1, \dots, n$.

To prove the inclusion $R_i(G \backslash A) \subset G \backslash A$ observe that it is equivalent to the condition

$$(\forall g \in G)\,(g \in G \backslash A \longrightarrow R_ig \in G \backslash A),$$

which can be written in the form

$$(\forall g \in G)\,(R_ig \in A \longrightarrow g \in A). \tag{4.2.11}$$

In the case $A = C_H[H]$ this last condition means that

$$(\forall g \in G)(\forall m \in \mathbb{N}) \left(R_ig \in \overset{m}{C_H}(H) \longrightarrow (\exists n \in \mathbb{N})\, g \in \overset{n}{C_H}(H)\right). \tag{4.2.12}$$

Let $R_ig \in \overset{m}{C_H}(H)$ for some $g \in G$ and $m \in \mathbb{N}$. Considering $R_ig = R_i(R_ig)$ and (4.2.10), we conclude $R_ig \in H$. Thus $R_ig \leqslant R_ig$, $R_ig \sqsubset g$ and $R_ig \in H$ and therefore $g \in C_H(H)$, i.e., $g \in \overset{n}{C_H}(H)$ for some $n \in \mathbb{N}$. This proves (4.2.12). So, $R_i(G \backslash A) \subset G \backslash A$ for $A = C_H[H]$ and $i = 1, \dots, n$.

To prove (4.1.1), observe that for $x, y \in H$ and $t(x) \in A = C_H[H]$, the just proved inclusion implies $R_it(x) \in R_iA \subset H$. Thus $x, y, R_it(x) \in H$. But H is a normal $v^\star$-complex, hence $R_it(y) \in H$. So, for $A = C[H]$, the condition (4.1.1) is satisfied. The condition (4.1.2) is also valid, because for $A = C_H[H]$ it coincides with (4.2.9). Since the subset $C_H[H]$ is C_H-closed, condition (4.1.3) holds too. This means that H satisfies all the conditions of Theorem 4.2.15. Hence, H is a stabilizer. □

As it is not difficult to see, condition (4.2.9) is equivalent to the system of conditions $(A'_m)_{m \in \mathbb{N}}$, where

$$A'_m: \;\; x = y[R_1x \cdots R_nx] \wedge x \in \overset{m}{C_H}(H) \wedge u[\bar{w}\,|_i y] \in H \longrightarrow u[\bar{w}\,|_i x] \in H$$

for all $x, y, u \in G$, $\bar{w} \in G^n$, $i = 1, \dots, n$. Since, $x \in \overset{m}{C_H}(H)$ means that system of conditions $\mathfrak{M}_H(H, m, x)$ is satisfied, condition A'_m can be written as

$$A_m: \;\; x = y[R_1x \cdots R_nx] \wedge \mathfrak{M}_H(H, m, x) \wedge u[\bar{w}\,|_i y] \in H \longrightarrow u[\bar{w}\,|_i x] \in H.$$

So, the last theorem can be rewritten in the following form.

Theorem 4.2.17. *A nonempty subset H of G is a stabilizer of a functional Menger system $\mathcal{G}$ of rank n if and only if it is a quasi-stable l-unitary normal $v^\star$-complex such that $R_iH \subset H$ for every $i = 1, \dots, n$ and the system of conditions $(A_m)_{m \in \mathbb{N}}$ is satisfied.*

Let $\mathcal{G}_\curlywedge = (G, o, \curlywedge, R_1, \ldots, R_n)$ be a functional Menger $\curlywedge$-algebra of rank n. A subset H of G is called $\curlywedge$-*stable* if the implication

$$x \in H \wedge y \in H \longrightarrow x \curlywedge y \in H$$

is valid for all $x, y \in G$.

Theorem 4.2.18. *A nonempty subset H of G is a stabilizer of a functional Menger $\curlywedge$-algebra $\mathcal{G}_\curlywedge$ of rank n if and only if*
(a) *it is a quasi-stable, $\curlywedge$-stable and v-unitary subset of G,*
(b) *there exists a subset A of G such that $H \subset A$, $R_i(G \setminus A) \subset G \setminus A$ and $R_i A \subset H$ for every $i = 1, \ldots, n$,*
(c) *the following two implications*

$$x \in A \wedge y \in H \longrightarrow y[R_1 x \cdots R_n x] \in H, \qquad (4.2.13)$$

$$x \in A \wedge y \in A \longrightarrow y[R_1 x \cdots R_n x] \in A \qquad (4.2.14)$$

are true for all $x, y \in G$.

Proof. The proof of the necessity of the conditions of the theorem is very similar to the proof of the necessity of the conditions of Lemma 4.1.1. So, we only prove the sufficiency of these conditions.

We shall however show first that the implication

$$x \curlywedge y \in A \wedge \, u[\bar{w}\,|_i(y \curlywedge z)] \in A \longrightarrow u[\bar{w}\,|_i(x \curlywedge y \curlywedge z)] \in A, \qquad (4.2.15)$$

where the symbol $u[\bar{w}\,|_i\ ]$ may be empty, is valid for all $x, y, z \in G$ and $i = 1, \ldots, n$. For this purpose suppose that the premise of this implication is fulfilled. Then, according to (4.2.14), we have

$$u[\bar{w}\,|_i(y \curlywedge z)][R_1(x \curlywedge y) \cdots R_n(x \curlywedge y)] \in A.$$

Whence, by (3.7.6), we obtain $u[\bar{w}\,|_i(y \curlywedge z)[R_1(x \curlywedge y) \cdots R_n(x \curlywedge y)]] \in A$. From this, by (3.7.12), we obtain $u[\bar{w}\,|_i(z \curlywedge y[R_1(x \curlywedge y) \cdots R_n(x \curlywedge y)])] \in A$, which, according to (3.7.13), implies $u[\bar{w}\,|_i(x \curlywedge y \curlywedge z)] \in A$. This completes the proof of (4.2.15).

Furthermore, in the same way as in the proof of Theorem 4.2.15, we can show that conditions (4.2.3), (4.2.4), (4.2.5) are satisfied and $G \setminus A$ is an l-ideal. We consider next the relation:

$$\mathcal{E}[A] = \{(g_1, g_2) \in G \times G \,|\, g_1 \curlywedge g_2 \in A \vee g_1, g_2 \in A'\}, \qquad (4.2.16)$$

where $A' = G \setminus A$. The reflexivity and symmetry of this relation are obvious. Let $(g_1, g_2), (g_2, g_3) \in \mathcal{E}[A]$. If $g_1, g_2, g_3 \in A'$, then obviously $(g_1, g_3) \in \mathcal{E}[A]$. If $g_1 \curlywedge g_2, g_2 \curlywedge g_3 \in A$, then, according to (4.2.15), we have $g_1 \curlywedge g_2 \curlywedge g_3 \in A$.

Whence, in view of (4.2.4) and the fact that the relation $\leqslant$ is contained in the relation $\sqsubset$, $g_1 \curlywedge g_2 \curlywedge g_3 \leqslant g_1 \curlywedge g_3$ and $g_1 \curlywedge g_2 \curlywedge g_3 \sqsubset g_1 \curlywedge g_3$, we conclude $g_1 \curlywedge g_3 \in A$, i.e., $(g_1, g_3) \in \mathcal{E}[A]$. So, $\mathcal{E}[A]$ is also transitive. Thus $\mathcal{E}[A]$ is an equivalence relation.

It is v-regular too. Indeed, for $(g_1, g_2) \in \mathcal{E}[A]$ we have $g_1, g_2 \in A'$ or $g_1 \curlywedge g_2 \in A$. Since A' is an l-ideal, in the first case $u[\bar{w}\,|_i g_1], u[\bar{w}\,|_i g_2] \in A'$, i.e.,

$$u[\bar{w}\,|_i g_1] \equiv u[\bar{w}\,|_i g_2](\mathcal{E}[A]). \tag{4.2.17}$$

In the case $g_1 \curlywedge g_2 \in A$, $u[\bar{w}\,|_i g_1], u[\bar{w}\,|_i g_2] \in A'$, (4.2.17) holds too. In the case $g_1 \curlywedge g_2 \in A$, $u[\bar{w}\,|_i g_1] \in A$, according to (4.2.14), we have

$$u[\bar{w}\,|_i g_1][R_1(g_1 \curlywedge g_2) \cdots R_n(g_1 \curlywedge g_2)] \in A,$$

whence, applying (3.7.6), we obtain

$$u[\bar{w}\,|_i g_1[R_1(g_1 \curlywedge g_2) \cdots R_n(g_1 \curlywedge g_2)]] \in A,$$

which, by (3.7.13), implies $u[\bar{w}\,|_i (g_1 \curlywedge g_2)] \in A$. This, together with (4.2.4) and $u[\bar{w}\,|_i (g_1 \curlywedge g_2)] \sqsubset u[\bar{w}\,|_i g_2]$, gives us $u[\bar{w}\,|_i g_2] \in A$.

Similarly, $g_1 \curlywedge g_2 \in A$ and $u[\bar{w}\,|_i g_2] \in A$ imply $u[\bar{w}\,|_i g_1] \in A$ and therefore $u[\bar{w}\,|_i g_1]$ and $u[\bar{w}\,|_i g_2]$ simultaneously belong or do not belong to A.

Let $u[\bar{w}\,|_i g_1], u[\bar{w}\,|_i g_2] \in A$. Since

$$u[\bar{w}\,|_i (g_1 \curlywedge g_2)] \leqslant u[\bar{w}\,|_i g_j]$$

for $j = 1, 2$, from the above we obtain

$$u[\bar{w}\,|_i (g_1 \curlywedge g_2)] \sqsubset u[\bar{w}\,|_i g_1] \curlywedge u[\bar{w}\,|_i g_2],$$

which, after application of (4.2.4), gives us $u[\bar{w}\,|_i (g_1 \curlywedge g_2)] \in A$. Thus (4.2.17) is satisfied in any case. So, the relation $\mathcal{E}[A]$ is i-regular for every $i = 1, \ldots, n$, i.e., it is v-regular.

H is an equivalence class of $\mathcal{E}[A]$. Indeed, if $g_1, g_2 \in H$, then $g_1 \curlywedge g_2 \in H$ by the $\curlywedge$-stability of H. Consequently $g_1 \curlywedge g_2 \in A$, i.e., $g_1 \equiv g_2(\mathcal{E}[A])$. On the other hand, if $g_1 \equiv g_2(\mathcal{E}[A])$ and $g_1 \in H$, then $g_1 \curlywedge g_2 \in A$, whence, by (4.2.13), we get

$$g_1[R_1(g_1 \curlywedge g_2) \cdots R_n(g_1 \curlywedge g_2)] \in H.$$

From this, applying (3.7.13), we deduce $g_1 \curlywedge g_2 \in H$, which, in view of $g_1 \curlywedge g_2 \leqslant g_2$ and (4.2.3), implies $g_2 \in H$. So, H is an equivalence class of $\mathcal{E}[A]$.

Consider the representation $P_{(\mathcal{E}[A],A')}$ of (G, o) induced by the pair $(\mathcal{E}[A], A')$. We shall prove that

$$P_{(\mathcal{E}[A],A')}(g_1 \curlywedge g_2) = P_{(\mathcal{E}[A],A')}(g_1) \cap P_{(\mathcal{E}[A],A')}(g_2), \tag{4.2.18}$$

$$P_{(\mathcal{E}[A],A')}(R_i g) = \mathcal{R}_i P_{(\mathcal{E}[A],A')}(g) \tag{4.2.19}$$

for all $g, g_1, g_2 \in G$ and $i = 1, \ldots, n$.

Let $P_{(\mathcal{E}[A],A')}(g_1 \curlywedge g_2)(a_1, \ldots, a_n) = b$, where H_{a_i}, H_b are equivalence classes of $\mathcal{E}[A]$ containing the elements a_i and $y \in A$, respectively. Then we have $(g_1 \curlywedge g_2)[x_1 \cdots x_n] \equiv y(\mathcal{E}[A])$, whence, in view of (3.7.14), we obtain

$$y \curlywedge g_1[x_1 \cdots x_n] \curlywedge g_2[x_1 \cdots x_n] \in A. \tag{4.2.20}$$

So, $y \curlywedge g_1[x_1 \cdots x_n] \in A$ and $y \curlywedge g_2[x_1 \cdots x_n] \in A$, i.e., $g_1[x_1 \cdots x_n] \equiv y(\mathcal{E}[A])$ and $g_2[x_1 \cdots x_n] \equiv y(\mathcal{E}[A])$. Therefore

$$P_{(\mathcal{E}[A],A')}(g_1)(a_1, \ldots, a_n) = b \ \text{ and } \ P_{(\mathcal{E}[A],A')}(g_2)(a_1, \ldots, a_n) = b. \tag{4.2.21}$$

This proves the inclusion $P_{(\mathcal{E}[A],A')}(g_1 \curlywedge g_2) \subset P_{(\mathcal{E}[A],A')}(g_1) \cap P_{(\mathcal{E}[A],A')}(g_2)$.

To prove the inverse inclusion assume (4.2.21). Then $g_1[x_1 \cdots x_n] \equiv y(\mathcal{E}[A])$ and $g_2[x_1 \cdots x_n] \equiv y(\mathcal{E}[A])$, i.e., $y \curlywedge g_1[x_1 \cdots x_n]$ and $y \curlywedge g_2[x_1 \cdots x_n]$ are in A. From this, applying (4.2.15), we obtain (4.2.20), whence, as it was shown above, we have $P_{(\mathcal{E}[A],A')}(g_1 \curlywedge g_2)(a_1, \ldots, a_n) = b$. Thus, $P_{(\mathcal{E}[A],A')}(g_1) \cap P_{(\mathcal{E}[A],A')}(g_2) \subset P_{(\mathcal{E}[A],A')}(g_1 \curlywedge g_2)$. This completes the proof of (4.2.18).

Further, using the same method as in the proof of condition (4.2.7), we prove that

$$\mathrm{pr}_1\, P_{(\mathcal{E}[A],A')}(R_i g) = \mathrm{pr}_1\, P_{(\mathcal{E}[A],A')}(g) = \mathrm{pr}_1\, \mathcal{R}_i P_{(\mathcal{E}[A],A')}(g)$$

for every $g \in G$. Now, if $(a_1, \ldots, a_n) \in \mathrm{pr}_1\, P_{(\mathcal{E}[A],A')}(R_i g)$, then obviously $(R_i g)[x_1 \cdots x_n] \in A$ for all $x_i \in H_{a_i}$, $i = 1, \ldots, n$. Since $(R_i g)[x_1 \cdots x_n] \leqslant x_i$ implies $(R_i g)[x_1 \cdots x_n] \curlywedge x_i = (R_i g)[x_1 \cdots x_n]$, from the above we conclude $(R_i g)[x_1 \cdots x_n] \curlywedge x_i \in A$. Hence $(R_i g)[x_1 \cdots x_n] \equiv x_i(\mathcal{E}[A])$. Therefore,

$$P_{(\mathcal{E}[A],A')}(R_i g)(a_1, \ldots, a_n) = a_i$$

for all $i = 1, \ldots, n$. This proves (4.2.19). So, the representation $P_{(\mathcal{E}[A],A')}$ of (G, o) is also a representation of a functional Menger $\curlywedge$-algebra $\mathcal{G}_\curlywedge$.

Observe that

$$g \in H \longleftrightarrow P_{(\mathcal{E}[A],A')}(g)(a, \ldots, a) = a, \tag{4.2.22}$$

where a is an element used as an index of the $\mathcal{E}[A]$-class H. In fact, for $g \in H$ the quasi-stability of H implies $g[g \cdots g] \in H$. Whence, $g[H \cdots H] \subset H$ because H is an $\mathcal{E}[A]$-class and the relation $\mathcal{E}[A]$ is v-regular. So, $P_{(\mathcal{E}[A],A')}(g)(a, \ldots, a) = a$. Conversely, if $g[H \cdots H] \subset H$, then $g[h \cdots h] \in H$ for every $h \in H$. From this, by the l-unitarity of H, we get $g \in H$, which completes the proof of (4.2.22).

Since the algebra $\mathcal{G}_\curlywedge$ satisfies the axioms (3.7.4)–(3.7.10), from Theorem 3.7.6, it follows that there exists an isomorphism of the algebra $\mathcal{G}$ onto some functional Menger $\cap$-algebra of n-place functions. Denote this isomorphism by P_1 and consider the representation $P = P_1 + P_{(\mathcal{E}[A],A')}$. It is clear that P is a faithful representation of the algebra $\mathcal{G}_\curlywedge$ by n-place functions and $H = H_P^a$. □

Theorem 4.2.19. *A nonempty subset H of a functional Menger $\curlywedge$-algebra $\mathcal{G}_\curlywedge$ of rank n is its stabilizer if and only if it is a stable, $\curlywedge$-stable and v-unitary subset of G such that $R_iH \subset H$ for every $i = 1, \ldots, n$.*

Proof. The necessity of these conditions is obvious, therefore we shall prove only their sufficiency.

Recall that on p. 168 we noted that relation $\leqslant$ coincides with α_d and relation $\sqsubset$ coincides with α_s. Moreover, as it is not difficult to see, for any subset H of G satisfying all the conditions of the theorem, we have $\alpha_d(H) \subset H$.[5] This inclusion is equivalent to (4.2.3). Indeed, for $x \in H$ and $x \leqslant y$ we have $y[R_1x \cdots R_nx] = x \in H$ and $R_ix \in H$ for every $i = 1, \ldots, n$, whence, according to the v-unitarity of H, we obtain $y \in H$. This proves (4.2.3).

Using this condition we shall prove that $A = \alpha_s(H)$ and H satisfy all conditions of Theorem 4.2.18. The stability of H implies its quasi-stability. Since α_s is a quasi-order we have also $H \subset \alpha_s(H) = A$. Moreover, for every $x \in A$ there exists $h \in H$ such that $(h, x) \in \alpha_s$, i.e., $R_ih \leqslant R_ix$ for every $i = 1, \ldots, n$. Since $R_ih \in H$ for every $h \in H$ and $i = 1, \ldots, n$, the statement above, according to (4.2.3), implies that $R_ix \in H$ for $i = 1, \ldots, n$. Therefore $R_iA \subset H$ for every $i = 1, \ldots, n$.

The inclusion $R_i(G \setminus A) \subset G \setminus A$, where $i = 1, \ldots, n$, is equivalent to the implication $(\forall x \in G)(x \in G \setminus A \longrightarrow R_ix \in G \setminus A$, which, by contraposition, means that

$$R_ix \in A \longrightarrow x \in A$$

for every $x \in G$. But $A = \alpha_s(H)$, so, for every $x \in G$ and every $h \in H$

$$(h, R_ix) \in \alpha_s \longrightarrow x \in \alpha_s(H). \tag{4.2.23}$$

Let $(h, R_ix) \in \alpha_s$ for some $x \in G$ and $h \in H$. Then

$$H \ni h = h[R_1R_ix \cdots R_nR_ix] = h[R_1x \cdots R_nx],$$

whence $(h, x) \in \alpha_s$ and therefore $x \in \alpha_s(H)$. This means that the implication (4.2.23) is valid. So, $R_i(G \setminus A) \subset G \setminus A$ for every $i = 1, \ldots, n$. In this way we have proved that A and H satisfy the first two conditions of Theorem 4.2.18.

To prove that A and H satisfy the third condition of the theorem, we must show that

$$x \in A \wedge y \in H \longrightarrow y[R_1x \cdots R_nx] \in H, \tag{4.2.24}$$

$$x \in A \wedge y \in A \longrightarrow y[R_1x \cdots R_nx] \in A. \tag{4.2.25}$$

Let $x \in A = \alpha_s(H)$ and $y \in H$. Then also $(h, x) \in \alpha_s$, i.e., $h = h[R_1x \cdots R_nx]$ for some $h \in H$. Since from $h \in H$ follows $R_ih \in H$ for all $i = 1, \ldots, n$, we

[5] Recall that for any relation $\rho \subset X \times Y$ and any subset A of X by $\rho(A)$ we denote the set $\{y \in Y \mid (\exists x \in A)(x, y) \in \rho\}$.

have $y \in H$ and $R_i h \in H$, $i = 1, \ldots, n$. This, by the stability of H, implies $y[R_1 h \cdots R_n h] \in H$. Whence, applying (3.7.7), (3.7.4) and (3.7.8), we obtain

$$\begin{aligned} y[R_1 h \cdots R_n h] &= y[R_1 h[R_1 x \cdots R_n x] \cdots R_n h[R_1 x \cdots R_n x]] \\ &= y[(R_1 h)[R_1 x \cdots R_n x] \cdots (R_n h)[R_1 x \cdots R_n x]] \\ &= y[R_1 h \cdots R_n h][R_1 x \cdots R_n x] \\ &= y[R_1 x \cdots R_n x][R_1 h \cdots R_n h]. \end{aligned}$$

Thus $y[R_1 x \cdots R_n x][R_1 h \cdots R_n h] \in H$. But H is v-unitary and $R_i h \in H$ for every $i = 1, \ldots, n$, so, $y[R_1 x \cdots R_n x] \in H$. This proves (4.2.24).

Now let $x, y \in A = \alpha_s(H)$. Then there exist $a, b \in H$ such that $(a, x) \in \alpha_s$ and $(b, y) \in \alpha_s$, i.e., $a = a[R_1 x \cdots R_n x]$ and $b = b[R_1 y \cdots R_n y]$. Because $b \in H$ implies $R_i b \in H$ for $i = 1, \ldots, n$, we have $a, R_1 b, \ldots, R_n b \in H$. Whence, $a[R_1 b \cdots R_n b] \in H$ according to the stability of H. Moreover, applying (3.7.8), (3.7.7) and (3.7.4), we obtain

$$\begin{aligned} a[R_1 b \cdots R_n b] &= a[R_1 x \cdots R_n x][R_1 b \cdots R_n b] \\ &= a[R_1 b \cdots R_n b][R_1 x \cdots R_n x] \\ &= a[R_1 b[R_1 y \cdots R_n y] \cdots R_n b[R_1 y \cdots R_n y]] \\ &= a[(R_1 b)[R_1 y \cdots R_n y] \cdots (R_n b)[R_1 y \cdots R_n y]][R_1 x \cdots R_n x] \\ &= a[R_1 b \cdots R_n b][R_1 y \cdots R_n y][R_1 x \cdots R_n x] \\ &= a[R_1 b \cdots R_n b][(R_1 y)[R_1 x \cdots R_n x] \cdots (R_n y)[R_1 x \cdots R_n x]] \\ &= a[R_1 b \cdots R_n b][R_1(y[R_1 x \cdots R_n x]) \cdots R_n(y[R_1 x \cdots R_n x])]. \end{aligned}$$

Thus, $(a[R_1 b \cdots R_n b], y[R_1 x \cdots R_n x]) \in \alpha_s$, i.e., $y[R_1 x \cdots R_n x] \in \alpha_s(H)$. This completes the proof of (4.2.25).

In this way, we have shown that A and H satisfy all the conditions of Theorem 4.2.18. Therefore, H is the stabilizer of a functional Menger $\curlywedge$-algebra $\mathcal{G}_\curlywedge$. □

4.3 Stationary subsets

The *stationary subset* of Φ, where $\Phi \subseteq \mathcal{F}(A^n, A)$, is the set of all functions from Φ for which there exists at least one fixed point, i.e., the set

$$St(\Phi) = \{\varphi \in \Phi \,|\, (\exists a \in A)\, \varphi(a, \ldots, a) = a\}.$$

Proposition 4.3.1. *The stationary subset $St(\Phi)$ of the algebra (Φ, O) has the following properties:*

$$\varphi \subset \psi \ \wedge \ \varphi \in St(\Phi) \longrightarrow \psi \in St(\Phi), \tag{4.3.1}$$

$$\varphi \in St(\Phi) \longrightarrow \varphi[\varphi \cdots \varphi] \in St(\Phi), \tag{4.3.2}$$

$$\varphi \in St(\Phi) \longrightarrow \mathcal{R}_i\varphi \in St(\Phi), \tag{4.3.3}$$

$$\varphi[\psi \cdots \psi] = \psi \ \wedge \ \psi \in St(\Phi) \longrightarrow \varphi \in St(\Phi), \tag{4.3.4}$$

$$\varphi[\psi \cdots \psi] = \psi \neq 0 \longrightarrow \varphi \in St(\Phi), \tag{4.3.5}$$

$$\varphi[\psi \cdots \psi] \neq 0 \longrightarrow \mathcal{R}_i\varphi \in St(\Phi), \tag{4.3.6}$$

$$\varphi[\psi \cdots \psi] \cap \psi \neq 0 \longrightarrow \varphi \in St(\Phi), \tag{4.3.7}$$

$$0 \notin St(\Phi) \longrightarrow \mathcal{R}_i 0 = 0, \tag{4.3.8}$$

where $\varphi, \psi \in \Phi$, $i = 1, \ldots, n$ *and* 0 *is a zero of* (Φ, O).[6]

Proof. If $\varphi \in St(\Phi)$, then $\varphi(a, \ldots, a) = a$, whence, by $\varphi \subset \psi$, we obtain $\psi(a, \ldots, a) = a$. Thus $\psi \in St(\Phi)$. This proves (4.3.1).

For $\varphi \in St(\Phi)$ we also have

$$\varphi[\varphi \cdots \varphi](a, \ldots, a) = \varphi(\varphi(a, \ldots, a), \ldots, \varphi(a, \ldots, a)) = \varphi(a, \ldots, a) = a.$$

This implies $\varphi[\varphi \cdots \varphi] \in St(\Phi)$. So, (4.3.2) is valid too. The proof of (4.3.3) is analogous.

If $\psi \in St(\Phi)$ and $\varphi[\psi \cdots \psi] = \psi$, then $\psi(a, \ldots, a) = a$ for some $a \in A$. Thus, $\varphi(a, \ldots, a) = \varphi(\psi(a, \ldots, a), \ldots, \psi(a, \ldots, a)) = \varphi[\psi \cdots \psi](a, \ldots, a) = \psi(a, \ldots, a) = a$, i.e., $\varphi \in St(\Phi)$. This proves (4.3.4).

Let now $\varphi[\psi \cdots \psi] = \psi \neq 0$, where 0 is a zero of (Φ, O). Then $\psi \neq \varnothing$ and thus there exists $\bar{a} = (a_1, \ldots, a_n) \in \mathrm{pr}_1\, \psi$. Therefore $\varphi(\psi(\bar{a}) \cdots \psi(\bar{a})) = \varphi[\psi \cdots \psi](\bar{a}) = \psi(\bar{a})$, which implies $\varphi \in St(\Phi)$. The condition (4.3.5) is proved. Similarly we can prove (4.3.6) and (4.3.7).

Observe that

$$0 \neq \varnothing \longleftrightarrow St(\Phi) = \Phi. \tag{4.3.9}$$

Indeed, if $0 \neq \varnothing$, then $0(\bar{a}) = b$ for some $\bar{a} \in A^n$ and $b \in A$. Since $0 = \varphi[0 \cdots 0]$ for every $\varphi \in \Phi$, we have $b = 0(\bar{a}) = \varphi(0(\bar{a}), \ldots, 0(\bar{a})) = \varphi(b, \ldots, b)$. Thus $\varphi \in St(\Phi)$. Consequently, $St(\Phi) = \Phi$. Conversely, if $St(\Phi) = \Phi$, then $0 \in St(\Phi)$. Therefore $0(a, \ldots, a) = a$ for some $a \in A$. So, $0 \neq \varnothing$.

Using the equivalence just proved we can see that in the case where $0 \notin St(\Phi)$, i.e., $St(\Phi) \neq \Phi$, we must have $0 = \varnothing$. Therefore $\mathcal{R}_i 0 = \mathcal{R}_i \varnothing = \varnothing = 0$ for every $i = 1, \ldots, n$. This proves (4.3.8). □

[6] A zero of (G, o) (if it exists) is an element 0 such that $0[yz \cdots u] = x[0yz \cdots u] = x[y0z \cdots u] = \cdots = x[yz \cdots 0] = 0$ for all $x, y, z, \cdots u \in G$. In the algebras of functions, the role of zero is played the empty set, i.e., the empty function. But in some algebras of functions, there exist a nonempty function satisfying the above identities defining zero.

Definition 4.3.2. A nonempty subset H of G is called a *stationary subset* of a Menger algebra (G, o) of rank n if it has a faithful representation P by n-place functions such that

$$g \in H \longleftrightarrow P(g) \in \mathbf{St}(P(G)) \tag{4.3.10}$$

for every $g \in G$, where $P(G) = \{P(g) \mid g \in G\}$.

First we shall describe all stationary subsets of f.o.p., f.o. and p.q-o. Menger algebras. For this purpose we will use the relation

$$\theta_h = \{(g_1, g_2) \in G \times G \mid \exists t \in T_n(G)\ g_1 = t(h),\ g_2 = t(h[h \cdots h])\}.$$

Using this relation, we can consider on a f.o.p. Menger algebra (G, o, ζ, χ) of rank n the new binary relation $\rho_h = \zeta \cup \theta_h$ and the (ρ_h, χ)-closure of any subset $X \subset G$, for every $h \in G$. The (ρ_h, χ)-closure of X will be denoted by $[X]_h$. In the case when $X = \{g, g[g \cdots g]\}$ instead of $[X]_h$ we shall write $[g]_h$.

Theorem 4.3.3. *For a nonempty subset H of a f.o.p. Menger algebra (G, o, ζ, χ) of rank n to be a stationary subset it is necessary and sufficient that for all $m \in \mathbb{N}$, $g, h, x_i, z_i, x_{m+1} \in G$ and $i = 0, 1, \ldots, m$, the following implications are true:*

$$0 \notin H \longrightarrow 0 \leqslant g, \tag{4.3.11}$$

$$\left.\begin{array}{c} \bigwedge\limits_{i=0}^{m} \Big(0 \neq h \sqsubset x_i \leqslant z_i \ \wedge\ 0 \neq h \sqsubset x_{i+1} \leqslant z_i\Big), \\ g[x_0 \cdots x_0] = x_{m+1} \end{array}\right\} \longrightarrow g \in H, \tag{4.3.12}$$

$$\left.\begin{array}{c} \bigwedge\limits_{i=0}^{m} \left(\begin{array}{c} (x_i \leqslant z_i \ \wedge\ x_{i+1} \leqslant z_i) \vee \\ \{x_i, x_{i+1}\} \subset \{t_i(h),\ t_i(h[h \cdots h])\} \end{array}\right), \\ \{x_0, \ldots, x_{m+1}\} \subset [h]_h \ \wedge\ h \in H, \\ g[x_0 \cdots x_0] = x_{m+1} \end{array}\right\} \longrightarrow g \in H, \tag{4.3.13}$$

where 0 is a zero of (G, o), $x \leqslant y \longleftrightarrow (x, y) \in \zeta$, $x \sqsubset y \longleftrightarrow (x, y) \in \chi$ and $h \sqsubset x \leqslant z$ means $h \sqsubset x \wedge x \leqslant z$.

Proof. Necessity. Let $(\Phi, O, \zeta_\Phi, \chi_\Phi)$ be some f.o.p. Menger algebra of n-place functions defined on the set A. Recall that if φ is a zero of (Φ, O), then according to (4.3.9) we have

$$\varphi \neq \varnothing \longleftrightarrow St(\Phi) = \Phi. \tag{4.3.14}$$

Now if $\varphi \notin St(\Phi)$, then $St(\Phi) \neq \Phi$. Hence, according to (4.3.14) we get $\varphi = \varnothing$. Therefore $\varphi \subset \psi$ for every $\psi \in \Phi$. This proves (4.3.11).

To prove (4.3.12), assume that for some $\alpha, \beta_i, \chi_i \in \mathbf{\Phi}$, $i = 0, 1, \ldots, m+1$ the conditions

$$\bigwedge_{i=0}^{m} \big(\varphi \neq \psi \ \wedge \ \mathrm{pr}_1\psi \subset \mathrm{pr}_1\chi_i \cap \mathrm{pr}_1\chi_{i+1} \ \wedge \ \chi_i \subset \beta_i \ \wedge \ \chi_{i+1} \subset \beta_i\big)$$

and $\alpha[\chi_0 \cdots \chi_0] = \chi_{m+1}$, where φ is a zero of $(\mathbf{\Phi}, O)$, are satisfied. Since $\psi \neq \varphi$, then, obviously, $\psi \neq \varnothing$, i.e., $\psi\langle\bar{a}\rangle \neq \varnothing$ for some $\bar{a} \in A^n$. From $\mathrm{pr}_1\psi \subset \mathrm{pr}_1\chi_i \cap \mathrm{pr}_1\chi_{i+1}$ we conclude that $\chi_i\langle\bar{a}\rangle \neq \varnothing$ for all $i = 0, 1, \ldots, m+1$. But $\chi_i \subset \beta_i$, $\chi_{i+1} \subset \beta_i$, so, $\beta_i\langle\bar{a}\rangle \neq \varnothing$ and $\chi_i(\bar{a}) = \beta_i(\bar{a}) = \chi_{i+1}(\bar{a})$ for $i = 0, 1, \ldots, m$. Thus, $\chi_{m+1}(\bar{a}) = \alpha[\chi_0 \cdots \chi_0](\bar{a}) = \alpha(\chi_0(\bar{a}), \ldots, \chi_0(\bar{a})) = \alpha(\chi_{m+1}(\bar{a}), \ldots, \chi_{m+1}(\bar{a}))$. Hence $\alpha \in St(\mathbf{\Phi})$, which proves (4.3.12).

The proof of (4.3.13) is based on the condition

$$\alpha(a, \ldots, a) = a \ \wedge \ \beta \in \overset{m}{E}_\alpha(\alpha) \longrightarrow (a, \ldots, a) \in \mathrm{pr}_1\beta, \tag{4.3.15}$$

where $\alpha, \beta \in \mathbf{\Phi}$, $a \in A$, $m \in \mathbb{N}$ and $E_\alpha(X)$ means $E(X)$ for $\rho = \zeta_{\mathbf{\Phi}} \cup \theta_\alpha$ and $\sigma = \chi_{\mathbf{\Phi}}$.[7] It is clear that for $m = 0$ the above condition is true. Assume that it is valid for some m and consider $\beta \in \overset{m+1}{E}_\alpha(\alpha)$, where $\alpha(a, \ldots, a) = a$. Then $(\varphi_1, \varphi_2) \in \zeta_{\mathbf{\Phi}} \cup \theta_\alpha$, $\mathrm{pr}_1 f[\bar{\psi}|_i\varphi_1] \subset \mathrm{pr}_1\beta$ and $\varphi_1, f[\bar{\psi}|_i\varphi_2] \in \overset{m}{E}_\alpha(\alpha)$. According to our assumption, the last condition implies $(a, \ldots, a) \in \mathrm{pr}_1\varphi_1$ and $(a, \ldots, a) \in \mathrm{pr}_1 f[\bar{\psi}|_i\varphi_2]$. Thus, if $\varphi_1 \subset \varphi_2$, then $\varphi_1(a, \ldots, a) = \varphi_2(a, \ldots, a)$; if $(\varphi_1, \varphi_2) \in \theta_\alpha$, then $\varphi_1 = t(\psi_1)$, $\varphi_2 = t(\psi_2)$ for some $t \in T_n(\mathbf{\Phi})$ and $\psi_1, \psi_2 \in \{\alpha, \alpha[\alpha^n]\}$. Of course $\psi_i(a, \ldots, a) = a$ for $i = 1, 2$. Whence $\varphi_1(a, \ldots, a) = t(\psi_1)(a, \ldots, a) = t(\psi_2)(a, \ldots, a) = \varphi_2(a, \ldots, a)$. Thus, $f[\bar{\psi}|_i\varphi_2](a, \ldots, a) = f(\bar{\psi}(a, \ldots, a)|_i\varphi_2(a, \ldots, a)) = f[\bar{\psi}|_i\varphi_1](a, \ldots, a)$. So, $(a, \ldots, a) \in \mathrm{pr}_1\beta$. This means that (4.3.15) is valid for $m+1$. Hence, by induction, it is valid for every $m \in \mathbb{N}$.

Now let $\alpha \in St(\mathbf{\Phi})$, $\beta[\chi_0 \cdots \chi_0] = \chi_{m+1}$,

$$\bigwedge_{i=0}^{m} \Big((\chi_i \subset \varphi_i \ \wedge \ \chi_{i+1} \subset \varphi_i) \ \vee \ \{\chi_i, \chi_{i+1}\} \subset \{t_i(\alpha), t_i(\alpha[\alpha^n])\}\Big)$$

and $\{\chi_0, \chi_1, \ldots, \chi_{m+1}\} \subset [\alpha]_\alpha$. Because $\alpha(a, \ldots, a) = a$ for some $a \in A$, then, according to (4.3.15), we have $(a, \ldots, a) \in \mathrm{pr}_1\chi_0 \cap \cdots \cap \mathrm{pr}_1\chi_{m+1}$. Further, from

$$\chi_0(a, \ldots a) = \chi_1(a, \ldots a) = \cdots = \chi_{m+1}(a, \ldots, a),$$

we obtain

$$\begin{aligned} \chi_{m+1}(a, \ldots, a) &= \beta[\chi_0 \cdots \chi_0](a, \ldots, a) = \beta(\chi_0(a, \ldots, a), \ldots, \chi_0(a, \ldots, a)) \\ &= \beta(\chi_{m+1}(a, \ldots, a), \ldots, \chi_{m+1}(a, \ldots, a)), \end{aligned}$$

[7] For the definition of $E(X)$ see p. 34.

which implies $\beta \in St(\Phi)$ and proves (4.3.13).

Sufficiency. Let (G, o, ζ, χ) be a f.o.p. Menger algebra of rank n and let H be its nonempty subset satisfying all the conditions of the theorem. Observe that if (G, o) has a zero 0 and 0 belongs to H, then $H = G$. Indeed, according to (4.3.13), for $g \in G$ and $t \in T_n(G)$, from $0 \in H$, $g[0^n] = 0$, $0 \in \{t(0), t(0[0^n])\}$ and $0 \in [0]_0$, it follows that $g \in H$, which was to be proved.

Consider two families of determining pairs $\big((\varepsilon_g^*, W_g)\big)_{g\in G}$ and $((\pi_h^*, V_h))_{h\in H}$ on a Menger algebra (G, o), where

$$\varepsilon_g^* = \varepsilon(\zeta^{-1} \circ \zeta,\, \chi\langle g\rangle) \cup \{(e_1, e_1), \dots, (e_n, e_n)\},$$

$$\pi_h^* = \varepsilon(\zeta^{-1} \circ \zeta \cup \theta_h,\, [h]_h) \cup \{(e_1, e_1), \dots, (e_n, e_n)\},$$

$W_g = G \setminus \chi\langle g\rangle$, $V_h = G \setminus [h]_h$, $e_1, \dots, e_n$ — the selectors in (G^*, o^*), and the following representations of (G, o) by n-place functions:

$$P_1 = \sum_{g\in G} P_{(\varepsilon_g^*, W_g)}\,, \qquad P_1^0 = \sum_{g\in G\setminus\{0\}} P_{(\varepsilon_g^*, W_g)}\,, \qquad P_2 = \sum_{h\in H} P_{(\pi_h^*, V_h)}.$$

Let $P = P_1 + P_2$ in the case when $0 \in H$, and $P = P_1^0 + P_2$ in the case when $0 \notin H$. P, defined in such a way is an isomorphism of (G, o, ζ, χ) onto some f.o.p. Menger algebra of n-place functions for which H is the preimage of the stationary subset. In fact, for $g_1 \leqslant g_2$ we have $g_1[\bar{x}] \leqslant g_2[\bar{x}]$ for any $\bar{x} \in B = G^n \cup \{(e_1, \dots, e_n)\}$. If $g_1[\bar{x}] \notin W_g$, i.e., $g \sqsubset g_1[\bar{x}]$, then

$$(g_1[\bar{x}],\, g_2[\bar{x}]) \in \zeta^{-1} \circ \zeta \cap \chi\langle g\rangle \times \chi\langle g\rangle \subset \varepsilon(\zeta^{-1} \circ \zeta,\, \chi\langle g\rangle).$$

If $g_1[\bar{x}] \notin V_h$, where $h \in H$, then

$$(g_1[\bar{x}],\, g_2[\bar{x}]) \in \zeta^{-1} \circ \zeta \cap [h]_h \times [h]_h \subset \varepsilon(\zeta^{-1} \circ \zeta, [h]_h).$$

Thus, $\zeta \subset \zeta_{(\varepsilon_g^*, W_g)}$ and $\zeta \subset \zeta_{(\pi_h^*, V_h)}$ for all $g \in G$ and $h \in H$. Whence $\zeta \subset \zeta_{P_1} \cap \zeta_{P_2} \subset \zeta_{P_2^0} \cap \zeta_{P_2}$.

Assume now that $(g_1, g_2) \in \zeta_{P_1}$, i.e., that for all $g \in G$ and $\bar{x} \in B$ $g \sqsubset g_1[\bar{x}]$ implies $(g_1[\bar{x}],\, g_2[\bar{x}]) \in \varepsilon(\zeta^{-1} \circ \zeta, \chi\langle g\rangle)$. This gives us, for $g = g_1$ and $\bar{x} = (e_1, \dots, e_n)$

$$(g_1, g_2) \in \varepsilon(\zeta^{-1} \circ \zeta, \chi\langle g_1\rangle),$$

which means that

$$\{(g_1, x_1), (x_1, x_2), \dots, (x_{n-1}, g_2)\} \subset \zeta^{-1} \circ \zeta \cap \chi\langle g_1\rangle \times \chi\langle g_1\rangle$$

for some $x_1, \dots, x_{n-1} \in G$. So, one can find $y_1, \dots, y_n \in G$ such that $g_1 \leqslant y_1$, $x_1 \leqslant y_1$, $x_1 \leqslant y_2$, $x_2 \leqslant y_2, \dots, x_{n-1} \leqslant y_n$, $g_2 \leqslant y_n$. From the conditions $x_1 \leqslant y_1$, $g_1 \sqsubset x_1$ and $g_1 \leqslant y_1$ we conclude $g_1 \leqslant x_1$. Consequently $g_1 \leqslant y_2$.

In the same way $x_2 \leqslant y_2$, $g_1 \sqsubset x_2$ and $g_1 \leqslant y_2$ imply $g_1 \leqslant x_2$. Iterating this procedure, after a finite number of steps, we obtain $g_1 \leqslant g_2$, which proves $\zeta_{P_1} \subset \zeta$. Hence $\zeta = \zeta_{P_1}$.

Now let $0 \notin H$ and $(g_1, g_2) \in \zeta_{P_1^0}$. If $g_1 = 0$, then, according to (4.3.11), $(g_1, g_2) \in \zeta$. If $g_1 \neq 0$, then, similarly as in the previous case, we obtain $(g_1, g_2) \in \zeta$. Thus, if $0 \notin H$, then $\zeta_{P_1^0} \subset \zeta$. So, in this case $\zeta = \zeta_{P_1^0}$.

Summarizing the above discussion we see that $\zeta_P = \zeta_{P_1} \cap \zeta_{P_2} = \zeta \cap \zeta_{P_2} = \zeta$ if $0 \in H$, and $\zeta_P = \zeta_{P_1^0} \cap \zeta_{P_2} = \zeta \cap \zeta_{P_2} = \zeta$ if $0 \notin H$. Hence $\zeta = \zeta_P$ in both cases, which, in particular, proves that the representation P is faithful, because ζ is an ordering relation.

Let $g_1 \sqsubset g_2$. Then $g_1[\bar{x}] \sqsubset g_2[\bar{x}]$ for every $\bar{x} \in B$. If $g \sqsubset g_1[\bar{x}]$, then $g \sqsubset g_2[\bar{x}]$. If $g_1[\bar{x}] \in [h]_h$ for some $h \in H$, then also $g_2[\bar{x}] \in [h]_h$, because the subset $[h]_h$ is χ-saturated. Thus $\chi \subset \chi_{(\varepsilon_g^*, W_g)}$ and $\chi \subset \chi_{(\pi_h^*, V_h)}$ for all $g \in G$, $h \in H$. Hence $\chi \subset \chi_{P_1} \cap \chi_{P_2} \subset \chi_{P_1^0} \cap \chi_{P_2}$. Further, from $(g_1, g_2) \in \chi_{P_1^0}$, i.e., from $g \sqsubset g_1[\bar{x}]$, it follows that $g \sqsubset g_2[\bar{x}]$ for any $g \in G \setminus \{0\}$ and $\bar{x} \in B$, which for $g = g_1$ and $\bar{x} = (e_1, \ldots, x_n)$ gives $g_1 \sqsubset g_2$. For $g_1 = 0$, the v-negativeness of χ implies $0 = 0[g_2 \cdots g_2] \sqsubset g_2$. Therefore $\chi_{P_1^0} \subset \chi$, whence $\chi = \chi_{P_1^0}$, and consequently $\chi \subset \chi_{P_1} \subset \chi_{P_1^0} = \chi$. Thus $\chi = \chi_{P_1}$. Finally, $\chi_P = \chi_{P_1} \cap \chi_{P_2} = \chi \cap \chi_{P_2} = \chi$ for $0 \in H$ and $\chi_P = \chi_{P_1^0} \cap \chi_{P_2} = \chi \cap \chi_{P_2} = \chi$ for $0 \notin H$. So, $\chi = \chi_P$ in both cases. This means that P is an isomorphism of (G, o, ζ, χ) onto some f.o.p. Menger algebra of n-place functions.

For every $g \in H$, the elements g and $g[g^n]$ are in the same π_g^*-class, let us say X, different from V_g. Because the relation $\varepsilon(\zeta^{-1} \circ \zeta \cup \theta_g, [g]_g)$ is v-regular, we have $g[X \cdots X] \subset X$. This means that $P_{(\pi_g^*, V_g)}(g)(a, \ldots, a) = a$, where a is an element used for the indexation of the class X. Hence $P(g) \in St(P(G))$. Conversely, if $P(g)$ has some fixed point and G has a zero 0 and $0 \in H$, then, as it was mentioned above, $H = G$. Whence $g \in H$. Assume that $0 \notin H$. Because $P(g) = P_1^0(g) \cup P_2(g) \in St(P(G))$, then there exists (a) $g_1 \neq 0$ such that $P_{(\varepsilon_{g_1}^*, W_{g_1})}(g)$ has the fixed point, or (b) $h \in H$ such that $P_{(\pi_h^*, V_h)}(g)$ has the fixed point. If (a) holds and X is the $\varepsilon_{g_1}^*$-class different from W_{g_1} such that $g[X \cdots X] \subset X$, then $g[x_0 \cdots x_0] \equiv x_0(\varepsilon_{g_1}^*)$ for some $x_0 \in X$. Thus, for some $x_i, x_{m+1}, z_i \in G$, $i = 0, 1, \ldots, m$ we have

$$\bigwedge_{i=0}^{m} (0 \neq g_1 \sqsubset x_i \leqslant z_i \wedge 0 \neq g_1 \sqsubset x_{i+1} \leqslant z_i)$$

and $g[x_0 \cdots x_0] = x_{m+1}$, which together with (4.3.12) implies $g \in H$. If (b) holds and X is the π_h^*-class different from V_h and $g[X \cdots X] \subset X$, then for $x_0 \in X$ we have $g[x_0 \cdots x_0] \equiv x_0(\pi_h^*)$, i.e., there are $x_i, z_i, x_{m+1} \in G$, $t_i \in T_n(G)$, $i = 0, 1, \ldots, m$ such that $g[x_0 \cdots x_0] = x_{m+1}$ and

$$\bigwedge_{i=1}^{m} \Big((x_i \leqslant z_i \wedge x_{i+1} \leqslant z_i) \vee \{x_i, x_{i+1}\} \subset \{t_i(h),\, t_i(h[h^n])\} \Big).$$

This, according to (4.3.13), implies $g \in H$. So, $g \in H$ if and only if $P(g)$ has the fixed point. □

Replacing in the conditions (4.3.12) and (4.3.13) the relation χ by relation $\delta \circ \zeta$ we obtain two new conditions which will be denoted by (4.3.12 a) and (4.3.13 a), respectively.

Corollary 4.3.4. *A nonempty subset H of a f.o. Menger algebra (G, o, ζ) of rank n is stationary if and only if for all $m \in \mathbb{N}$, $g, h, x_i, z_i, x_{m+1} \in G$, $i = 0, 1, \ldots, m$, it satisfies the conditions* (4.3.11), (4.3.12 a), *and* (4.3.13 a), *where* 0 *is a zero of* (G, o).

Proof. Indeed, if H is the stationary subset of a f.o. Menger algebra (G, o, ζ), then, of course, it is the stationary subset of a f.o.p. Menger algebra (G, o, ζ, χ_P), where P is a faithful representation of (G, o, ζ) by n-place functions such that $H = P^{-1}(St(P(G)))$. Moreover, $\delta \circ \zeta \subset \chi_P$ and the premises of (4.3.12 a) and (4.3.13 a) imply the premises of (4.3.12) and (4.3.13), and consequently, the conclusions of (4.3.12 a) and (4.3.13 a). So, the conditions (4.3.12 a) and (4.3.13 a) are satisfied.

Conversely, if the subset H of a f.o. Menger algebra (G, o, ζ) satisfies all the conditions of the corollary, then, as it is known, the system $(G, o, \zeta, \delta \circ \zeta)$ is a f.o.p. Menger algebra for which the conditions (4.3.12 a) and (4.3.13 a) correspond to (4.3.12) and (4.3.13), respectively. Thus, by Theorem 4.3.3, H is the stationary subset of (G, o, ζ). □

Let $[X]^0_h$ be the (ρ^0_h, χ)-closure of the subset X of a p.q-o. Menger algebra (G, o, χ) of rank n, where $h \in G$ and $\rho^0_h = \triangle_G \cup \theta_h$. Using this closure, we can give the following characterization of stationary subsets of (G, o, χ).

Theorem 4.3.5. *A nonempty subset H of a p.q-o. Menger algebra (G, o, χ) of rank n is stationary if and only if the following conditions are satisfied:*

$$0 \notin H \wedge g \sqsubset 0 \longrightarrow g = 0, \tag{4.3.16}$$

$$g[x \cdots x] = x \neq 0 \longrightarrow g \in H, \tag{4.3.17}$$

$$\left.\begin{array}{r} \bigwedge\limits_{i=0}^{m} \Big(x_i = x_{i+1} \vee \{x_i, x_{i+1}\} \subset \{t_i(h), t_i(h[h^n])\} \Big), \\ \{x_0, \ldots, x_{m+1}\} \subset [h]^0_h \wedge h \in H, \\ g[x_0 \cdots x_0] = x_{m+1} \end{array}\right\} \longrightarrow g \in H \tag{4.3.18}$$

for all $m \in \mathbb{N}$, $g, h, x_i, x_{m+1} \in G$, $t_i \in T_n(G)$, $i = 0, 1, \ldots, m$, where 0 *is a zero element of* (G, o).

Proof. It is not difficult to check that the conditions of the theorem are necessary, so, we verify only the sufficiency. For this, let H be the subset of a p.q-o. Menger algebra (G, o, χ) of rank n satisfying all the conditions of the theorem. Consider the canonical representation Λ of (G, o) by translations. Obviously $\chi_\Lambda = G \times G$. For every element $g \in G$ we define the set

$$H_g = \{\bar{x} \in B \,|\, g[\bar{x}] \neq 0\},$$

where $B = G^n \cup \{(e_1, \ldots, e_n)\}$ and 0 is a zero in (G, o). We can show that Λ^0, where $\Lambda^0(g) = \Lambda(g) \cap (H_g \times G)$ for every $g \in G$, is a faithful representation of (G, o). Let $g_1 \sqsubset g_2$ and $g_1[\bar{x}] \neq 0$, where $\bar{x} \in B$, then $g_1[\bar{x}] \sqsubset g_2[\bar{x}]$. Thus, according to (4.3.16), $0 \notin H$ implies $g_2[\bar{x}] \neq 0$. Hence $(g_1, g_2) \in \chi_{\Lambda^0}$ for $0 \notin H$, and consequently, $\chi \subset \chi_{\Lambda^0}$. Let a be some fixed element from G and let $\Lambda_a(g) = \Lambda(g) \cap (B \times \chi\langle a\rangle)$ for every $g \in G$. It is clear that Λ_a is a representation of (G, o) by n-place functions and $\chi = \bigcap_{a \in G} \chi_{\Lambda_a} = \bigcap_{a \in G\setminus\{0\}} \chi_{\Lambda_a}$.

Consider now the following representations of a Menger algebra (G, o) by n-place functions:

$$P_2 = \sum_{a \in G} \Lambda_a\,, \qquad P_2^0 = \sum_{a \in G\setminus\{0\}} \Lambda_a\,, \qquad P_3 = \sum_{h \in H} P_{(\varepsilon_h^0,\, W_h)}\,,$$

where $\varepsilon_h^0 = \varepsilon(\rho_h^0, [h]_h^0)$, $W_h = G \setminus [h]_h^0$. Note that $\chi \subset \chi_{P_3}$, because the subset $[h]_h^0$ is χ-saturated. Now putting $P = \Lambda + P_2 + P_3$ when $0 \in H$, and $P = \Lambda^0 + P_2^0 + P_3$ when $0 \notin H$, we can see that P is a faithful representation of (G, o). Furthermore, for $0 \in H$ we have

$$\chi_P = \chi_\Lambda \cap \chi_{P_2} \cap \chi_{P_3} = (G \times G) \cap \chi \cap \chi_{P_3} = \chi.$$

A similar equality holds for $0 \notin H$. So, P is an isomorphism of the system (G, o, χ).

Finally, if $g \in H$, then, analogously as in the proof of the previous theorem, we show that $P(g)$ has a fixed point. Conversely, let $P(g)$ have some fixed point. If $0 \in H$, then $H = G$, according to (4.3.18). Whence $g \in H$. In the case $0 \notin H$ we have three possibilities: (a) $\Lambda^0(g) \in St(\Lambda^0(G))$, (b) $P_2^0(g) \in St(P_2^0(G))$ and (c) $P_3(g) \in St(P_3(G))$. In the cases (a) and (b), by (4.3.17), and in the case (c), by (4.3.18), we get $g \in H$. So, $g \in H$ if and only if $P(g)$ has a fixed point, which completes the proof. □

Corollary 4.3.6. *A nonempty subset H of a Menger algebra (G, o) of rank n is stationary if and only if*

$$\left.\begin{array}{c} \bigwedge_{i=0}^{m} \Big(x_i = x_{i+1} \vee \{x_i, x_{i+1}\} \subset \{t_i(h), t_i(h[h^n])\}\Big), \\ \{x_0, \ldots, x_{m+1}\} \subset [h]_h^{0,\delta} \wedge g[x_0 \cdots x_0] = x_{m+1}, \\ h \in H \end{array}\right\} \longrightarrow g \in H \qquad (4.3.19)$$

for all $m \in \mathbb{N}$, $g, h, x_i, x_{m+1} \in G$, $t_i \in T_n(G)$, $i = 0, 1, \ldots, m$, *where* $[h]_h^{0,\delta}$ *is the* (ρ_h^0, δ)*-closure of the set* $\{h, h[h^n]\}$.

Proof. Indeed, according to Theorem 4.3.5, condition (4.3.19) means that H is a stationary subset of a p.q-o. Menger algebra (G, o, δ), which implies our corollary. □

Let us come to the description of reversive stationary subsets. Let X be a subset of a strong f.o.p. Menger algebra (G, o, ζ, χ) of rank $n \geqslant 2$. By $\langle X \rangle_h$, where $h \in G$, we will denote the $\langle \zeta^{-1} \circ \zeta \cup \theta_h, \chi \rangle$-closure of X. In the case $X = \{g, g[g^n]\}$ we shall write $\langle g \rangle_h$.

Theorem 4.3.7. *A nonempty subset* H *of a strong f.o.p. Menger algebra* (G, o, ζ, χ) *of rank* $n \geqslant 2$ *is its stationary subset if and only if it satisfies* (4.1.16) *and for any* $m \in \mathbb{N}$, $g, h, x, y \in G$ *the following two implications are true:*

$$y \neq 0 \wedge (g[x^n], x) \in \varepsilon_m \langle \zeta^{-1} \circ \zeta, \chi\langle y \rangle \rangle \longrightarrow g \in H, \tag{4.3.20}$$

$$h \in H \wedge (g[x^n], x) \in \varepsilon_m \langle \zeta^{-1} \circ \zeta \cup \theta_h, \langle h \rangle_h \rangle \longrightarrow g \in H, \tag{4.3.21}$$

where 0 *is the zero element of* (G, o).

Proof. Let $(\Phi, O, \zeta_\Phi, \chi_\Phi)$ be a f.o.p. Menger algebra of reversive n-place functions and φ be the zero of (Φ, O). If $(\alpha[\beta^n], \beta) \in \varepsilon_m \langle \zeta_\Phi^{-1} \circ \zeta_\Phi, \chi_\Phi \langle \gamma \rangle \rangle$ and $\gamma \neq \varphi$ and for some $\alpha, \beta, \gamma \in \Phi$, then $\alpha[\beta^n] \circ \triangle_{\mathrm{pr}_1 \gamma} = \beta \circ \triangle_{\mathrm{pr}_1 \gamma}$, according to (3.3.1). From $\gamma \neq \varphi$ it follows that $\gamma \neq \varnothing$. So, there exists $\bar{a} \in \mathrm{pr}_1 \gamma$, which together with $\mathrm{pr}_1 \gamma \subset \mathrm{pr}_1 \beta \cap \mathrm{pr}_1\, \alpha[\beta^n]$, implies

$$\beta(\bar{a}) = \alpha[\beta^n](\bar{a}) = \alpha(\beta(\bar{a}), \ldots, \beta(\bar{a})).$$

Consequently $\alpha \in St(\Phi)$. This proves (4.3.20).

Now let $\gamma \in St(\Phi)$ and $(\alpha[\beta^n], \beta) \in \varepsilon_m \langle \zeta_\Phi^{-1} \circ \zeta_\Phi, \langle \gamma \rangle_\gamma \rangle$, where $\alpha, \beta, \gamma \in \Phi$. From this, by induction, we obtain $\alpha[\beta^n]\langle a, \ldots, a \rangle = \beta \langle a, \ldots, a \rangle \neq \varnothing$, where $a = \gamma(a, \ldots, a)$. Thus

$$\beta(a, \ldots, a) = \alpha[\beta^n](a, \ldots, a) = \alpha(\beta(a, \ldots, a), \ldots, \beta(a, \ldots, a)).$$

Hence $\alpha \in St(\Phi)$, which proves (4.3.21). The necessity is proved.

We shall not give the proof of the sufficiency, because it is analogous to the proof of the sufficiency of Theorem 4.3.3. Note only that in this case the representations P_1, P_1^0 and P_2 of (G, o) by reversive n-place functions are determined by the families of determining pairs $\big((\varepsilon_g^*, W_g)\big)_{g \in G}$ and $((\pi_h^*, V_h))_{h \in H}$, where

$$\varepsilon_g^* = \varepsilon \langle \zeta^{-1} \circ \zeta, \chi\langle g \rangle \rangle \cup \{(e_1, e_1), \ldots, (e_n, e_n)\},$$
$$\pi_h^* = \varepsilon \langle \zeta^{-1} \circ \zeta \cup \theta_h, \langle h \rangle_h \rangle \cup \{(e_1, e_1), \ldots, (e_n, e_n)\},$$
$$W_g = G \setminus \chi\langle g \rangle\,, \qquad V_h = G \setminus \langle h \rangle_h.$$

The theorem is proved. □

Theorem 4.3.8. *A nonempty subset H of a strong f.o. Menger algebra (G,o,ζ) of rank $n \geqslant 2$ is stationary if and only if* (4.1.16) *holds and for all $m \in \mathbb{N}$, $t \in T_n(G)$, $x,y,h,g \in G$ it satisfies the conditions*

$$x \neq 0 \wedge x \leqslant t(y) \wedge x \leqslant t(g[y^n]) \longrightarrow g \in H, \tag{4.3.22}$$

$$h \in H \wedge (g[x^n],x) \in \varepsilon_m\langle \zeta^{-1} \circ \zeta \cup \theta_h, \langle h\rangle_h^{\delta}\rangle \longrightarrow g \in H, \tag{4.3.23}$$

where $\langle h\rangle_h^{\delta}$ is the $\langle \zeta^{-1} \circ \zeta \cup \theta_h, \delta \circ \zeta\rangle$-closure of $\{h, h[h^n]\}$.

Proof. The proof of the necessity is analogous to the corresponding parts of the proofs of previous theorems. To prove the sufficiency consider the sum $P = P_1 + P_2$ if $0 \in H$, and $P = P_1^0 + P_2$ if $0 \notin H$, of representations of (G,o) by reversive n-place functions, where

$$P_1 = \sum_{g\in G} P_{(\varepsilon_v^*(\zeta\langle g\rangle), W_v(\zeta\langle g\rangle))}, \quad P_1^0 = \sum_{g\in G\setminus\{0\}} P_{(\varepsilon_v^*(\zeta\langle g\rangle), W_v(\zeta\langle g\rangle))},$$

$$P_2 = \sum_{h\in H} P_{(\pi_h^*, V_h)}, \quad \pi_h^* = \varepsilon\langle \overset{-1}{\zeta} \circ \zeta \cup \theta_h, \langle h\rangle_h^{\delta}\rangle, \quad V_h = G \setminus \langle h\rangle_h^{\delta}.$$

It is not difficult to see that the representation P_1 is faithful and $\zeta = \zeta_{P_1}$. Similarly, using (4.1.16), we show that the representation P_1^0 is faithful and $\zeta = \zeta_{P_1^0}$. Moreover $\zeta \subset \zeta_{P_2}$. Thus $\zeta_P = \zeta \cap \zeta_{P_2} = \zeta$. So, P is an isomorphism of a strong f.o. Menger algebra (G,o,ζ).

Further, using the above, we prove that $g \in H$ implies $P(g) \in St(P(G))$. Let $P(g)$, where $g \in G$, has some fixed point. If $0 \in H$, then, evidently, $H = G$. Thus, in this case, $g \in H$. If $0 \notin H$ and $P_2(g)$ has the fixed point, then $g \in H$, by (4.3.23). So, in both cases $g \in H$. For $P_1^0(g) \in St(P_1^0(G))$ there are $g_1 \neq 0$ and $x \notin W_v(\zeta\langle g_1\rangle)$ such that $g[x^n] \equiv x(\varepsilon_v(\zeta\langle g_1\rangle))$. The last statement is equivalent to the condition

$$(\forall t \in T_n(G)) \ (g_1 \leqslant t(g[x^n]) \longleftrightarrow g_1 \leqslant t(x)) \,.$$

Since $x \notin W_v(\zeta\langle g_1\rangle)$, for some $t_1 \in T_n(G)$ must be $g_1 \leqslant t_1(x)$, and consequently $g_1 \leqslant t_1(g[x^n])$. Thus $g_1 \neq 0$, $g_1 \leqslant t_1(x)$ and $g_1 \leqslant t_1(g[x^n])$. Hence, according to (4.3.22), we get $g \in H$. In this way we showed that $H = P^{-1}(St(P(G)))$. □

Theorem 4.3.7 for $\zeta = \triangle_G$ gives rise to the following Corollary.

Corollary 4.3.9. *A nonempty subset H of a strong p.q-o. Menger algebra (G,o,χ) of rank $n \geqslant 2$ is its reversive stationary subset if and only if it satisfies* (4.3.16) *and for all $g,x,y,h \in G$, $m \in \mathbb{N}$ and the implications*

$$y \neq 0 \wedge (g[x^n],x) \in \varepsilon_m\langle \triangle_G, \chi\langle y\rangle\rangle \longrightarrow g \in H, \tag{4.3.24}$$

$$h \in H \wedge (g[x^n],x) \in \varepsilon_m\langle \triangle_G \cup \theta_h, \langle h\rangle_h^0\rangle \longrightarrow g \in H, \tag{4.3.25}$$

where $\langle h\rangle_h^0$ is the $\langle \triangle_G \cup \theta_h, \chi\rangle$-closure of $\{h, h[h^n]\}$, are valid.

For a strong Menger algebra of rank $n \geqslant 2$ with the strong order $\overset{\wedge}{\zeta}$ defined by (3.3.6), Theorem 4.1.10 implies:

Corollary 4.3.10. *A nonempty subset H of a strong Menger algebra (G, o) of rank $n \geqslant 2$ is its reversive stationary subset if and only if it satisfies the conditions*

$$x \neq 0 \wedge t(y) \in \hat{x} \wedge t(g[y^n]) \in \hat{x} \longrightarrow g \in H, \tag{4.3.26}$$

$$h \in H \wedge (g[x^n], x) \in \varepsilon_m \langle \overset{\wedge}{\zeta}^{-1} \circ \overset{\wedge}{\zeta} \cup \theta_h, \langle h\rangle_h^{\wedge}\rangle \longrightarrow g \in H \tag{4.3.27}$$

for all $m \in \mathbb{N}$, $x, y, g, h \in G$, $t \in T_n(G)$, where $\hat{x}$ is the strong closure of $\{x\}$, and $\langle h\rangle_h^{\wedge}$ is the $\langle \overset{\wedge}{\zeta}^{-1} \circ \overset{\wedge}{\zeta} \cup \theta_h, \delta \circ \zeta\rangle$-closure of $\{h, h[h^n]\}$.

Let $(G, o, \curlywedge, \blacktriangleright)$ be a restrictive $\curlywedge$-Menger algebra of rank n. Consider the family of pairs $\{(\varepsilon_g, W_g)\}_{g \in G}$, where ε_g and W_g are defined in the following way:

$$W_g = \{x \in G \mid x \blacktriangleright g \neq g\},$$

$$\varepsilon_g = \{(x, y) \in G \times G \mid x \curlywedge y \notin W_g \vee x, y \in W_g\}.$$

Proposition 4.3.11. *For every element $g \in G$ of a restrictive $\curlywedge$-Menger algebra $(G, o, \curlywedge, \blacktriangleright)$ of rank n the ordered pair (ε_g, W_g) has the following properties:*

(i) *any nonempty W_g is an l-ideal of $(G, o, \curlywedge, \blacktriangleright)$,*
(ii) *ε_g is a v-regular equivalence,*
(iii) *any nonempty W_g is an ε_g-class,*
(iv) *a mapping $P_{(\varepsilon_g, W_g)} : g \mapsto P_{(\varepsilon_g, W_g)}(g)$ is a representation of this restrictive $\curlywedge$-Menger algebra by n-place functions.*

Proof. (i) Let $W_g \neq \varnothing$. If $x \in W_g$, then $x \blacktriangleright g \neq g$. Suppose that for some $u \in G$, $\bar{w} \in G^n$ and $i \in \{1, \ldots, n\}$ the equality $u[\bar{w}|_i x] \blacktriangleright g = g$ is true. Applying to this equality (3.6.1), (3.6.3) and the idempotency of the operation $\blacktriangleright$, we obtain $x \blacktriangleright u[\bar{w}|_i x] \blacktriangleright g = g$, which gives $x \blacktriangleright g = g$. Thus, $u[\bar{w}|_i x] \blacktriangleright g \neq g$, hence $u[\bar{w}|_i x] \in W_g$. So, W_g is an l-ideal of $(G, o, \curlywedge, \blacktriangleright)$.

(ii) The reflexivity and symmetry of ε_g are obvious. To prove the transitivity let $(x, y) \in \varepsilon_g$ and $(y, z) \in \varepsilon_g$. If $x, y, z \in W_g$, then, obviously, $(x, z) \in \varepsilon_g$, therefore suppose that $x \curlywedge y \notin W_g$ and $y \curlywedge z \notin W_g$. These last conditions mean that $(x \curlywedge y) \blacktriangleright g = g$ and $(y \curlywedge z) \blacktriangleright g = g$, whence $(x \curlywedge y) \blacktriangleright g \blacktriangleright (y \curlywedge z) \blacktriangleright g = g \blacktriangleright g = g$, hence, $(x \curlywedge y) \blacktriangleright (y \curlywedge z) \blacktriangleright g = g$. Now, using (3.6.14), we obtain $y \blacktriangleright (x \curlywedge y \curlywedge z) \blacktriangleright g = g$, whence, by (3.6.3), we have $(x \curlywedge y \curlywedge z) \blacktriangleright y \blacktriangleright g = g$. The last equality, by means of (3.6.15), is equivalent to $(x \curlywedge y \curlywedge z) \blacktriangleright g = g$. Since $x \curlywedge y \curlywedge z \leqslant x \curlywedge z$, then, evidently, $x \curlywedge y \curlywedge z = (x \curlywedge y \curlywedge z) \blacktriangleright (x \curlywedge z)$.

Thus $(x \curlywedge y \curlywedge z) \blacktriangleright (x \curlywedge z) \blacktriangleright g = g$, whence, by (3.6.3), it follows $(x \curlywedge z) \blacktriangleright (x \curlywedge y \curlywedge z) \blacktriangleright g = g$, i.e., $(x \curlywedge z) \blacktriangleright g = g$. So, $x \curlywedge z \notin W_g$, i.e., $(x, y) \in \varepsilon_g$, which proves the transitivity of ε_g. Thus ε_g is an equivalence relation.

Let $x \equiv y(\varepsilon_g)$. If $x, y \in W_g$, then $u[\bar{w}|_i x]$ and $u[\bar{w}|_i y]$, where $u \in G$, $\bar{w} \in G^n$, $i \in \{1, \dots, n\}$, are in W_g, because W_g is an l-ideal. Whence $u[\bar{w}|_i x] \equiv u[\bar{w}|_i y](\varepsilon_g)$. If $x \notin W_g$ and $y \notin W_g$, then, obviously, $x \curlywedge y \notin W_g$, i.e., $(x \curlywedge y) \blacktriangleright g = g$. Suppose that $u[\bar{w}|_i x] \notin W_g$, i.e., $u[\bar{w}|_i x] \blacktriangleright g = g$. Then $(x \curlywedge y) \blacktriangleright u[\bar{w}|_i x] \blacktriangleright g = (x \curlywedge y) \blacktriangleright g = g$, whence, by (3.6.1) and (3.6.15), we obtain $u[\bar{w}|_i (x \curlywedge y)] \blacktriangleright g = g$. But $x \curlywedge y \leqslant y$ gives $(x \curlywedge y) \blacktriangleright y = x \curlywedge y$, so, $u[\bar{w}|_i ((x \curlywedge y) \blacktriangleright y)] \blacktriangleright g = g$. Whence $(x \curlywedge y) \blacktriangleright u[\bar{w}|_i y] \blacktriangleright g = g$, which, by (3.6.3), implies $u[\bar{w}|_i y] \blacktriangleright (x \curlywedge y) \blacktriangleright g = g$. Hence $u[\bar{w}|_i y] \blacktriangleright g = g$. So, $u[\bar{w}|_i y] \notin W_g$. Similarly we can show that $u[\bar{w}|_i y] \notin W_g$ implies $u[\bar{w}|_i x] \notin W_g$. In this way we have shown that $u[\bar{w}|_i x], u[\bar{w}|_i y]$ simultaneously belong or not belong to W_g. In the case when these elements belong to W_g, we have $u[\bar{w}|_i x] \equiv u[\bar{w}|_i y](\varepsilon_g)$. In the case when these elements do not belong to W_g it is not difficult to see that $x \leqslant y$ implies $u[\bar{w}|_i x] \leqslant u[\bar{w}|_i y]$. Therefore, $x \curlywedge y \leqslant x$ and $x \curlywedge y \leqslant y$ imply $u[\bar{w}|_i (x \curlywedge y)] \leqslant u[\bar{w}|_i x]$ and $u[\bar{w}|_i (x \curlywedge y)] \leqslant u[\bar{w}|_i y]$, whence $u[\bar{w}|_i (x \curlywedge y)] \leqslant u[\bar{w}|_i x] \curlywedge u[\bar{w}|_i y]$. The last inequality means that $u[\bar{w}|_i (x \curlywedge y)] \blacktriangleright (u[\bar{w}|_i x] \curlywedge u[\bar{w}|_i y]) = u[\bar{w}|_i (x \curlywedge y)]$. Further we obtain $u[\bar{w}|_i (x \curlywedge y)] \blacktriangleright (u[\bar{w}|_i x] \curlywedge u[\bar{w}|_i y]) \blacktriangleright g = u[\bar{w}|_i (x \curlywedge y)] \blacktriangleright g$. Hence, by (3.6.3), we have

$$(u[\bar{w}|_i x] \curlywedge u[\bar{w}|_i y]) \blacktriangleright u[\bar{w}|_i (x \curlywedge y)] \blacktriangleright g = u[\bar{w}|_i (x \curlywedge y)] \blacktriangleright g. \quad (4.3.28)$$

It has been shown above that if $u[\bar{w}|_i x] \notin W_g$, then $u[\bar{w}|_i (x \curlywedge y)] \blacktriangleright g = g$, therefore equality (4.3.28) is equivalent to $(u[\bar{w}|_i x] \curlywedge u[\bar{w}|_i y]) \blacktriangleright g = g$. Thus, $u[\bar{w}|_i x] \curlywedge u[\bar{w}|_i y] \notin W_g$, hence $u[\bar{w}|_i x] \equiv u[\bar{w}|_i y](\varepsilon_g)$. So, ε_g is i-regular for every $i = 1, \dots, n$, whence it is v-regular.

(iii) Let $W_g \neq \varnothing$. If $x, y \in W_g$, then, obviously, $x \equiv y(\varepsilon_g)$. This means that W_g is contained in some ε_g-class. In order to show that W_g coincides with an ε_g-class, it is necessary to verify that the condition

$$x \in W_g \wedge x \equiv y(\varepsilon_g) \longrightarrow y \in W_g \quad (4.3.29)$$

holds. Indeed, let the premise of (4.3.29) be valid. Since $x \equiv y(\varepsilon_g)$, we have two possibilities: (a) $x, y \in W_g$ and (b) $x \curlywedge y \notin W_g$. In the first case condition (4.3.29) is evident. In the second case we have $(x \curlywedge y) \blacktriangleright g = g$, i.e., $(g, x \curlywedge y) \in \alpha_s$. But $(x \curlywedge y, x) \in \alpha_d$ and $\alpha_d \subset \alpha_s$ imply $(x \curlywedge y, x) \in \alpha_s$. Hence, $(g, x) \in \alpha_s$, i.e., $x \blacktriangleright g = g$. So, $x \notin W_g$. We have obtained a contradiction by assuming (4.3.29). Thus, the case (b) is impossible.

(iv) To prove this fact it is sufficient to verify that for each $g \in G$ the pair (ε_g, W_g) satisfies the conditions (3.4.8), (3.4.9), (3.4.10) of Proposition 3.4.2 and the conditions (3.6.7), (3.6.8) of Proposition 3.6.4. The first condition means that $x \notin W_g$ follows from $x \curlywedge y \notin W_g$. This fact have been proved above. So, (3.4.8) is true. The condition (3.4.9) is fulfilled by the definition of ε_g. Let $x \notin W_g$ and $x \equiv y(\varepsilon_g)$, then $x \curlywedge y \notin W_g$, whence $x \curlywedge (x \curlywedge y) \notin W_g$. Hence, $x \curlywedge y \equiv x(\varepsilon_g)$. The condition (3.4.10) is also true. Suppose that $x \blacktriangleright y \notin W_g$, i.e., $(x \blacktriangleright y) \blacktriangleright g = g$. The last equality means $(g, x \blacktriangleright y) \in \alpha_s$. Since $(x \blacktriangleright y, x) \in \alpha_s$, then, evidently, $(g, x) \in \alpha_s$. Therefore $x \blacktriangleright g = g$, i.e., $x \notin W_g$. Next, by (3.6.14), we have $y \curlywedge (x \blacktriangleright y) = x \blacktriangleright (y \curlywedge y) = x \blacktriangleright y$. Hence $y \curlywedge (x \blacktriangleright y) \notin W_g$, whence $x \blacktriangleright y \equiv y(\varepsilon_g)$. This proves (3.6.7). Now let $x \notin W_g$ and $y \notin W_g$. In this case $x \blacktriangleright g = g$, $y \blacktriangleright g = g$ and $g = (x \blacktriangleright g) \curlywedge (y \blacktriangleright g) = y \blacktriangleright ((x \blacktriangleright g) \curlywedge g) = y \blacktriangleright (x \blacktriangleright g) = (x \blacktriangleright y) \blacktriangleright g$, whence $x \blacktriangleright y \notin W_g$. But $x \blacktriangleright y = y \curlywedge (x \blacktriangleright y)$, so $y \curlywedge (x \blacktriangleright y) \notin W_g$, i.e., $x \blacktriangleright y \equiv y(\varepsilon_g)$. This proves (3.6.8). So, $P_{(\varepsilon_g, W_g)}$ induced by the pair (ε_g, W_g) is a representation of a restrictive $\curlywedge$-Menger algebra $(G, o, \curlywedge, \blacktriangleright)$ by n-place functions. □

Let $(\Phi, O, \cap, \rhd)$ be a restrictive $\cap$-Menger algebra of n-place functions on a set A, which does not have a zero. Therefore $\varphi \neq \varnothing$ for each $\varphi \in \Phi$, hence for all $\varphi, \psi \in \Phi$ the condition $\varphi \cap \psi[\varphi^n] \neq \varnothing$ is true. Thus, there exists a vector $(a_1, \ldots, a_n) \in A^n$ such that $\varphi(a_1, \ldots, a_n) = \psi[\varphi^n](a_1, \ldots, a_n)$, i.e., $\psi(\varphi(a_1, \ldots, a_n), \ldots, \varphi(a_1, \ldots, a_n)) = \varphi(a_1, \ldots, a_n)$. So, $\psi(b, \ldots, b) = b$, where $b = \varphi(a_1, \ldots, a_n)$, hence $\psi \in St(\Phi)$. We have proved that if a restrictive $\cap$-Menger algebra $(\Phi, O, \cap, \rhd)$ of n-place functions does not have a zero, then it coincides with the stationary subset $St(\Phi)$. For this reason in the future we will consider only algebras of type $(\Phi, O, \cap, \rhd)$ for which the operation O has a zero.

Let $(G, o, \curlywedge, \blacktriangleright)$ be a restrictive $\curlywedge$-Menger algebra of rank n. A subset H of G is called $\curlywedge$-*quasi-stable* if the implication

$$x \in H \longrightarrow x[x \cdots x] \curlywedge x \in H$$

is true for all $x \in G$.

Theorem 4.3.12. *For a nonempty subset H of a restrictive $\curlywedge$-Menger algebra $(G, o, \curlywedge, \blacktriangleright)$ of rank n to be a stationary subset it is necessary and sufficient that it is an α_d-saturated, $\curlywedge$-quasi-stable and satisfies the conditions*

$$0 \notin H \longrightarrow 0 \curlywedge x = 0, \tag{4.3.30}$$

$$x[y^n] = y \in H \longrightarrow x \in H, \tag{4.3.31}$$

$$x[y^n] \curlywedge y \neq 0 \longrightarrow x \in H \tag{4.3.32}$$

for all $x, y \in G$, where 0 is a zero of the algebra (G, o).

Proof. Necessity. Let $(\Phi, O, \cap, \rhd)$ be some restrictive $\cap$-Menger algebra of n-place functions defined on A, $St(\Phi)$ be its stationary subset. In this algebra, the order α_d coincides with the set-theoretic inclusion, hence $St(\Phi)$ is an

α_d-saturated subset. Now let $\varphi \in St(\Phi)$, then $\varphi(a,\dots,a) = a$ for some $a \in A$, whence $\varphi[\varphi^n](a,\dots,a) = \varphi(\varphi(a,\dots,a),\dots,\varphi(a,\dots,a)) = \varphi(a,\dots,a) = a$, hence $(a^n, a) \in \varphi[\varphi^n] \cap \varphi$. Thus $\varphi[\varphi^n] \cap \varphi \in St(\Phi)$. So, $St(\Phi)$ is $\cap$-quasi-stable.

Let φ be a zero of (Φ, O). If $\varphi \notin St(\Phi)$, then $St(\Phi) \neq \Phi$. Thus $\varphi = \varnothing$, by (4.3.14). Therefore $\varphi \subset \psi$ for every $\psi \in \Phi$, i.e., $\varphi \cap \psi = \varphi$. This proves (4.3.30). The rest is a consequence of our Proposition 4.3.1.

Sufficiency. Let a nonempty subset H of a restrictive $\curlywedge$-Menger algebra $(G, o, \curlywedge, \blacktriangleright)$ of rank n satisfy all the conditions of the theorem. For every element $g \in G$, consider the mapping $P_{(\varepsilon_g, W_g)}$, which is a representation of our restrictive $\curlywedge$-Menger algebra by n-place functions. Next, let P denotes the sum of the family of representations $\{P_{(\varepsilon_g, W_g)}\}_{g \in G_0}$, where $G_0 = G \setminus \{0\}$ for $0 \notin H$ and $G_0 = G$ for $0 \in H$, where 0 is a zero of (G, o). Obviously, P is a representation of this algebra by n-place functions, therefore we must show only that P is a faithful representation such that $H = P^{-1}(St(\Phi))$, where $\Phi = P(G)$. So, let $g_1 \in H$. Suppose that $0 \in H$, then $G_0 = G$, hence $0 \in G_0$. Consider the representation $P_{(\varepsilon_0, W_0)}$. It is easy to see that $g \notin W_g$ for every $g \in G$, therefore $0 \notin W_0$. Let X denote the ε_0-class containing 0. It is clear that $X \neq W_0$. Since $g_1[0^n] = 0$, the v-regularity of ε_g implies $g_1[X^n] \subset X$. The last condition means that $P_{(\varepsilon_0, W_0)}(g_1)(X,\dots,X) = X$. Hence $P(g_1)(X,\dots,X) = X$. If $0 \notin H$, then $G_0 = G \setminus \{0\}$. For $h = g_1 \curlywedge g_1[g_1 \cdots g_1]$ from $g_1 \in H$, by the $\curlywedge$-quasi-stability of H, we deduce $h \in H$, i.e., $h \neq 0$. Therefore $h \in G_0$. Consider now the representation $P_{(\varepsilon_h, W_h)}$. Since $h \notin W_h$, so $g_1 \curlywedge g_1[g_1 \cdots g_1] \notin W_h$. Whence $g_1 \equiv g_1[g_1 \cdots g_1](\varepsilon_h)$. Next, by (3.6.14) and (3.6.1), we obtain

$$\begin{aligned} g_1 \blacktriangleright h &= g_1 \blacktriangleright (g_1 \curlywedge g_1[g_1 \cdots g_1]) = g_1 \curlywedge (g_1 \blacktriangleright g_1[g_1 \cdots g_1]) \\ &= g_1 \curlywedge g_1[g_1 \blacktriangleright g_1 \cdots g_1 \blacktriangleright g_1] = g_1 \curlywedge g_1[g_1 \cdots g_1] = h. \end{aligned}$$

Therefore, $g_1 \curlywedge h = h$ and hence $g_1 \notin W_h$. Let Y be the ε_h-class containing g_1. It is clear that $Y \neq W_h$ and $g_1[Y^n] \subset Y$. Hence $P_{(\varepsilon_h, W_h)}(g_1)(Y,\dots,Y) = Y$, whence we have $P(g_1)(Y,\dots,Y) = Y$. So, $P(g_1)$ has a fixed point, i.e., $P(g_1) \in St(\Phi)$. Thus, we have proved that the implication

$$g_1 \in H \longrightarrow P(g_1) \in St(\Phi)$$

is satisfied. Conversely, let $P(g_1) \in St(\Phi)$ for some $g_1 \in G$. Since $P = \sum_{g \in G_0} P_{(\varepsilon_g, W_g)}$, then there exists $g \in G$ such that $P_{(\varepsilon_g, W_g)}(g_1)$ has a fixed point. Let $0 \notin H$. Then, obviously, $G_0 = G \setminus \{0\}$, whence $g \neq 0$. Suppose that X is an ε_g-class such that $P_{(\varepsilon_g, W_g)}(g_1)(X,\dots,X) = X$, i.e., $g_1[X^n] \subset X$, where $X \neq W_g$. Let x be some element of X, then $g_1[x^n] \equiv x(\varepsilon_g)$. Since $x \notin W_g$, then, evidently, $x \curlywedge g_1[x^n] \notin W_g$, i.e., $(x \curlywedge g_1[x^n]) \blacktriangleright g = g \neq 0$. Therefore $(x \curlywedge g_1[x^n]) \blacktriangleright g \neq 0$ is true. From (4.3.30) it follows that $0 \leqslant g$ for all $g \in G$, where $0 \notin H$. Hence $(x \curlywedge g_1[x^n]) \blacktriangleright g \neq 0$ implies $x \curlywedge g_1[x^n] \neq 0$, whence, by (4.3.32), we have $g_1 \in H$. If $0 \in H$, then for each $x \in G$ we have $x[0^n] = 0 \in H$, whence, by (4.3.31), it follows $x \in H$. So, $H = G$, hence $g_1 \in H$. Thus we have shown that

$$P(g_1) \in St(\Phi) \longrightarrow g_1 \in H.$$

We have proved that $H = P^{-1}(St(\Phi))$.

Finally we show that the representation P is faithful. For this, let the equality $P(g_1) = P(g_2)$ be true for some $g_1,\ g_2 \in G$. Then $P_{(\varepsilon_g, W_g)}(g_1) = P_{(\varepsilon_g, W_g)}(g_2)$ holds for every $g \in G_0$. Thus

$$g_1[X_1 \cdots X_n] \subset X \longleftrightarrow g_2[X_1 \cdots X_n] \subset X$$

for all $g \in G_0$, $X \in A^0_{\varepsilon_g}$, $(X_1, \ldots, X_n) \in A_{\varepsilon_g}$, where $A^0_{\varepsilon_g}$ denotes the set of all ε_g-classes different from W_g and $A_{\varepsilon_g} = A^0_{\varepsilon_g} \cup \{\{e_1\}, \ldots, \{e_n\}\}$, $e_1, \ldots, e_n$ — selectors in $(G^*; o^*)$. Setting $X_i = \{e_i\}$ for all $i = 1, \ldots, n$, we obtain

$$g_1 \in X \longleftrightarrow g_2 \in X \qquad (4.3.33)$$

for all $g \in G_0$, $X \in A^0_{\varepsilon_g}$. Let $0 \in H$, then $G_0 = G$, hence $g_1, g_2 \in G_0$. Since $g_1 \notin W_{g_1}$, from the above, taking $g = g_1$, $X = \langle g_1 \rangle$, where $\langle g_1 \rangle$ is the ε_{g_1}-class of g_1, we conclude that g_1, g_2 simultaneously belong or not belong to $\langle g_1 \rangle$. Therefore $g_1 \equiv g_2(\varepsilon_{g_1})$. Thus $g_1 \curlywedge g_2 \notin W_{g_1}$, i.e., $(g_1 \curlywedge g_2) \blacktriangleright g_1 = g_1$. But $(g_1 \curlywedge g_2) \blacktriangleright g_1 = g_1 \curlywedge g_2$, so, $g_1 \curlywedge g_2 = g_1$. Hence $g_1 \leqslant g_2$. Similarly we show that $g_2 \leqslant g_1$ and therefore $g_1 = g_2$. In the second case, when $0 \notin H$, we have $G_0 = G \setminus \{0\}$ and, by (4.3.28), $g_2 \neq 0$ for every $g \in G$. Suppose that $g_1 = 0$ and $g_2 \neq 0$, then $g_2 \in G_0$. But $g_2 \notin W_{g_2}$, so, by (4.3.33), we obtain $g_1 \equiv g_2(\varepsilon_{g_2})$, i.e., $(g_1 \curlywedge g_2) \blacktriangleright g_2 = g_2$. Therefore $g_1 \curlywedge g_2 = g_2$, whence $g_2 \leqslant g_1 = 0$. Thus $g_2 = 0$ and so, $g_1,\ g_2$ are simultaneously equal or not equal to 0. In the first case $g_1 = g_2$. In the second case $g_1, g_2 \in G_0$ and, consequently (as it was just proved) also $g_1 = g_2$. This means that P is one-to-one. So, P is a faithful representation. □

Theorem 4.3.13. *For a nonempty subset H of G to be a stationary subset of a functional Menger $\curlywedge$-algebra $\mathcal{G}_\curlywedge = (G, o, \curlywedge, R_1, \ldots, R_n)$ with a zero 0, it is necessary and sufficient to be a $\curlywedge$-quasi-stable subset satisfying for all $x, y \in G$ the conditions* (4.3.31), (4.3.32) *and*

$$0 \notin H \longrightarrow R_i 0 = 0, \qquad (4.3.34)$$

where $i = 1, \ldots, n$.

Proof. The necessity of these conditions is a consequence of Proposition 4.3.1, we shall therefore only prove their sufficiency. For this assume that a nonempty subset H of G satisfies all the conditions of the theorem and consider on G a binary relation $\mathcal{E}_g$ and a subset W_g defined in the following way:

$$\mathcal{E}_g = \{(x, y) \in G \times G \mid g \sqsubset x \curlywedge y \vee x, y \notin \chi\langle g \rangle\},$$
$$W_g = \{x \in G \mid x \notin \chi\langle g \rangle\},$$

where $g \in G$ is fixed. From the proof of Theorem 3.4.5, it follows that the relation $\mathcal{E}_g$ is a v-regular equivalence for which each nonempty subset W_g is an $\mathcal{E}_g$-class and an l-ideal simultaneously. Thus the pair $(\mathcal{E}_g, W_g)$ determines the simplest representation $P_g = P_{(\mathcal{E}_g,W_g)}$ of (G, o) by n-place functions. From the proofs of Theorems 3.6.6 and 3.7.7 it follows that P_g is also a representation of $\mathcal{G}_\curlywedge$. Hence $P = \sum_{h\in G_0} P_h$, where $G_0 = G \setminus \{0\}$ if $0 \notin H$, and $G_0 = G$, if $0 \in H$, is also a representation of $\mathcal{G}_\curlywedge$. We must only prove that P is a faithful representation satisfying condition (4.3.10).

First we shall show that H satisfies the condition

$$0 \in H \longrightarrow H = G. \tag{4.3.35}$$

Indeed, let $g \in G$ be any element of the algebra $\mathcal{G}_\curlywedge$, then $g[0\cdots 0] = 0 \in H$, whence by axiom (4.3.31) we obtain $g \in H$. So, $G \subseteq H \subseteq G$, hence, $H = G$.

Now we shall prove that H is a stationary subset of $\mathcal{G}_\curlywedge$. Let $g \in H$ and $P_0 = P_{(\mathcal{E}_0,W_0)}$. If $0 \in H$, then $G_0 = G = H$, whence $0 \in G_0$. Since $g \notin W_g$ for every $g \in G$, we have $0 \notin W_0$. Let X be the $\mathcal{E}_0$-class, indexed by a, which contains 0. Clearly $X \neq W_0$. From $g[0\cdots 0] = 0$, applying the v-regularity of $\mathcal{E}_0$, we obtain $g[X\cdots X] \subseteq X$. Consequently, $P_0(g)(a,\ldots,a) = a$. Hence, $P(g)(a,\ldots,a) = a$, which proves $P(g) \in St(P(G))$.

Now let $0 \notin H$. Then $G_0 = G \setminus \{0\}$ and $h = g \curlywedge g[g\cdots g] \in H$ for every $g \in H$, because H is $\curlywedge$-quasi-stable. Thus $h \neq 0$, whence $h \in G_0$. We shall consider the representation P_h. Since $h \notin W_h$, we have $g \curlywedge g[g\cdots g] \notin W_h$. Consequently, $g \equiv g[g\cdots g](\mathcal{E}_h)$. Further, using (3.7.12), (3.7.4), and (3.7.5) we obtain

$$\begin{aligned} h[R_1g\cdots R_ng] &= (g \curlywedge g[g\cdots g])[R_1g\cdots R_ng] = g \curlywedge g[g\cdots g][R_1g\cdots R_ng] \\ &= g \curlywedge g[g[R_1g\cdots R_ng]\cdots g[R_1g\cdots R_ng]] = g \curlywedge g[g\cdots g] = h, \end{aligned}$$

i.e., $h[R_1g\cdots R_ng] = h$. Therefore $h \sqsubset g$, which means that $g \notin W_h$. Let Y denote the $\mathcal{E}_h$-class containing g. Clearly, $Y \neq W_h$ and $g[Y\cdots Y] \subseteq Y$. Hence $P_h(g)(b,\ldots,b) = b$, where b is an element used for the indexation of Y. Thus $P(g)(b,\ldots,b) = b$. This means that also in this case $g \in H$ implies $P(g) \in St(P(G))$.

To prove the converse implication let $P(g) \in St(P(G))$ for some $g \in G$. Since $P = \sum_{h\in G_0} P_h$, there exists $h \in G$ such that $P_h(g)$ has a fixed point. If $0 \notin H$, then $G_0 = G \setminus \{0\}$ and $h \neq 0$. Let $X = H_a$ be the $\mathcal{E}_h$-class for which $P_h(g)(a,\ldots,a) = a$, i.e., $g[X\cdots X] \subseteq X$, where $X \neq W_h$. Obviously, for any $x \in X$ we have $g[x\cdots x] \equiv x(\mathcal{E}_h)$. This means that $x \curlywedge g[x\cdots x] \notin W_h$ for any $x \in X$ and therefore

$$h[R_1(x \curlywedge g[x\cdots x])\cdots R_n(x \curlywedge g[x\cdots x])] = h \neq 0,$$

whence $R_i(x \curlywedge g[x\cdots x]) \neq 0$ for every $i = 1,\ldots,n$. This, in view of (4.3.34), gives $x \curlywedge g[x\cdots x] \neq 0$ because in the opposite case $0 \in H$, which is impossible.

Applying (4.3.32) to $x \curlywedge g[x \cdots x] \neq 0$ we obtain $g \in H$. If $0 \in H$, then $H = G$, by (4.3.35). Therefore $g \in H$ in any case.

We have thus proved that g satisfies (4.3.10). So, H is a stationary subset of $\mathcal{G}_{\curlywedge}$.

To complete the proof we must show that the representation P is faithful. If $P(g_1) = P(g_2)$, then $P(g_1) \subseteq P(g_2)$ and $P(g_2) \subseteq P(g_1)$, whence $g_1 \leqslant g_2$ and $g_2 \leqslant g_1$. This implies $g_1 = g_2$. □

Note that the conditions used for the characterization of stationary subsets of functional Menger $\curlywedge$-algebras and the conditions used for the characterization of stationary subsets of restrictive $\curlywedge$-Menger algebras are not identical. Nevertheless, as it is proved below, stationary subsets of functional Menger $\curlywedge$-algebras with a zero have the same properties as stationary subsets of restrictive $\curlywedge$-Menger algebras.

Theorem 4.3.14. *For a stationary subset H of a functional Menger $\curlywedge$-algebra $\mathcal{G}_{\curlywedge}$ with a zero 0 the following implications:*

$$0 \notin H \longrightarrow 0 \leqslant x, \tag{4.3.36}$$

$$x \leqslant y \wedge x \in H \longrightarrow y \in H, \tag{4.3.37}$$

$$x \in H \longrightarrow x[x \cdots x] \in H, \tag{4.3.38}$$

$$x \in H \longrightarrow R_i x \in H, \tag{4.3.39}$$

$$x[y \cdots y] \neq 0 \longrightarrow R_i x \in H, \tag{4.3.40}$$

$$x[y \cdots y] = y \neq 0 \longrightarrow x \in H, \tag{4.3.41}$$

$$x \in H \wedge x \sqsubset y \longrightarrow R_i y \in H, \tag{4.3.42}$$

$$0 \notin H \wedge x \sqsubset 0 \longrightarrow x = 0 \tag{4.3.43}$$

are true for all $x, y \in G$ and $i = 1, \ldots, n$.

Proof. If $0 \notin H$, then, by (4.3.34), we obtain $R_i 0 = 0$ for all $i = 1, \ldots, n$. Hence, $0 = x[0 \cdots 0] = x[R_1 0 \cdots R_n 0]$, i.e., $0 = x[R_1 0 \cdots R_n 0]$ for any $x \in G$. So, $0 \leqslant x$. This proves (4.3.36).

Now, let the premise of (4.3.37) be satisfied, i.e., $x \leqslant y$ and $x \in H$ for some $x, y \in G$. If $0 \in H$, then $H = G$ by (4.3.35) and therefore $y \in H$. If $0 \notin H$, then, according to (4.3.36), for every $x \in G$ we have $0 \leqslant x$. Since $x \in H$, the $\curlywedge$-quasi-stability of H implies $x[x \cdots x] \curlywedge x \in H$. Hence $x[x \cdots x] \curlywedge x \neq 0$, because $0 \notin H$. From $x \leqslant y$, by the stability of $\leqslant$, we conclude $x[x \cdots x] \leqslant y[x \cdots x]$. Consequently, $x[x \cdots x] \curlywedge x \leqslant y[x \cdots x] \curlywedge x$. However $0 \leqslant x[x \cdots x] \curlywedge x$, therefore

$$0 \leqslant x[x \cdots x] \curlywedge x \leqslant y[x \cdots x] \curlywedge x.$$

Since $x[x \cdots x] \curlywedge x \neq 0$, the above gives us $y[x \cdots x] \curlywedge x \neq 0$, because in the opposite case, by antisymmetry of $\leqslant$, we obtain $x[x \cdots x] \curlywedge x = 0$, which is impossible. So, $y[x \cdots x] \curlywedge x \neq 0$, whence, according to (4.3.32), we conclude $y \in H$. This completes the proof of (4.3.37).

To prove (4.3.38) observe that for $x \in H$, by the $\curlywedge$-quasi-stability, we also have $x[x \cdots x] \curlywedge x \in H$, which in view of $x[x \cdots x] \curlywedge x \leqslant x[x \cdots x]$ and (4.3.37) implies $x[x \cdots x] \in H$. So, (4.3.38) is valid too.

Now, we shall verify (4.3.39). Let $x \in H$. If $0 \in H$ then, as it has been shown in the proof of Theorem 4.3.13, $H = G$. Thus, in this case, $R_i x \in H$ for every $i = 1, \ldots, n$. If $0 \notin H$, then, evidently, $x \neq 0$. Consequently, $R_i x \neq 0$ for every $i = 1, \ldots, n$, because in the opposite case from (3.7.5), it follows that $x = 0$. This, together with (4.3.38), gives us $x[x \cdots x] \neq 0$ and $R_i x[x \cdots x] \neq 0$. Now, applying (3.7.9), we obtain $0 \neq R_i((R_k x)[x \cdots x])$, whence, by (4.3.34), we deduce $(R_k x)[x \cdots x] \neq 0$ for every $k = 1, \ldots, n$. Since $(R_k x)[x \cdots x] \leqslant x$, we have

$$(R_k x)[x \cdots x] \curlywedge x = (R_k x)[x \cdots x] \neq 0,$$

which, by (4.3.32), implies $R_k x \in H$ for all $k = 1, \ldots, n$. The condition (4.3.39) is proved.

The proof of (4.3.40) is similar to the proof of (4.3.39). Namely, let $x[y \cdots y] \neq 0$ for some $x, y \in G$. If $0 \in H$, then, as in the previous case, $H = G$. Hence $R_i x \in H$ for every $i = 1, \ldots, n$. If $0 \notin H$, then $R_i x[y \cdots y] \neq 0$ for all $i = 1, \ldots, n$, because in the case $R_i x[y \cdots y] = 0$, by (3.7.5), we obtain $x[y \cdots y] = x[y \cdots y][R_1 x[y \cdots y] \cdots R_n x[y \cdots y]] = 0$ which contradicts our assumption. Next, applying (3.7.9), we get $0 \neq R_i x[y \cdots y] = R_i((R_k x)[y \cdots y])$, whence we deduce $(R_k x)[y \cdots y] \neq 0$ for each $k = 1, \ldots, n$. In fact, from the above, for $(R_k x)[y \cdots y] = 0$, it follows that $R_i 0 \neq 0$. This contradicts (4.3.34). Furthermore, $(R_k x)[y \cdots y] \leqslant y$ gives

$$(R_k x)[y \cdots y] \curlywedge y = (R_k x)[y \cdots y] \neq 0,$$

whence, by (4.3.32), we obtain $R_k x \in H$. This completes the proof of (4.3.40).

If $x[y \cdots y] = y \neq 0$ for some $x, y \in G$, then $x[y \cdots y] \curlywedge y = x[y \cdots y] \neq 0$, whence, according to (4.3.32), we have $x \in H$. This proves (4.3.41).

Now let $x \sqsubset y$ for some $x \in H$ and $y \in G$. Then, obviously, $R_i x \leqslant R_i y$ for each $i = 1, \ldots, n$. From this, applying (4.3.39) and (4.3.37), we obtain $R_i y \in H$. So, (4.3.42) is also true.

At last, let $0 \notin H$ and $x \sqsubset 0$. Then $0 \leqslant x$, by (4.3.36), and $0 = R_i 0$, by (4.3.34). Thus $0 = R_i 0 \leqslant R_i x$ for each $i = 1, \ldots, n$. But from $x \sqsubset 0$ we have also $R_i x \leqslant R_i 0 = 0$. Therefore $R_i x = 0$ for every $i = 1, \ldots, n$. Consequently, $x = x[R_1 x \cdots R_n x] = x[0 \cdots 0] = 0$. This completes the proof of (4.3.43) and the proof of Theorem 4.3.14. $\square$

4.4 Semi-compatibility relation

Let (Φ, O) be a Menger algebra of n-place functions, ξ_Φ – the semi-compatibility relation on it. The system (Φ, O, ξ_Φ) will be called the *transformative Menger algebra of n-place functions*. *Transformative f.o.p., f.o., p.q-o. Menger algebras* and *(p.q-o.) ∩-Menger algebras of n-place functions* are defined analogously.

We start with the following characterization of transformative f.o.p. Menger algebras.

Theorem 4.4.1. *An algebraic system (G, o, ξ, ζ, χ), where (G, o) is a Menger algebra of rank n and ξ, ζ, χ are binary relations on G, is isomorphic to a transformative f.o.p. Menger algebra of n-place functions if and only if*
(a) *ξ is an l-regular and i-regular quasi-equivalence for every $i = 1, \dots, n$,*
(b) *ζ is an l-regular order,*
(c) *χ is an l-regular and v-negative quasi-order,*
(d) *the following conditions are satisfied:*

$$\xi \cap \chi = \zeta, \tag{4.4.1}$$

$$g_1 \downarrow g_2 \ \wedge \ g \sqsubset g_1 \ \wedge \ g \sqsubset u[\bar{w}|_i g_2] \longrightarrow g \sqsubset u[\bar{w}|_i g_1], \tag{4.4.2}$$

$$g_0 \downarrow \cdots \downarrow g_m \ \wedge \ g_1, \dots, g_{m-1} \in \{g_0, g_m\}_{(\xi,\chi)} \longrightarrow g_0 \downarrow g_m \tag{4.4.3}$$

for all $m \in \mathbb{N}$, $i = 1, \dots, n$, $g, g_0, \dots, g_m \in G$, $u \in G \cup \{e_i\}$, $\bar{w} \in G^n$, where $g_1 \downarrow g_2$, $g_1 \sqsubset g_2$ and $g_0 \downarrow g_1 \downarrow \dots \downarrow g_m$ means respectively $(g_1, g_2) \in \xi$, $(g_1, g_2) \in \chi$, $(g_0, g_1) \in \xi \wedge \cdots \wedge (g_{m-1}, g_m) \in \xi$, $\{g_0, g_m\}_{(\xi,\chi)}$ is the (ξ, χ)-closure of the subset $\{g_0, g_m\}$ and $e_1, \dots, e_n$ are the selectors of (G^*, o^*).

Proof. Necessity. Let $(\Phi, O, \xi_\Phi, \zeta_\Phi, \chi_\Phi)$ be a transformative f.o.p. Menger algebra of n-place functions. It is clear that in this algebra the conditions (4.4.1) and (4.4.2) are satisfied. To prove condition (4.4.3), let $(\varphi_0, \varphi_1) \in \xi_\Phi, \dots, (\varphi_{m-1}, \varphi_m) \in \xi_\Phi$ and $\varphi_1, \dots, \varphi_{m-1} \in \{\varphi_0, \varphi_m\}_{(\xi_\Phi, \chi_\Phi)}$. We can prove by induction that for every $i = 1, \dots, m-1$, from $\varphi_i \in \{\varphi_0, \varphi_m\}_{(\xi_\Phi, \chi_\Phi)}$ follows $\mathrm{pr}_1\, \varphi_0 \cap \mathrm{pr}_1\, \varphi_m \subset \mathrm{pr}_1\, \varphi_i$. Similarly, for every $i = 0, 1, \dots, m-1$, from $(\varphi_i, \varphi_{i+1}) \in \xi_\Phi$ we conclude $\varphi_i \circ \triangle_{\mathrm{pr}_1 \varphi_{i+1}} = \varphi_{i+1} \circ \triangle_{\mathrm{pr}_1 \varphi_i}$. So,

$$\varphi_i \circ \triangle_{\mathrm{pr}_1 \varphi_0 \cap \mathrm{pr}_1 \varphi_m} = \varphi_{i+1} \circ \triangle_{\mathrm{pr}_1 \varphi_0 \cap \mathrm{pr}_1 \varphi_m}$$

for all $i = 0, 1, \dots, m-1$, whence $\varphi_0 \circ \triangle_{\mathrm{pr}_1 \varphi_m} = \varphi_m \circ \triangle_{\mathrm{pr}_1 \varphi_0}$. Thus $(\varphi_0, \varphi_m) \in \xi_\Phi$, which was to be proved.

Sufficiency. Let (G, o, ξ, ζ, χ) satisfy all the demands of the theorem. Obviously, the relation $\varepsilon(\xi, \{h_1, h_2\}_{(\xi,\chi)})$ and the subset $\{h_1, h_2\}_{(\xi,\chi)}$, where $h_1, h_2 \in G$, satisfy (a) of Proposition 2.1.14. Thus, the pairs $(\varepsilon^*_{(h_1, h_2)}, W_{(h_1, h_2)})$, where

$$\varepsilon^*_{(h_1,h_2)} = \varepsilon(\xi, \{h_1, h_2\}_{(\xi,\chi)}) \cup \{(e_1, e_1), \ldots, (e_n, e_n)\},$$
$$W_{(h_1,h_2)} = G \setminus \{h_1, h_2\}_{(\xi,\chi)},$$

$e_1, \ldots, e_n$ – the selectors in (G^*, o^*), are the determining pairs. Let P be the sum of the family of the simplest representations, which correspond to the determining pairs defined by $h_1, h_2 \in G$.

Let us show that $\zeta = \zeta_P$. Let $g_1 \leqslant g_2$, i.e., $(g_1, g_2) \in \zeta$. In this case also, $g_1[\bar{x}] \leqslant g_2[\bar{x}]$ for every $\bar{x} \in B = G^n \cup \{(e_1, \ldots, e_n)\}$. If $g_1[\bar{x}] \in \{h_1, h_2\}_{(\xi,\chi)}$ for $h_1, h_2 \in G$, then, from $\zeta \subset \xi$ and the ζ-saturability of $\{h_1, h_2\}_{(\xi,\chi)}$ we conclude

$$(g_1[\bar{x}], g_2[\bar{x}]) \in \xi \cap \{h_1, h_2\}_{(\xi,\chi)} \times \{h_1, h_2\}_{(\xi,\chi)} \subset \varepsilon^*_{(h_1,h_2)}.$$

Thus, $(g_1, g_2) \in \zeta_P$. So, $\zeta \subset \zeta_P$. Conversely, let $(g_1, g_2) \in \zeta_P$. Then for any $h_1, h_2 \in G$, $\bar{x} \in B$ from $g_1[\bar{x}] \in \{h_1, h_2\}_{(\xi,\chi)}$ it follows that $g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon^*_{(h_1,h_2)})$. This, in particular, for $\bar{x} = (e_1, \ldots, e_n)$ and $h_1 = h_2 = g_1$ shows that $(g_1, g_2) \in \varepsilon^*_{(g_1,g_1)}$. Since $\{g_1\}_{(\xi,\chi)} = \chi\langle g_1\rangle$, by (4.4.2), we have also $(g_1, g_2) \in \varepsilon(\xi, \chi\langle g_1\rangle)$. Moreover, in this case

$$x \leqslant y \wedge y \downarrow z \longrightarrow x \downarrow z \tag{4.4.4}$$

for all $x, y, z \in G$. Indeed, $x \leqslant y$ implies $x \downarrow y$ and similarly, $x \leqslant y$ and $x \in \{x, z\}_{(\xi,\chi)}$ imply $y \in \{x, z\}_{(\xi,\chi)}$. Consequently, according to (4.4.3), from $x \downarrow y \downarrow z$ and $y \in \{x, z\}_{(\xi,\chi)}$ it follows $x \downarrow z$. The condition $(g_1, g_2) \in \varepsilon(\xi, \chi\langle g_1\rangle)$ means that there are $x_0, x_1, \ldots, x_m \in G$ such that $x_0 = g_1$, $x_m = g_2$ and $(x_i, x_{i+1}) \in \xi \cap \chi\langle g_1\rangle \times \chi\langle g_1\rangle$ for all $i = 0, 1, \ldots, m-1$. As $g_1 \downarrow x_1$ and $g_1 \sqsubset x_1$, then, according to (4.4.1), is $g_1 \leqslant x_1$. But $x_1 \downarrow x_2$, so $g_1 \downarrow x_2$, by (4.4.4). Continuing a similar argumentation, after finite number of steps, we obtain $g_1 \downarrow g_2$. So, $g_1 \downarrow g_2$ and $g_1 \sqsubset g_2$ and consequently $g_1 \leqslant g_2$. This proves the inclusion $\zeta_P \subset \zeta$. Thus $\zeta = \zeta_P$. This means that the representation P is faithful.

The proofs of $\chi = \chi_P$ and $\xi = \xi_P$ are analogous. □

From Theorem 4.4.1, it follows that the relations ξ and ζ are stable. Indeed, let $x \downarrow y$ and $x_i \downarrow y_i$ for all $i = 1, \ldots, n$. Then

$$x[x_1 \cdots x_n] \downarrow y[x_1 \cdots x_n],$$

because the relation ξ is l-regular. Similarly, the i-regularity of χ and $x_i \downarrow y_i$, imply

$$y[y_1 \cdots y_{i-1} x_i \cdots x_n] \downarrow y[y_1 \cdots y_i x_{i+1} \cdots x_n]$$

for $i = 1, \ldots, n$. Therefore

$$x[x_1 \cdots x_n] \downarrow y[x_1 \cdots x_n] \downarrow y[y_1 x_2 \cdots x_n] \downarrow \cdots \downarrow y[y_1 \cdots y_n]. \tag{4.4.5}$$

Let H denote the (ξ,χ)-closure of the set $\{x[x_1\cdots x_n], y[y_1\cdots y_n]\}$. Since H is χ-saturated and $x[x_1\cdots x_n]\sqsubset x_i$ for $i=1,\ldots,n$, all $x_1,\ldots,x_n$ must be in H. Furthermore we have $x_n\downarrow y_n$, $y[y_1\cdots y_{n-1}x_n]\sqsubset y[y_1\cdots y_{n-1}x_n]$ and x_n, $y[y_1\cdots y_n]\in H$, which together with the fact that H is (ξ,χ)-closed gives us $y[y_1\cdots y_{n-1}x_n]\in H$. In a similar way, we obtain $y[y_1\cdots y_{n-2}x_{n-1}x_n]\in H$ and so on. Finally, $y[x_1\cdots x_n]\in H$. Therefore

$$y[x_1\cdots x_n]\,,\ y[y_1x_2\cdots x_n]\,,\ \ldots,\ y[y_1\cdots y_n]\in H.$$

This together with (4.4.3) and (4.4.5) gives $x[x_1\cdots x_n]\downarrow y[y_1\cdots y_n]$, which proves the stability of ξ.

The stability of ζ can be proved analogously.

Theorem 4.4.2. *An algebraic system (G,o,ξ,ζ), where (G,o) is a Menger algebra of rank n, ξ and ζ are binary relations on (G,o), is isomorphic to some transformative f.o. Menger algebra of n-place functions if and only if ξ is an l-regular and i-regular (for every $1\leqslant i\leqslant n$) quasi-equivalence, ζ is an l-regular order contained in ξ and the conditions*

$$g_1\downarrow g_2\ \wedge\ g\leqslant t_1(g_1)\ \wedge\ g\leqslant t_2(g_2)\longrightarrow g\leqslant t_2(g_1), \tag{4.4.6}$$

$$g_0\downarrow\cdots\downarrow g_m\ \wedge\ g_1,\ldots,g_{m-1}\in\{g_0,g_m\}_{(\xi,\,\delta\circ\zeta)}\longrightarrow g_0\downarrow g_m \tag{4.4.7}$$

are satisfied for all $g,g_0,g_1,\ldots,g_m\in G$, $t_1,t_2\in T_n(G)$.

Proof. We prove only the sufficiency. The necessity can be proved in the same way as in the previous theorem. Assume that a given algebraic system (G,o,ξ,ζ) satisfies all the conditions required by the theorem and consider an l-regular and v-negative quasi-order $\delta\circ\zeta$. Let $(g_1,g_2)\in\xi\cap\delta\circ\zeta$, i.e., $g_1\downarrow g_2$, $g_1\leqslant t(g_2)$ and $g_1\leqslant g_1$. Then, according to (4.4.6), we have $g_1\leqslant g_2$. Thus, $\xi\cap\delta\circ\zeta\subset\zeta$. But $\zeta\subset\xi$ and $\zeta\subset\delta\circ\zeta$, hence $\xi\cap\delta\circ\zeta=\zeta$. Further, putting in (4.4.6) $t_2(g_2)=t(u[\bar{w}|_i g_2])$ for some $u\in G\cup\{e_i\}$, $\bar{w}\in G^n$, $i=1,\ldots,n$, $t\in T_n(G)$, we obtain the condition which is analogous to condition (4.4.2) of Theorem 4.4.1. This means that $(G,o,\xi,\zeta,\delta\circ\zeta)$ satisfies all the conditions of our previous theorem, so there exists a faithful representation P of (G,o) by n-place functions, such that $\xi=\xi_P$ and $\zeta=\zeta_P$. □

Theorem 4.4.3. *An algebraic system (G,o,ξ,χ), where (G,o) is a Menger algebra of rank n, ξ and χ are binary relations on (G,o), is isomorphic to some transformative p.q-o. Menger algebra of n-place functions if and only if ξ is an l-regular and i-regular (for every $1\leqslant i\leqslant n$) quasi-equivalence, χ is an l-regular and v-negative quasi-order, the conditions* (4.4.2) *and* (4.4.3) *are satisfied, and*

$$g_1\downarrow g_2\ \wedge\ g_1\sqsubset g_2\ \wedge\ g_2\sqsubset g_1\longrightarrow g_1=g_2 \tag{4.4.8}$$

for all $g_1,g_2\in G$.

Proof. Indeed, for $\zeta_0 = \xi \cap \chi$ condition (4.4.8) is equivalent to the antisymmetry of ζ_0. Thus $(G, o, \xi, \zeta_0, \chi)$ satisfies all the conditions of Theorem 4.4.1, which means that the above theorem is true. □

Corollary 4.4.4. *An algebraic system (G, o, ξ), where (G, o) is a Menger algebra of rank n, $\xi \subset G \times G$, is isomorphic to some transformative Menger algebra of n-place functions if and only if ξ is an l-regular and i-regular (for every $1 \leqslant i \leqslant n$) quasi-equivalence and the following two implications*

$$g_1 \downarrow g_2 \wedge t_1(g_1) = t_2(g_2) \longrightarrow t_1(g_1) = t_2(g_1), \tag{4.4.9}$$

$$g_0 \downarrow \cdots \downarrow g_m \wedge g_1, \ldots, g_{m-1} \in \{g_0, g_m\}_{(\xi,\,\delta)} \longrightarrow g_0 \downarrow g_m \tag{4.4.10}$$

are satisfied for all $m \in \mathbb{N}$, $g_0, g_1, \ldots, g_m \in G$, $t_1, t_2 \in T_n(G)$.

Proof. Indeed, (G, o, ξ, δ) satisfies all the conditions of Theorem 4.4.3. □

The semi-compatibility relation on Menger algebras of reversive n-place functions is characterized in the following theorem.

Theorem 4.4.5. *An algebraic system (G, o, ξ, ζ, χ), where (G, o) is a Menger algebra of rank $n \geqslant 2$, and ξ, ζ, χ are binary relations on G, is isomorphic to some transformative f.o.p. Menger algebra of reversive n-place functions if and only if*

(a) *ξ is an l-regular and i-regular (for every $1 \leqslant i \leqslant n$) quasi-equivalence,*
(b) *χ is an l-regular and v-negative quasi-order,*
(c) *ζ is an l-regular order contained in $\xi \cap \chi$,*
(d) *the following conditions*

$$(g_1, g_2) \in \varepsilon_m\langle \xi, \chi\langle g_1\rangle\rangle \longrightarrow g_1 \leqslant g_2, \tag{4.4.11}$$

$$(g_1, g_2) \in \varepsilon_m\langle \xi, \chi\langle g\rangle\rangle \wedge g \sqsubset x[\bar{y}|_i g_2] \longrightarrow g \sqsubset x[\bar{y}|_i g_1], \tag{4.4.12}$$

$$(g_1, g_2) \in \varepsilon_m\langle \xi, \{g_1, g_2\}_{\langle \xi,\,\chi\rangle}\rangle \longrightarrow g_1 \downarrow g_2 \tag{4.4.13}$$

are satisfied for all $m \in \mathbb{N}$, $g, g_1, g_2 \in G$, $x \in G \cup \{e_i\}$, $\bar{y} \in G^n$, $i = 1, \ldots, n$, where $\{g_1, g_2\}_{\langle \xi,\,\chi\rangle}$ is the $\langle \xi, \chi\rangle$-closure of $\{g_1, g_2\}$, $e_1, \ldots, e_n$ – the selectors in (G^*, o^*).

Proof. Necessity. Let $(\Phi, O, \xi_\Phi, \zeta_\Phi, \chi_\Phi)$ be a transformative f.o.p. Menger algebra of reversive n-place functions. First of all, observe that it is not difficult to prove by induction the following two implications

$$(\psi_1, \psi_2) \in \varepsilon_m\langle \xi_\Phi, H\rangle \longrightarrow \psi_1 \circ \triangle_{\mathrm{pr}_1 H} = \psi_2 \circ \triangle_{\mathrm{pr}_1 H}, \tag{4.4.14}$$

$$\psi \in F_m(H) \longrightarrow \mathrm{pr}_1\, H \subset \mathrm{pr}_1\, \psi, \tag{4.4.15}$$

where $H \subset \mathbf{\Phi}$, $m \in \mathbb{N}$, $\psi, \psi_1, \psi_2 \in \mathbf{\Phi}$ are arbitrary, $\mathrm{pr}_1 H$ designates the set $\bigcap\{\mathrm{pr}_1 \varphi \mid \varphi \in H\}$, and $F_m(H) \subset \mathbf{\Phi}$ is defined as in Section 2.1 (p. 35) for $\rho = \xi_{\mathbf{\Phi}}$ and $\sigma = \chi_{\mathbf{\Phi}}$. The proofs of (4.4.11) and (4.4.12) are similar to the proof of the analogous conditions of Theorem 3.3.1. To prove (4.4.13), let

$$(\psi_1, \psi_2) \in \varepsilon_m \langle \xi_{\mathbf{\Phi}}, \{\psi_1, \psi_2\}_{\langle \xi_{\mathbf{\Phi}},\, \chi_{\mathbf{\Phi}} \rangle} \rangle,$$

where $H = \{\psi_1, \psi_2\}_{\langle \xi_{\mathbf{\Phi}},\, \chi_{\mathbf{\Phi}} \rangle}$. Since H is the union of all $F_m(\{\psi_1, \psi_2\})$, $m = 0, 1, \ldots$, then, according to (4.4.15), we have $\mathrm{pr}_1 \psi_1 \cap \mathrm{pr}_1 \psi_2 \subset \mathrm{pr}_1 H$. So,

$$\psi_1 \circ \triangle_{\mathrm{pr}_1 \psi_1 \cap \mathrm{pr}_1 \psi_2} = \psi_2 \circ \triangle_{\mathrm{pr}_1 \psi_1 \cap \mathrm{pr}_1 \psi_2},$$

whence $\psi_1 \circ \triangle_{\mathrm{pr}_1 \psi_2} = \psi_2 \circ \triangle_{\mathrm{pr}_1 \psi_1}$, i.e., $(\psi_1, \psi_2) \in \xi_{\mathbf{\Phi}}$, which was to be proved.

Sufficiency. Let (G, o, ξ, ζ, χ) satisfy all the conditions of the theorem and let P be the sum of the family of simplest representations of (G, o), which correspond to the determining pairs $(\varepsilon^*_{\langle h_1,\, h_2 \rangle}, W_{\langle h_1,\, h_2 \rangle})$, where $h_1, h_2 \in G$,

$$\varepsilon^*_{\langle h_1,\, h_2 \rangle} = \varepsilon \langle \xi, \{h_1, h_2\}_{\langle \xi,\, \chi \rangle} \rangle \cup \{(e_1, e_1), \ldots, (e_n, e_n)\},$$

$$W_{\langle h_1,\, h_2 \rangle} = G \setminus \{h_1, h_2\}_{\langle\, \xi,\, \chi \rangle},$$

$e_1, \ldots, e_n$ are selectors in (G^*, o^*). Since these determining pairs satisfy condition (2.7.1), P is a representation of (G, o) by reversive n-place functions.

Let $g_1 \downarrow g_2$, $\bar{x} \in B = G^n \cup \{(e_1, \ldots, e_n)\}$ and $g_1[\bar{x}], g_2[\bar{x}] \in \{h_1, h_2\}_{\langle \xi,\, \chi \rangle}$ for some $h_1, h_2 \in G$. Since the relation ξ is l-regular, $g_1 \downarrow g_2$ implies $g_1[\bar{x}] \downarrow g_2[\bar{x}]$. Therefore $(g_1[\bar{x}], g_2[\bar{x}]) \in \varepsilon_0 \langle \xi, \{h_1, h_2\}_{\langle\, \xi,\, \chi \rangle} \rangle \subset \varepsilon^*_{\langle h_1,\, h_2 \rangle}$. Thus, $(g_1, g_2) \in \xi_{(\varepsilon^*_{\langle h_1,\, h_2 \rangle}, W_{\langle h_1,\, h_2 \rangle})}$ for all $h_1, h_2 \in G$. Hence $(g_1, g_2) \in \xi_P$. Conversely, if $(g_1, g_2) \in \xi_P$, then, according to Section 2.7 (pp. 79, 80), we see that $\{g_1[\bar{x}], g_2[\bar{x}]\} \subset \{h_1, h_2\}_{\langle \xi,\, \chi \rangle}$ implies $(g_1[\bar{x}], g_2[\bar{x}]) \in \varepsilon \langle \xi, \{h_1, h_2\}_{\langle \xi,\, \chi \rangle} \rangle$ for all $h_1, h_2 \in G$, $\bar{x} \in B$. This applied to $\bar{x} = (e_1, \ldots, e_n)$, $h_1 = g_1$, $h_2 = g_2$ gives $(g_1, g_2) \in \varepsilon \langle \xi, \{g_1, g_2\}_{\langle \xi,\, \chi \rangle} \rangle$. Hence, by (4.4.13), we obtain $g_1 \downarrow g_2$. So, $\xi_P \subset \xi$, and consequently $\xi = \xi_P$. The proofs of $\zeta = \zeta_P$ and $\chi = \chi_P$ are similar.

Now, in view of the assumption that ζ is an order, we conclude that P is one-to-one. □

Starting from the formulas given in Section 2.1, each of the conditions (4.4.11), (4.4.12), and (4.4.13) can be written as a system of elementary conditions. Moreover, the conditions of Theorem 4.4.1 can be deduced from the conditions of the above theorem. So, the relations ξ and ζ are stable.

Theorem 4.4.6. *Let (G, o) be a Menger algebra of rank $n \geqslant 2$ and ξ, ζ be binary relations on it. An algebraic system (G, o, ξ, ζ) is isomorphic to some transformative f.o. Menger algebra of reversive n-place functions if and only if*

ξ is an l-regular and i-regular (for all $1 \leqslant i \leqslant n$) quasi-equivalence, ζ is a stable l-regular order contained in ξ, the conditions (4.4.6) *and*

$$(g_1, g_2) \in \varepsilon_m\langle \xi, \{g_1, g_2\}_{\langle \xi,\, \delta\circ\zeta\rangle}\rangle \longrightarrow g_1 \downarrow g_2 \tag{4.4.16}$$

are satisfied for all $m \in \mathbb{N}$, $g, g_1, g_2 \in G$.

Proof. The necessity of the above conditions is obvious. To prove their sufficiency assume that (G, o, ξ, ζ) satisfies all these conditions and consider the relation $\chi_0 = \delta \circ \zeta$, which, as it is known, is an l-regular and v-negative quasi-order containing ζ. By induction, it is not difficult to see that

$$\varepsilon_m\langle \xi, \chi_0\langle g\rangle\rangle \subset \varepsilon_v(\zeta\langle g\rangle)$$

for all $m \in \mathbb{N}$ and $g \in G$. Thus, from $(g_1, g_2) \in \varepsilon_m\langle \xi, \chi_0\langle g_1\rangle\rangle$ it follows $(g_1, g_2) \in \varepsilon_v(\zeta\langle g_1\rangle)$. Hence

$$(\forall t \in T_n(G))\ \ (g_1 \leqslant t(g_1) \longleftrightarrow g_1 \leqslant t(g_2))\,,$$

whence for $t = \triangle_G$ we conclude $g_1 \leqslant g_2$. This proves (4.4.11). Now let $(g_1, g_2) \in \varepsilon_m\langle \xi, \chi_0\langle g\rangle\rangle$ and $g \leqslant t_1(x[\bar{y}|_i g_2])$. Then $(g_1, g_2) \in \varepsilon_v(\zeta\langle g\rangle)$, which implies $g \leqslant t_1(x[\bar{y}|_i g_1])$. So condition (4.4.12) is fulfilled. This, together with (4.4.16), means that $(G, o, \xi, \zeta, \chi_0)$ satisfies all the conditions of Theorem 4.4.5. Therefore it is isomorphic to some transformative f.o. Menger algebra of reversive n-place functions. □

It is not difficult to see that an algebraic system $(G, o, \xi, \zeta_0, \chi)$, where $\zeta_0 = \xi \cap \chi$, satisfies all the conditions of Theorem 4.4.5. So, as a simple consequence of this theorem, we obtain

Corollary 4.4.7. *An algebraic system (G, o, ξ, χ), where (G, o) is a Menger algebra of rank $n \geqslant 2$ and ξ, χ are binary relations on G, is isomorphic to some transformative p.q-o. Menger algebra of reversive n-place functions if and only if ξ is an l-regular and i-regular (for all $1 \leqslant i \leqslant n$) quasi-equivalence, χ is an l-regular and v-negative quasi-order, and the conditions* (4.4.8), (4.4.12), (4.4.13) *are satisfied.*

Similarly, an algebraic system (G, o, ξ, δ) satisfies all the conditions of Theorem 4.4.7. Thus, as a consequence of the theorem, we have

Corollary 4.4.8. *An algebraic system (G, o, ξ), where (G, o) is a Menger algebra of rank $n \geqslant 2$, $\xi \subset G \times G$, is isomorphic to some transformative Menger algebra of reversive n-place functions if and only if ξ is an l-regular and i-regular (for every $1 \leqslant i \leqslant n$) quasi-equivalence, and for all $m \in \mathbb{N}$, $t_1, t_2, t \in T_n(G)$, $g, g_1, g_2 \in G$ the conditions* (4.4.9) *and*

$$(g_1, g_2) \in \varepsilon_m\langle \xi, \delta\langle g\rangle\rangle \,\wedge\, g = t(g_2) \longrightarrow g = t(g_1), \tag{4.4.17}$$

$$(g_1, g_2) \in \varepsilon_m\langle \xi, \{g_1, g_2\}_{\langle \xi,\delta\rangle}\rangle \longrightarrow g_1 \downarrow g_2 \tag{4.4.18}$$

are satisfied.

Now we will consider transformative $\cap$-Menger algebras of n-place functions.

Proposition 4.4.9. *In a transformative $\cap$-Menger algebra $(\Phi, O, \cap, \xi_\Phi)$ of n-place functions for all $\varphi, \psi, \varphi_i, \psi_i, \gamma, \alpha, \beta \in \Phi$, $\bar{\chi} \in \Phi^n$, $i = 1, \ldots, n$, $t_1, t_2 \in T_n(\Phi)$ the following implications:*

$$\varphi \subset \psi \longrightarrow (\varphi, \psi) \in \xi_\Phi, \tag{4.4.19}$$

$$\varphi \subset \psi \,\wedge\, \alpha \subset \beta \,\wedge\, (\psi, \beta) \in \xi_\Phi \longrightarrow (\alpha, \varphi) \in \xi_\Phi, \tag{4.4.20}$$

$$\bigwedge_{i=1}^{n} (\varphi_i, \psi_i) \in \xi_\Phi \longrightarrow \begin{cases} \alpha[(\varphi_1 \cap \psi_1) \cdots (\varphi_n \cap \psi_n)] \\ = \alpha[\varphi_1 \cdots \varphi_n] \cap \alpha[\psi_1 \cdots \psi_n], \end{cases} \tag{4.4.21}$$

$$\left.\begin{array}{r} (\varphi, \psi) \in \xi_\Phi, \\ pr_1\,\gamma \subset pr_1\,\varphi, \\ pr_1\,\gamma \subset pr_1\,\alpha[\bar{\chi}\,|_i\,\psi], \end{array}\right\} \longrightarrow pr_1\,\gamma \subset pr_1\,\alpha[\bar{\chi}\,|_i(\varphi \cap \psi)], \tag{4.4.22}$$

$$(\varphi, \psi) \in \xi_\Phi \longrightarrow t_1(\varphi) \cap t_2(\psi) = t_1(\varphi \cap \psi) \cap t_2(\psi) \tag{4.4.23}$$

are satisfied.

Proof. The conditions (4.4.19) and (4.4.20) are obvious. To prove (4.4.21), assume $(\varphi_i, \psi_i) \in \xi_\Phi$ for every $i = 1, \ldots, n$. Then $\varphi_i \circ \triangle_{\mathrm{pr}_1 \psi_i} = \psi_i \circ \triangle_{\mathrm{pr}_1 \varphi_i}$, and, consequently,

$$\begin{aligned} \varphi_i \cap \psi_i &= (\varphi_i \cap \psi_i) \circ \triangle_{\mathrm{pr}_1 \psi_i} = \varphi_i \circ \triangle_{\mathrm{pr}_1 \psi_i} \cap \psi_i \\ &= \psi_i \circ \triangle_{\mathrm{pr}_1 \varphi_i} \cap \psi_i = \psi_i \circ \triangle_{\mathrm{pr}_1 \varphi_i}\ (= \varphi_i \circ \triangle_{\mathrm{pr}_1 \psi_i}), \end{aligned}$$

for all $i = 1, \ldots, n$. Therefore

$$\begin{aligned} &\alpha[(\varphi_1 \cap \psi_1) \cdots (\varphi_n \cap \psi_n)] \\ &= \alpha[(\varphi_1 \circ \triangle_{\mathrm{pr}_1 \psi_1}) \cdots (\varphi_n \circ \triangle_{\mathrm{pr}_1 \psi_n})] \cap\, \alpha[(\psi_1 \circ \triangle_{\mathrm{pr}_1 \varphi_1}) \cdots (\psi_n \circ \triangle_{\mathrm{pr}_1 \varphi_n})] \\ &= \alpha[\varphi_1 \cdots \varphi_n] \circ \triangle_{\mathrm{pr}_1 \psi_1} \circ \cdots \circ \triangle_{\mathrm{pr}_1 \psi_n} \cap\, \alpha[\psi_1 \cdots \psi_n] \circ \triangle_{\mathrm{pr}_1 \varphi_1} \circ \cdots \circ \triangle_{\mathrm{pr}_1 \varphi_n} \\ &= \alpha[\varphi_1 \cdots \varphi_n] \circ \triangle_{\mathrm{pr}_1 \varphi_1} \circ \cdots \circ \triangle_{\mathrm{pr}_1 \varphi_n} \cap\, \alpha[\psi_1 \cdots \psi_n] \circ \triangle_{\mathrm{pr}_1 \psi_1} \circ \cdots \circ \triangle_{\mathrm{pr}_1 \psi_n} \\ &= \alpha[\varphi_1 \cdots \varphi_n] \cap\, \alpha[\psi_1 \cdots \psi_n], \end{aligned}$$

i.e., $\alpha[(\varphi_1 \cap \psi_1) \cdots (\varphi_n \cap \psi_n)] = \alpha[\varphi_1 \cdots \varphi_n] \cap \alpha[\psi_1 \cdots \psi_n]$, which proves (4.4.21).

Assume now that all the conditions of the premise of (4.4.22) are satisfied. From $(\varphi, \psi) \in \xi_\Phi$ we deduce $\varphi \cap \psi = \varphi \circ \triangle_{\mathrm{pr}_1 \psi} = \psi \circ \triangle_{\mathrm{pr}_1 \varphi}$, whence

$$\mathrm{pr}_1\,(\varphi \cap \psi) = \mathrm{pr}_1\,(\varphi \circ \triangle_{\mathrm{pr}_1 \psi}) = \mathrm{pr}_1\,\varphi \cap \mathrm{pr}_1\,\psi.$$

But, by assumption, $\mathrm{pr}_1\,\gamma \subset \mathrm{pr}_1\,\varphi$ and $\mathrm{pr}_1\,\gamma \subset \mathrm{pr}_1\,\alpha[\bar{\chi}\,|_i\,\psi]$. Thus

$$\mathrm{pr}_1\,\gamma \subset \mathrm{pr}_1\,\varphi \cap \mathrm{pr}_1\,\psi = \mathrm{pr}_1\,(\varphi \cap \psi).$$

Hence

$$\begin{aligned}\mathrm{pr}_1\,\gamma &= \triangle_{\mathrm{pr}_1\,(\varphi\cap\psi)}(\mathrm{pr}_1\,\gamma) \subset \triangle_{\mathrm{pr}_1\,(\varphi\cap\psi)}(\mathrm{pr}_1\,\alpha[\bar{\chi}\,|_i\,\psi]) \\ &= \mathrm{pr}_1\,\alpha[\bar{\chi}\,|_i\,(\psi \circ \triangle_{\mathrm{pr}_1\,(\varphi\cap\psi)})] = \mathrm{pr}_1\,\alpha[\bar{\chi}\,|_i(\varphi \cap \psi)].\end{aligned}$$

This proves (4.4.22).

If $(\varphi, \psi) \in \xi_\Phi$, then, evidently, $\varphi \circ \triangle_{\mathrm{pr}_1\,\psi} = \varphi \cap \psi$. Therefore

$$\begin{aligned}t_1(\varphi) \cap t_2(\psi) &= t_1(\varphi) \cap t_2(\psi \circ \triangle_{\mathrm{pr}_1\,\psi}) = t_1(\varphi) \cap t_2(\psi) \circ \triangle_{\mathrm{pr}_1\,\psi} \\ &= t_1(\varphi) \circ \triangle_{\mathrm{pr}_1\,\psi} \cap t_2(\psi) = t_1(\varphi \circ \triangle_{\mathrm{pr}_1\,\psi}) \cap t_2(\psi) \\ &= t_1(\varphi \cap \psi) \cap t_2(\psi),\end{aligned}$$

which proves (4.4.23) and completes the proof. □

Let $(G, o, \curlywedge)$ be an algebraic system such that (G, o) is a Menger algebra of rank n, $(G, \curlywedge)$ – a semilattice, ρ – a quasi-equivalence on G, σ – a v-negative quasi-order on (G, o), $\zeta \subset \rho \cap \sigma$ – an order in a semilattice $(G, \curlywedge)$. It can be shown that *if ζ is stable on (G, o), then for any subset $X \subset G$ the following two conditions*

$$(a, b) \in \rho \,\wedge\, (u[\bar{w}|_i a], c) \in \sigma \,\wedge\, a, u[\bar{w}|_i b] \in X \longrightarrow c \in X, \tag{4.4.24}$$

$$(a, b) \in \rho \,\wedge\, \{a, b\} \subset X \longrightarrow a \curlywedge b \in X \tag{4.4.25}$$

are simultaneously satisfied if and only if X satisfies the condition

$$(a, b) \in \rho \,\wedge\, (u[\bar{w}|_i(a \curlywedge b)], c) \in \sigma \,\wedge\, a, u[\bar{w}\,|_i\,b] \in X \longrightarrow c \in X, \tag{4.4.26}$$

for all $a, b, c \in G$, $u \in G \cup \{e_i\}$, $\bar{w} \in G^n$, $i = 1, \ldots, n$.

Let $X^{\curlywedge}_{(\rho,\sigma)}$ be the least (ρ, σ)-closure of $X \subset G$ satisfying condition (4.4.25). Clearly,

$$X^{\curlywedge}_{(\rho,\sigma)} = \bigcup_{n=0}^{\infty} \overset{n}{E}_{\curlywedge}(X),$$

where $g \in E_{\curlywedge}(X)$ if and only if the premise of (4.4.26) is valid for some $a, b, u, \bar{w}, i$. One can show by induction that $g \in \overset{m}{E}_{\curlywedge}(X)$ if and only if the first formula of Proposition 2.1.15, in which the expression of the form $(u[\bar{w}\,|_k\,a], g) \in \sigma$ is replaced by the expression $(u[\bar{w}\,|_k\,a \curlywedge b], g) \in \sigma$, is true in $(G, o, \curlywedge)$.

Proposition 4.4.10. *In a $\cap$-Menger algebra $(\Phi, O, \cap)$ of n-place functions for any $\varphi, \psi \in \Phi$ the following implications*

$$\varphi \cap \psi \in \{\varphi, \psi\}^{\curlywedge}_{(\xi_\Phi,\, \chi_\Phi)} \longrightarrow (\varphi, \psi) \in \xi_\Phi, \tag{4.4.27}$$

$$\varphi \cap \psi \in \{\varphi, \psi\}^{\curlywedge}_{(\xi_\Phi,\, \delta_\Phi \circ \zeta_\Phi)} \longrightarrow (\varphi, \psi) \in \xi_\Phi, \tag{4.4.28}$$

where $\delta_\Phi = \{(\varphi, \psi) \,|\, (\exists t \in T_n(\Phi))\ \varphi = t(\psi)\}$, *are true.*

Proof. Let $X \subset \Phi$. For any $\varphi \in E_\curlywedge(X)$ there are $\chi, \psi_1, \psi_2 \in \Phi$, $\bar\omega \in \Phi^n$, $i \in \{1, \ldots, n\}$ such that $(\psi_1, \psi_2) \in \xi_\Phi$, $\mathrm{pr}_1\, \chi[\bar\omega\,|_i(\psi_1 \cap \psi_2)] \subset \mathrm{pr}_1\, \varphi$ and $\psi_1, \chi[\bar\omega\,|_i\,\psi_2] \in X$. Then

$$\mathfrak{A} = \bigcap\{\mathrm{pr}_1\, \psi \,|\, \psi \in X\} \subset \mathrm{pr}_1\, \psi_1 \cap \mathrm{pr}_1\, \chi[\bar\omega\,|_i\,\psi_2] \subset \mathrm{pr}_1\, \psi_1 \cap \mathrm{pr}_1\, \psi_2.$$

Since $(\psi_1, \psi_2) \in \xi_\Phi$, we have $\psi_1 \circ \triangle_{\mathrm{pr}_1 \psi_2} = \psi_2 \circ \triangle_{\mathrm{pr}_1 \psi_1} = \psi_1 \cap \psi_2$, whence $\psi \circ \triangle_{\mathfrak{A}} = \psi_2 \circ \triangle_{\mathfrak{A}} = (\psi_1 \cap \psi_2) \circ \triangle_{\mathfrak{A}}$. Thus

$$\begin{aligned}\mathfrak{A} &\subset \triangle_{\mathfrak{A}}(\mathrm{pr}_1\, \chi[\bar\omega\,|_i\,\psi_2]) = \mathrm{pr}_1\, \chi[\bar\omega\,|_i(\psi_2 \circ \triangle_{\mathfrak{A}})] \\ &= \mathrm{pr}_1\, \chi[\bar\omega\,|_i((\psi_1 \cap \psi_2) \circ \triangle_{\mathfrak{A}})] \subset \mathrm{pr}_1\, \chi[\bar\omega\,|_i(\psi_1 \cap \psi_2)] \subset \mathrm{pr}_1\, \varphi.\end{aligned}$$

If $\varphi \in \{\psi_1, \psi_2\}^{\curlywedge}_{(\xi_\Phi,\, \chi_\Phi)}$, then $\varphi \in \overset{m}{E}_\curlywedge(\{\psi_1, \psi_2\})$ for some $m \in \mathbb{N}$. Therefore

$$\begin{aligned}\mathrm{pr}_1\, \varphi &\supset \bigcap\{\mathrm{pr}_1\, \psi \,|\, \psi \in \overset{m-1}{E}_\curlywedge(\{\psi_1, \psi_2\})\} \supset \cdots \\ &\supset \bigcap\{\mathrm{pr}_1\, \psi \,|\, \psi \in E_\curlywedge(\{\psi_1, \psi_2\})\} \supset \mathrm{pr}_1\, (\psi_1 \cap \psi_2).\end{aligned}$$

Now let $\varphi \cap \psi \in \{\varphi, \psi\}^{\curlywedge}_{(\xi_\Phi, \chi_\Phi)}$, then $\mathrm{pr}_1\, \varphi \cap \mathrm{pr}_1\, \psi \subset \mathrm{pr}_1\, (\varphi \cap \psi)$, and consequently $\mathrm{pr}_1\, (\varphi \cap \psi) = \mathrm{pr}_1\, \varphi \cap \mathrm{pr}_1\, \psi$. So,

$$\varphi \circ \triangle_{\mathrm{pr}_1 \psi} = \varphi \circ \triangle_{\mathrm{pr}_1 \varphi \cap \mathrm{pr}_1 \psi} = \varphi \circ \triangle_{\mathrm{pr}_1 (\varphi \cap \psi)} = \varphi \cap \psi.$$

In the same way, we obtain $\psi \circ \triangle_{\mathrm{pr}_1 \varphi} = \varphi \cap \psi$, $\varphi \circ \triangle_{\mathrm{pr}_1 \psi} = \psi \circ \triangle_{\mathrm{pr}_1 \varphi}$ and $(\varphi, \psi) \in \xi_\Phi$. This proves (4.4.27).

The proof of (4.4.28) is similar. □

Theorem 4.4.11. *An algebraic system $(G, o, \curlywedge, \xi, \chi)$, where $(G, o, \curlywedge)$ is an algebra of type $(n+1, 2)$, ξ, χ are binary relations on G, is isomorphic to some transformative p.q-o. $\cap$-Menger algebra of n-place functions if and only if (G, o) is a Menger algebra of rank n, $(G, \curlywedge)$ is a semilattice, ξ is an l-regular relation, χ is an l-regular and v-negative quasi-order, and the conditions*

$$(x \curlywedge y)[z_1 \cdots z_n] = x[z_1 \cdots z_n] \curlywedge y[z_1 \cdots z_n], \tag{4.4.29}$$

$$x \curlywedge y = x \longrightarrow x \downarrow y, \tag{4.4.30}$$

$$x \curlywedge y = x \wedge u \curlywedge v = u \wedge y \downarrow v \longrightarrow u \downarrow x, \tag{4.4.31}$$

$$\bigwedge_{i=1}^{n} x_i \downarrow y_i \longrightarrow \begin{cases} u[(x_1 \curlywedge y_1) \cdots (x_n \curlywedge y_n)] \\ = u[x_1 \cdots x_n] \curlywedge u[y_1 \cdots y_n], \end{cases} \tag{4.4.32}$$

$$x \downarrow y \wedge z \sqsubset x \wedge z \sqsubset u[\bar{w}\,|_i\, y] \longrightarrow z \sqsubset u[\bar{w}|_i(x \curlywedge y)], \tag{4.4.33}$$

$$x \curlywedge y = x \longrightarrow x \sqsubset y, \tag{4.4.34}$$

$$x \sqsubset x \curlywedge y \longrightarrow x = x \curlywedge y, \tag{4.4.35}$$

$$x \curlywedge y \in \{x, y\}^{\curlywedge}_{(\xi,\chi)} \longrightarrow x \downarrow y \tag{4.4.36}$$

hold for all $x, y, z, u, v, x_i, y_i, z_i \in G,\ i = 1, \ldots, n,$ *and* $\bar{w} \in G^n$*, where* $u[\bar{w}\,|_i\ ]$ *may be the empty symbol.*[8]

Proof. The necessity of these conditions follows from the two propositions above. To prove their sufficiency consider an arbitrary algebraic system $(G, o, \curlywedge, \xi, \chi)$ in which all these conditions are valid. Then from (4.4.30) and (4.4.34), it follows that an order ζ of a semilattice $(G, \curlywedge)$ is contained in ξ and χ. From (4.4.29) and (4.4.21), we obtain the stability of ζ in (G, o). By (4.4.31) ξ is symmetric.

Let X be a subset of $(G, o, \curlywedge)$ for which condition (4.4.26) is satisfied for $\rho = \xi$ and $\sigma = \chi$. Since X is χ-saturated, $X' = G \setminus X$ is an l-ideal of (G, o). It is not difficult to see that the relation

$$\varepsilon(X) = \{(g_1, g_2) \,|\, g_1 \curlywedge g_2 \in X \vee g_1, g_2 \in X'\} \subset G \times G$$

is a v-congruence on (G, o). Moreover, $\varepsilon(X)$ and X' satisfy the conditions (3.4.8)–(3.4.10). Thus, $(\varepsilon^*(X), X')$ is the determining pair satisfying condition (3.4.7).

Consider the representation P of (G, o) such that

$$P = \sum_{(h_1,h_2)\in G\times G} P_{(\varepsilon^*_{(h_1,h_2)}, W_{(h_1,h_2)})}, \tag{4.4.37}$$

where $\varepsilon_{(h_1,h_2)} = \varepsilon(\{h_1, h_2\}^{\curlywedge}_{(\xi,\chi)})$, $W_{(h_1,h_2)} = G \setminus \{h_1, h_2\}^{\curlywedge}_{(\xi,\chi)}$. According to (3.4.7), for any $g_1, g_2 \in G$ we have $P(g_1 \curlywedge g_2) = P(g_1) \cap P(g_2)$. Let $(g_1, g_2) \in \xi_P$. Then for any $h_1, h_2 \in G$, $\bar{x} \in B = G^n \cup \{(e_1, \ldots, e_n)\}$ from $g_1[\bar{x}], g_2[\bar{x}] \in \{h_1, h_2\}^{\curlywedge}_{(\xi,\chi)}$, it follows that $g_1[\bar{x}] \curlywedge g_2[\bar{x}] \in \{h_1, h_2\}^{\curlywedge}_{(\xi,\chi)}$. This inclusion applied to $\bar{x} = (e_1, \ldots, e_n)$, $h_1 = g_1$, $h_2 = g_2$ implies $g_1 \curlywedge g_2 \in \{g_1, g_2\}^{\curlywedge}_{(\xi,\chi)}$, which, by (4.4.36), gives us $g_1 \downarrow g_2$. Conversely, if $g_1 \downarrow g_2$ and $g_1[\bar{x}], g_2[\bar{x}] \in \{h_1, h_2\}^{\curlywedge}_{(\xi,\chi)}$, then $g_1[\bar{x}] \downarrow g_2[\bar{x}]$, whence $g_1[\bar{x}] \curlywedge g_2[\bar{x}] \in \{h_1, h_2\}^{\curlywedge}_{(\xi,\chi)}$, according to (4.4.25). Thus $(g_1, g_2) \in \xi_P$, and consequently $\xi = \xi_P$.

Similarly we can prove that $\zeta = \zeta_P$ and $\chi = \chi_P$. Since ζ is an order, the above inclusion means that the representation P is faithful. □

[8] Recall that if $u[\bar{w}\,|_i\ ]$ is the empty symbol then $u[\bar{w}\,|_i x]$ is equal to x.

Corollary 4.4.12. *An algebraic system $(G, o, \curlywedge, \xi)$, where $(G, o, \curlywedge)$ is an algebra of type $(n+1, 2)$, $\xi \subset G \times G$, is isomorphic to some transformative $\cap$-Menger algebra of n-place functions if and only if (G, o) is a Menger algebra of rank n, $(G, \curlywedge)$ is a semilattice, ξ – an l-regular relation satisfying the conditions (4.4.29)–(4.4.31) and*

$$x \downarrow y \longrightarrow t_1(x) \curlywedge t_2(y) = t_1(x \curlywedge y) \curlywedge t_2(y), \tag{4.4.38}$$

$$x \curlywedge y \in \{x, y\}^{\curlywedge}_{(\xi,\, \delta\circ\zeta)} \longrightarrow x \downarrow y \tag{4.4.39}$$

where ζ is a semilattice order and $x, y \in G$, $t_1, t_2 \in T_n(G)$ are arbitrary.

Proof. Indeed, the necessity of (4.4.39) follows from Proposition 4.4.10. The condition (4.4.38) can be proved directly by checking. To prove the sufficiency, it is enough to show that the system $(G, o, \curlywedge, \xi, \chi(\zeta))$, where $\chi(\zeta) = \delta \circ \zeta$, satisfies all the conditions of Theorem 4.4.11. □

Substituting condition (4.4.32) in Theorem 4.4.11 by (3.4.3), we obtain a system of conditions that is an abstract characterization of the class of all transformative p.q-o. $\cap$-Menger algebras of reversive n-place functions. Adding (3.4.3) to the conditions of Corollary 4.4.12, we obtain a system of conditions characterizing the class of all transformative $\cap$-Menger algebras of reversive n-place functions. Starting from the first condition of Proposition 2.1.15 and condition (4.4.26) it is not difficult to write (4.4.36) and (4.4.39) as a system of elementary formulas.

4.5 Co-definability relation

Let ρ be a binary relation on a (f.o.p., f.o., p.q-o.) Menger algebra (G, o) of rank n. If there exists an isomorphism P of (G, o) onto some (respectively: f.o.p., f.o., p.q-o.) Menger algebra of (reversive) n-place functions such that $\rho = \gamma_P$, then ρ is called a (*reversive*) *co-definability relation* on a (respectively: f.o.p., f.o., p.q-o.) Menger algebra (G, o). If (G, o) has a zero 0 and $(g, g) \in \rho$ for any non-zero element $g \in G$, then we say that ρ is 0-*reflexive*. A symmetric relation ρ which is reflexive when $0 \in \mathrm{pr}_1\, \rho$, and 0-reflexive when $0 \notin \mathrm{pr}_1\, \rho$, is called a 0-*quasi-equivalence*.

Theorem 4.5.1. *For a binary relation γ of a f.o.p. Menger algebra (G, o, ζ, χ) of rank n to be its co-definability relation it is necessary and sufficient that γ is an l-cancellative* 0-*quasi-equivalence, and the following two conditions*

$$(0, 0) \notin \gamma \longrightarrow 0 \leqslant g, \tag{4.5.1}$$

$$h_1 \top h_2 \,\wedge\, g_1, g_2 \in \{h_1, h_2\}_{(\zeta, \chi)} \longrightarrow g_1 \top g_2 \tag{4.5.2}$$

are satisfied for all $g, g_1, g_2, h_1, h_2 \in G$*, where* $h_1 \top h_2$ *means* $(h_1, h_2) \in \gamma$*,* 0 *is a zero of* (G, o) *and* $\{h_1, h_2\}_{(\zeta,\chi)}$ *is the* (ζ, χ)*-closure of* $\{h_1, h_2\}$*.*

Proof. Necessity. Let $(\Phi, O, \zeta_\Phi, \chi_\Phi)$ be a f.o.p. Menger algebra of n-place functions with the co-definability relation γ_Φ defined as in Section 1.2. Assume that φ is a zero of (Φ, O). If $\varphi \neq \varnothing$, then $\operatorname{pr}_1 \varphi \neq \varnothing$. Thus $(\varphi, \varphi) \in \gamma_\Phi$. If $\varphi = \varnothing$, then $\operatorname{pr}_1 \varphi = \varnothing$. Therefore $(\varphi, \varphi) \notin \gamma_\Phi$ and $\varphi \subset \psi$ for any $\psi \in \Phi$. Thus γ_Φ is φ-reflexive and satisfies condition (4.5.1). Since it is also symmetric, it is a φ-quasi-equivalence.

Now let $(\alpha[\bar{\psi}], \beta[\bar{\psi}]) \in \gamma_\Phi$ for some $\alpha, \beta \in \Phi$ and $\bar{\psi} \in \Phi^n$. Then $\operatorname{pr}_1 \alpha[\bar{\psi}] \cap \operatorname{pr}_1 \beta[\bar{\psi}] \neq \varnothing$. Thus there exists $\bar{a}$ such that $\alpha[\bar{\psi}]\langle\bar{a}\rangle \neq \varnothing$ and $\beta[\bar{\psi}]\langle\bar{a}\rangle \neq \varnothing$. So, $\{\alpha(\psi_1(\bar{a}), \ldots, \psi_n(\bar{a}))\} \neq \varnothing$ and $\{\beta(\psi_1(\bar{a}), \ldots, \psi_n(\bar{a}))\} \neq \varnothing$, whence $(\psi_1(\bar{a}), \ldots, \psi_n(\bar{a})) \in \operatorname{pr}_1 \alpha \cap \operatorname{pr}_1 \beta$, i.e., $\operatorname{pr}_1 \alpha \cap \operatorname{pr}_1 \beta \neq \varnothing$. This implies $(\alpha, \beta) \in \gamma_\Phi$. So, the relation γ_Φ is l-cancellative.

Similarly $(\alpha, \beta) \in \gamma_\Phi$ and $\varphi, \psi \in \{\alpha, \beta\}_{(\zeta_\Phi, \chi_\Phi)}$ imply $\operatorname{pr}_1 \alpha \cap \operatorname{pr}_1 \beta \neq \varnothing$. Moreover, in this case $\operatorname{pr}_1 \alpha \cap \operatorname{pr}_1 \beta \subset \operatorname{pr}_1 \varphi \cap \operatorname{pr}_1 \psi$, which can be simply proved by induction starting from Proposition 2.1.15. So, $\operatorname{pr}_1 \varphi \cap \operatorname{pr}_1 \psi \neq \varnothing$, i.e., $(\varphi, \psi) \in \gamma_\Phi$. This proves (4.5.2) and completes the proof of the necessity.

Sufficiency. Assume that the relation $\gamma \subset G \times G$ satisfies all the conditions of the theorem. Let $\big((\varepsilon^*_{(h_1,h_2)}, W_{(h_1,h_2)})\big)_{(h_1,h_2)\in\gamma}$ be the family of the determining pairs such that

$$\varepsilon^*_{(h_1,h_2)} = \varepsilon(\zeta^{-1} \circ \zeta, \{h_1, h_2\}_{(\zeta,\chi)}) \cup \{(e_1, e_1), \ldots, (e_n, e_n)\},$$

$$W_{(h_1,h_2)} = G \setminus \{h_1, h_2\}_{(\zeta,\chi)},$$

$e_1, \ldots, e_n$ – selectors in (G^*, o^*). Consider the sum P of the family of the simplest representations determined by these pairs. If $g_1 \leqslant g_2$, $h_1 \top h_2$, $\bar{x} \in B = G^n \cup \{(e_1, \ldots, e_n)\}$ and $g_1[\bar{x}] \in \{h_1, h_2\}_{(\zeta,\chi)}$, then the stability of ζ implies $g_1[\bar{x}] \leqslant g_2[\bar{x}]$. Whence $g_2[\bar{x}] \in \{h_1, h_2\}_{(\zeta,\chi)}$, and consequently, $(g_1[\bar{x}], g_2[\bar{x}]) \in \zeta^{-1} \circ \zeta \cap (\{h_1, h_2\}_{(\zeta,\chi)})^2 \subset \varepsilon^*_{(h_1,h_2)}$. So, $\zeta \subset \zeta_{(\varepsilon^*_{(h_1,h_2)}, W_{(h_1,h_2)})}$ for all $(h_1, h_2) \in \gamma$, i.e., $\zeta \subset \zeta_P$. Conversely, if $(g_1, g_2) \in \zeta_P$, then for all $(h_1, h_2) \in \gamma$ and $\bar{x} \in B$ from $g_1[\bar{x}] \in \{h_1, h_2\}_{(\zeta,\chi)}$ it follows $(g_1[\bar{x}], g_2[\bar{x}]) \in \overline{\zeta^{-1} \circ \zeta \cap (\{h_1, h_2\}_{(\zeta,\chi)})^2}$, where $\overline{\rho}$ denotes a transitive closure of a binary relation ρ. This for $\bar{x} = (e_1, \ldots, e_n)$ means that the implication

$$g_1 \in \{h_1, h_2\}_{(\zeta,\chi)} \longrightarrow (g_1, g_2) \in \overline{\zeta^{-1} \circ \zeta \cap (\{h_1, h_2\}_{(\zeta,\chi)})^2}$$

is true for all $(h_1, h_2) \in \gamma$. Since $\{g_1\}_{(\zeta,\chi)} = \chi\langle g_1\rangle$, the above gives $(g_1, g_2) \in \overline{\zeta^{-1} \circ \zeta \cap \chi\langle g_1\rangle \times \chi\langle g_1\rangle}$ for all $(g_1, g_1) \in \gamma$. Thus $g_1 \leqslant g_2$. If $(g_1, g_1) \notin \gamma$, then $g_1 = 0$, whence $g_1 \leqslant g_2$, according to (4.5.1). Therefore $\zeta_P \subset \zeta$. So, $\zeta = \zeta_P$. The proof of $\chi = \chi_P$ is analogous.

Now let $g_1 \top g_2$. Then, obviously, $\{g_1[e_1 \cdots e_n], g_2[e_1 \cdots e_n]\} \cap W_{(g_1,g_2)} = \varnothing$, because $g_1, g_2 \in \{g_1, g_2\}_{(\zeta,\chi)}$. Thus $(g_1, g_2) \in \gamma_{(\varepsilon^*_{(g_1,g_2)}, W_{(g_1,\,g_2)})} \subset \gamma_P$, by results obtained in Section 2.7. So, $\gamma \subset \gamma_P$. Conversely, if $(g_1, g_2) \in \gamma_P$, then $g_1[\bar{x}], g_2[\bar{x}] \in \{h_1, h_2\}_{(\zeta,\chi)}$ for some $(h_1, h_2) \in \gamma$, $\bar{x} \in B$. From this, by (4.5.2), we obtain $g_1[\bar{x}] \top g_2[\bar{x}]$, whence, by the l-cancellativity of γ, we get $g_1 \top g_2$. Thus, $\gamma_P \subset \gamma$, i.e., $\gamma = \gamma_P$. Using the antisymmetry of ζ it is not difficult to see that the representation P is faithful. □

Putting $\rho = \zeta$ and $\sigma = \chi$ in the formula (2.1.12), one can prove that condition (4.5.2) is equivalent to the system of implications $(A'_m)_{m\in\mathbb{N}}$, where

$$A'_m:\ h_1 \top h_2 \ \wedge\ g_1, g_2 \in \overset{m}{E}\,(\{h_1, h_2\}) \longrightarrow g_1 \top g_2 .$$

Furthermore, from Proposition 2.1.15, it follows that each implication A'_m is equivalent to the formula

$$A_m:\ h_1 \top h_2 \ \wedge\ A_{1\,m} \ \wedge\ A_{2\,m} \longrightarrow g_1 \top g_2\,,$$

where

$$A_{s\,m}: \begin{cases} a_{s1} \leqslant b_{s1} \ \wedge\ u_{s1}[\bar{w}_{s1}\,|_{k_{s1}} a_{s1}] \sqsubset g_s\,, \\ \displaystyle\bigwedge_{i=1}^{2^{m-1}-1} \begin{pmatrix} a_{s,2i} \leqslant b_{s,2i} \ \wedge\ u_{s,2i}[\bar{w}_{s,2i}\,|_{k_{s,2i}} a_{s,2i}] \sqsubset a_{s,i}\,, \\ a_{s,2i+1} \leqslant b_{s,2i+1}\,, \\ u_{s,2i+1}[\bar{w}_{s,2i+1}\,|_{k_{s,2i+1}} a_{s,2i+1}] \sqsubset u_{s,i}[\bar{w}_{s,i}\,|_{k_{s,i}} b_{s,i}] \end{pmatrix}, \\ \displaystyle\bigwedge_{i=2^{m-1}}^{2^m-1} \Big(a_{s,i}, u_{s,i}[\bar{w}_{s,i}\,|_{k_{s,i}} b_{s,i}] \in \{h_1, h_2\}\Big) \end{cases}$$

for $s = 1, 2$.

In this way, we have showed that condition (4.5.2) is equivalent to the system of elementary formulas $(A_m)_{m\in\mathbb{N}}$. Thus, the following theorem has been proved.

Theorem 4.5.2. *A binary relation γ of a f.o.p. Menger algebra (G, o, ζ, χ) of rank n is its co-definability relation if and only if γ is an l-cancellative* 0*-quasi-equivalence and condition* (4.5.1) *together with the system of elementary formulas $(A_m)_{m\in\mathbb{N}}$ are satisfied.*

Replacing in each A_m the conditions of the form $\alpha_{s,j} \sqsubset \beta_s$, where $\alpha_{s,j} = u_{s,j}[\bar{w}_{s,j}\,|_{k_{s,j}} a_{s,j}]$ and β_s is one of the expressions g_s, $a_{s,i}$, $u_{s,i}[\bar{w}_{s,i}\,|_{k_{s,i}} b_{s,i}]$, by the conditions $\alpha_{s,j} \leqslant t_{s,j}(\beta_s)$, where $j \in \{1, 2i, 2i+1\}$, we obtain a system of formulas that will be denoted by $(C_m)_{m\in\mathbb{N}}$.

Using these formulas, we can give the following characterization of the co-definability relations in f.o. Menger algebras.

Corollary 4.5.3. *A binary relation γ of a f.o. Menger algebra (G, o, ζ) of rank n is its co-definability relation if and only if γ is an l-cancellative* 0*-quasi-equivalence and the condition* (4.5.1) *together with the system of formulas $(C_m)_{m\in\mathbb{N}}$ are satisfied.*

Indeed, it is easy to see that on a f.o.p. Menger algebra $(G, o, \zeta, \chi(\zeta))$, where $\chi(\zeta) = \delta \circ \zeta$, such γ satisfies all the conditions of Theorem 4.5.2.

Theorem 4.5.4. *A binary relation γ on a p.q-o. Menger algebra (G, o, χ) of rank n is its co-definability relation if and only if it is an l-cancellative* 0*-quasi-equivalence such that*

$$(0, 0) \notin \gamma \wedge g \sqsubset 0 \longrightarrow g = 0, \tag{4.5.3}$$

$$h_1 \top h_2 \wedge h_1 \sqsubset g_1 \wedge h_2 \sqsubset g_2 \longrightarrow g_1 \top g_2 \tag{4.5.4}$$

hold for all $h_1, h_2, g_1, g_2, g \in G$.

Proof. Since the necessity of these conditions is obvious, we only prove their sufficiency. For this consider on a p.q-o. Menger algebra (G, o, χ) an l-cancellative 0-quasi-equivalence γ satisfying the conditions (4.5.3) and (4.5.4). It is easy to check that the relation

$$\zeta_0 = \begin{cases} \triangle_G & \text{if } (0,0) \in \gamma, \\ \triangle \cup \{(0, g) \,|\, g \in G\} & \text{if } (0,0) \notin \gamma, \end{cases}$$

is the stable order on (G, o). Since χ is v-negative, then for all $g \in G$ we have $0 \sqsubset g$. Thus $\zeta_0 \subset \chi$. Also it is not difficult to see that a subset H of (G, o) is (ζ_0, χ)-closed if and only if it is χ-saturated. So, $(G, O, \zeta_0, \chi, \gamma)$ satisfies all the conditions of Theorem 4.5.1. Therefore, γ is the co-definability relation on (G, o, χ). □

Corollary 4.5.5. *A binary relation γ on a Menger algebra (G, o) of rank n is its co-definability relation if and only if it is an l-cancellative and v-negative* 0*-quasi-equivalence.*

Indeed, on a p.q-o. Menger algebra (G, o, δ) γ satisfies all the conditions of Theorem 4.5.4. We will now characterize the reversive co-definability relation on a strong Menger algebra.

Theorem 4.5.6. *For a binary relation γ on a strong f.o.p. Menger algebra (G, o, ζ, χ) of rank $n \geqslant 2$ to be its reversive co-definability relation, it is necessary and sufficient that γ is an l-cancellative* 0*-quasi-equivalence satisfying for all $h_1, h_2, g_1, g_2 \in G$ the condition*

$$h_1 \top h_2 \wedge g_1, g_2 \in \{h_1, h_2\}_{\langle \zeta^{-1} \circ \zeta, \chi \rangle} \longrightarrow g_1 \top g_2\,, \tag{4.5.5}$$

where $\{h_1, h_2\}_{\langle \zeta^{-1} \circ \zeta, \chi \rangle}$ denotes the $\langle \zeta^{-1} \circ \zeta, \chi \rangle$-closure of $\{h_1, h_2\}$.

Proof. Necessity. Let $(\Phi, O, \zeta_\Phi, \chi_\Phi)$ be some f.o.p. Menger algebra of reversive n-place functions. Let us show that γ_Φ satisfies condition (4.5.5). Indeed, it is not difficult to see that for all $m \in \mathbb{N}$, $\psi \in \Phi$, $H \subset \Phi$ the following implication is true

$$\psi \in F_m(H) \longrightarrow \mathrm{pr}_1 H \subset \mathrm{pr}_1 \psi, \tag{4.5.6}$$

where $\mathrm{pr}_1 H = \bigcap\{\mathrm{pr}_1 \varphi \,|\, \varphi \in H\}$ and $F_m(H) \subset \Phi$ is defined as in Section 2.1 on p. 35 for $\rho = \zeta_\Phi^{-1} \circ \zeta_\Phi$, $\sigma = \chi_\Phi$. Thus, according to (4.5.6), for $(\psi_1, \psi_2) \in \gamma_\Phi$ and $\varphi_1, \varphi_2 \in \{\psi_1, \psi_2\}_{\langle\zeta_\Phi^{-1}\circ\zeta_\Phi, \chi_\Phi\rangle}$, we have $\varnothing \neq \mathrm{pr}_1 \psi_1 \cap \mathrm{pr}_1 \psi_2 \subset \mathrm{pr}_1 \varphi \cap \mathrm{pr}_1 \varphi_2$, whence $(\varphi_1, \varphi_2) \in \gamma_\Phi$, which was to be proved.

Sufficiency. Let γ satisfy the conditions of the theorem. First observe that $0 \leqslant g$ for any $g \in G$, where 0 is a zero of (G, o). Indeed, χ is v-negative, so $0[g^n] \sqsubset g$. But, $0[g^n] \leqslant 0$, $0 \leqslant 0$ and $0[0g \cdots g] \leqslant 0$. Thus

$$(0[g^n], 0) \in \varepsilon_0\langle\zeta^{-1} \circ \zeta, \chi\langle 0\rangle\rangle \quad \text{and} \quad (0, 0[0g \cdots g]) \in \varepsilon_0\langle\zeta^{-1} \circ \zeta, \chi\langle 0\rangle\rangle,$$

whence $(0, g) \in \varepsilon_1\langle\zeta^{-1} \circ \zeta, \chi\langle 0\rangle\rangle$, which, by (3.3.2), proves $0 \leqslant g$.

Let P be the sum of these simplest representations of a Menger algebra (G, o), which correspond to the family of the determining pairs

$$(\varepsilon^*_{(h_1,h_2)}, W_{(h_1,h_2)})_{(h_1,h_2)\in\gamma}\,,$$

where

$$\varepsilon^*_{(h_1,h_2)} = \varepsilon\langle\zeta^{-1} \circ \zeta, \{h_1, h_2\}_{\langle\zeta^{-1}\circ\zeta, \chi\rangle}\rangle \cup \{(e_1, e_1), \ldots, (e_n, e_n)\},$$

$$W_{(h_1, h_2)} = G \setminus \{h_1, h_2\}_{\langle\zeta^{-1}\circ\zeta, \chi\rangle}\,,$$

$e_1, \ldots, e_n$ – the selectors in (G^*, o^*). Since these determining pairs satisfy (2.7.1), the sum P is a representation of (G, o) by reversive n-place functions.

Let $g_1 \leqslant g_2$, $\bar{x} \in B = G^n \cup \{(e_1, \ldots, e_n)\}$ and $g_1[\bar{x}] \in \{h_1, h_2\}_{\langle\zeta^{-1}\circ\zeta,\chi\rangle}$, where $h_1 \top h_2$. Then $g_1[\bar{x}] \leqslant g_2[\bar{x}]$, because ζ is l-regular. From this, we conclude $g_2[\bar{x}] \in \{h_1, h_2\}_{\langle\zeta^{-1}\circ\zeta,\chi\rangle}$ and

$$(g_1[\bar{x}], g_2[\bar{x}]) \in \varepsilon_0\langle\zeta^{-1} \circ \zeta, \{h_1, h_2\}_{\langle\zeta^{-1}\circ\zeta,\chi\rangle}\rangle \subset \varepsilon^*_{(h_1,h_2)}.$$

So, $\zeta \subset \zeta_P$. Conversely, let $(g_1, g_2) \in \zeta_P$. If $g_1 = 0$, then $g_1 \leqslant g_2$. If $g_1 \neq 0$, then $g_1 \top g_1$, and consequently $(g_1, g_2) \in \zeta_{(\varepsilon^*_{(g_1,g_1)}, W_{(g_1,g_1)})}$. Whence $(g_1, g_2) \in \varepsilon_m\langle\zeta^{-1} \circ \zeta, \chi\langle g_1\rangle\rangle$ for some $m \in \mathbb{N}$. This, according to (3.3.2), implies $g_1 \leqslant g_2$. So, $\zeta_P \subset \zeta$ and therefore $\zeta = \zeta_P$. The equality obtained proves that the representation P is faithful.

The proof of $\chi = \chi_P$ is similar. The equality $\gamma = \gamma_P$ can be proved in the same way as in the proof of Theorem 4.5.1. □

Corollary 4.5.7. *For a binary relation γ on a strong f.o. Menger algebra (G, o, ζ) of rank $n \geqslant 2$ to be its reversive co-definability relation, it is necessary and sufficient to be an l-cancellative* 0*-quasi-equivalence satisfying for all $h_1, h_2, g_1, g_2 \in G$ the condition*

$$h_1 \top h_2 \wedge g_1, g_2 \in \{h_1, h_2\}_{\langle \zeta^{-1} \circ \zeta, \delta \circ \zeta \rangle} \longrightarrow g_1 \top g_2 . \qquad (4.5.7)$$

Indeed, such γ is a reversive co-definability relation on a strong f.o.p. Menger algebra $(G, o, \zeta, \chi(\zeta))$, where $\chi(\zeta) = \delta \circ \zeta$.

Theorem 4.5.8. *A binary relation γ on a strong f.o. Menger algebra (G, o, χ) of rank $n \geqslant 2$ is its reversive co-definability relation if and only if it is an l-cancellative* 0*-quasi-equivalence satisfying for all $h_1, h_2, g_1, g_2 \in G$ the condition*

$$h_1 \top h_2 \wedge g_1, g_2 \in \{h_1, h_2\}_{\langle \triangle_G, \chi \rangle} \longrightarrow g_1 \top g_2 . \qquad (4.5.8)$$

Proof. The necessity of these conditions can be checked in the same way as in the previous theorem. To prove the sufficiency, we consider the representation

$$P = \sum_{(h_1,h_2) \in \gamma} P_{(\varepsilon^*_{(h_1,h_2)}, W_{(h_1,h_2)})},$$

of (G, o) by reversive n-place functions, where

$$\varepsilon^*_{(h_1,h_2)} = \varepsilon \langle \triangle_G, \{h_1, h_2\}_{\langle \triangle_G, \chi \rangle} \rangle \cup \{(e_1, e_1), \ldots, (e_n, e_n)\},$$

$$W_{(h_1,h_2)} = G \setminus \{h_1, h_2\}_{\langle \triangle_G, \chi \rangle},$$

$e_1, \ldots, e_n$ – the selectors in (G^*, o^*), and prove that $\chi = \chi_P$ and $\gamma = \gamma_P$. Thus, from the proof of Theorem 3.3.3 we conclude that this representation is faithful. □

Theorem 4.5.9. *A binary relation γ on a strong Menger algebra (G, o) of rank $n \geqslant 2$ is its reversive co-definability relation if and only if it is an l-cancellative* 0*-quasi-equivalence satisfying for all $h_1, h_2, g_1, g_2 \in G$ the condition*

$$h_1 \top h_2 \wedge g_1, g_2 \in \{h_1, h_2\}_{\left\langle \overset{\wedge}{\zeta}^{-1} \circ \overset{\wedge}{\zeta}, \, \delta \circ \overset{\wedge}{\zeta} \right\rangle} \longrightarrow g_1 \top g_2 , \qquad (4.5.9)$$

where $\overset{\wedge}{\zeta}$ is a strong order on (G, o).

Proof. Let P be a faithful representation of (G, o) by reversive n-place functions such that $\gamma = \gamma_P$. Since $\overset{\wedge}{\zeta} \subset \zeta_P$ and $\delta \circ \overset{\wedge}{\zeta} \subset \delta \circ \zeta_P \subset \chi_P$, then

$$\{h_1, h_2\}_{\left\langle \overset{\wedge}{\zeta}^{-1} \circ \overset{\wedge}{\zeta}, \, \delta \circ \overset{\wedge}{\zeta} \right\rangle} \subset \{h_1, h_2\}_{\left\langle \zeta_P^{-1} \circ \zeta_P, \, \chi_P \right\rangle}.$$

Thus, from the premise of (4.5.9) we get $(g_1, g_2) \in \{h_1, h_2\}_{\left\langle \zeta_P^{-1} \circ \zeta_P, \chi_P \right\rangle}$ and $h_1 \top h_2$, which, according to (4.5.5), implies $g_1 \top g_2$. Conversely, if γ satisfies the conditions of the theorem, then, evidently, it satisfies the conditions of Corollary 4.5.7 for a strong f.o. Menger algebra $(G, o, \overset{\wedge}{\zeta})$. Whence γ is a reversive co-definability relation on (G, o). □

Using the same argumentation as in Section 2.1, one can see that condition (4.5.9) is equivalent to a system of elementary formulas and each one of them can be obtained by a finite number of steps.

4.6 Connectivity relation

A binary relation κ on a Menger algebra (G, o) of rank n is called a (*reversive*) *connectivity relation*, if there exists an isomorphism P of (G, o) onto some Menger algebra of (reversive) n-place functions such that $\kappa = \kappa_P$, where κ_P is defined as in Section 2.7, p. 79.

For any $h_1, h_2 \in G$ by $\theta(h_1, h_2)$ we shall denote the set of pairs $(g_1, g_2) \in G \times G$ for which there exist $x, y \in \{h_1, h_2\}$ and $t \in T_n(G)$ such that $g_1 = t(x)$, $g_2 = t(y)$. By $[h_1, h_2]_{(\zeta,\chi)}$, where $\rho(h_1, h_2) = \zeta \cup \theta(h_1, h_2)$, we shall denote the $(\rho(h_1, h_2), \chi)$-closure of the set $\{h_1, h_2\}$ in a f.o.p. Menger algebra (G, o, ζ, χ).

The following theorem gives the characterization of the connectivity relation in the class of all f.o.p. Menger algebras.

Theorem 4.6.1. *For a relation $\kappa \subset G \times G$ to be a connectivity relation on a f.o.p. Menger algebra (G, o, ζ, χ) of rank n it is necessary and sufficient that it is 0-reflexive, l-cancellative and for all $h_1, h_2, g_1, g_2, g \in G$ the following two conditions hold:*

$$(0, 0) \notin \kappa \longrightarrow 0 \leqslant g, \tag{4.6.1}$$

$$\left.\begin{array}{r} (h_1, h_2) \in \kappa \\ (g_1, g_2) \in \varepsilon(h_1, h_2) \\ g_1 \in [h_1, h_2]_{(\zeta,\chi)} \end{array}\right\} \longrightarrow (g_1, g_2) \in \kappa\,, \tag{4.6.2}$$

where 0 is a zero of (G, o), $x \leqslant y \longleftrightarrow (x, y) \in \zeta$ and $\varepsilon(h_1, h_2)$ denotes the relation $\varepsilon(\zeta^{-1} \circ \zeta \cup \theta(h_1, h_2), [h_1, h_2]_{(\zeta,\chi)})$ defined according to (2.1.8).

Proof. Necessity. We only verify condition (4.6.2), because the rest is obvious. Let $(\Phi, O, \zeta_\Phi, \chi_\Phi)$ be a fixed f.o.p. Menger algebra of n-place functions. If $(f_1, f_2) \in \kappa_\Phi$, $(\varphi_1, \varphi_2) \in \varepsilon(f_1, f_2)$ and $\varphi_1 \in [f_1, f_2]_{(\zeta_\Phi,\chi_\Phi)}$, then the last two conditions give $\varphi_2 \in [f_1, f_2]_{(\zeta_\Phi,\chi_\Phi)}$. As $f_1 \cap f_2 \neq \varnothing$, there are $\bar{a}, b$ such that $(\bar{a}, b) \in f_1 \cap f_2$, whence $f_1(\bar{a}) = f_2(\bar{a})$. Next, by induction we can show that

$\varphi_1, \varphi_2 \in [f_1, f_2]_{(\zeta_\Phi,\chi_\Phi)}$ implies $\bar{a} \in \mathrm{pr}_1\,\varphi_1 \cap \mathrm{pr}_1\,\varphi_2$. Furthermore, from $(\varphi_1, \varphi_2) \in \varepsilon(f_1, f_2)$ we deduce

$$\{(\varphi_1, \chi_1), (\chi_1, \chi_2), \ldots, (\chi_m, \varphi_2)\} \subset \zeta_\Phi^{-1} \circ \zeta_\Phi \cup \theta(f_1, f_2),$$

$$\{\chi_1, \chi_2, \ldots, \chi_m\} \subset [f_1, f_2]_{(\zeta_\Phi,\chi_\Phi)}$$

for some $m \in \mathbb{N}$ and $\chi_1, \ldots, \chi_m \in \mathbf{\Phi}$. The last inclusion gives us, of course, $\bar{a} \in \mathrm{pr}_1\,\chi_i$ for all $i = 1, \ldots, m$. Moreover, for the pair (φ_1, χ_1) we have

$$(\varphi_1, \chi_1) \in \zeta_\Phi^{-1} \circ \zeta_\Phi \quad \text{or} \quad (\varphi_1, \chi_1) \in \theta(f_1, f_2).$$

In the first case, $\varphi_1 \subset \gamma$ and $\chi_1 \subset \gamma$ for some $\gamma \in \mathbf{\Phi}$. Thus $\varphi_1(\bar{a}) = \gamma(\bar{a}) = \chi_1(\bar{a})$. In the second case, $\{\varphi_1, \chi_1\} \subset \{t(f_1), t(f_2)\}$ for some $t \in T_n(\mathbf{\Phi})$. Since from $f_1(\bar{a}) = f_2(\bar{a})$, it follows that $t(f_1)(\bar{a}) = t(f_2)(\bar{a})$, we have also $\varphi_1(\bar{a}) = \chi_1(\bar{a})$. Repeating this procedure we obtain $\chi_1(\bar{a}) = \chi_2(\bar{a})$ and so on. Thus, $\varphi_1(\bar{a}) = \chi_1(\bar{a}) = \chi_2(\bar{a}) = \cdots = \chi_m(\bar{a}) = \varphi_2(\bar{a})$. So, $\varphi_1 \cap \varphi_2 \neq \varnothing$, i.e., $(\varphi_1, \varphi_2) \in \kappa_\Phi$, which was to be proved.

Sufficiency. Assume that a binary relation κ on a f.o.p. Menger algebra (G, o, ζ, χ) of rank n satisfies all the conditions of the theorem. Let P be the sum of these simplest representations of a Menger algebra (G, o), which correspond to the family of the determining pairs $(\varepsilon^*_{(h_1,h_2)}, W_{(h_1,h_2)})$, where $(h_1, h_2) \in \kappa$ and

$$\varepsilon^*_{(h_1,h_2)} = \varepsilon(h_1, h_2) \cup \{(e_1, e_1), \ldots, (e_n, e_n)\},$$

$$W_{(h_1,h_2)}) = G \setminus [h_1, h_2]_{(\zeta,\chi)},$$

$e_1, \ldots, e_n$ – are the selectors in (G^*, o^*). Let us show that P is a faithful representation such that $\zeta = \zeta_P$, $\chi = \chi_P$ and $\kappa = \kappa_P$.

Let $g_1 \leqslant g_2$. Then, using the stability of ζ, we have $g_1[\bar{x}] \leqslant g_2[\bar{x}]$ for all $\bar{x} \in B = G^n \cup \{(e_1, \ldots, e_n)\}$. Thus, for $g_1[\bar{x}] \in [h_1, h_2]_{(\zeta,\chi)}$, where $(h_1, h_2) \in \kappa$, we have $g_2[\bar{x}] \in [h_1, h_2]_{(\zeta,\chi)}$. Hence,

$$(g_1[\bar{x}], g_2[\bar{x}]) \in \zeta \cap \left([h_1, h_2]_{(\zeta,\chi)}\right)^2 \subset \varepsilon(h_1, h_2).$$

So, $\zeta \subset \zeta_P$.

Conversely, if $(g_1, g_2) \in \zeta_P$, then, evidently, for all $(h_1, h_2) \in \kappa$, $\bar{x} \in B$ from $g_1[\bar{x}] \in [h_1, h_2]_{(\zeta,\chi)}$, it follows that $(g_1[\bar{x}], g_2[\bar{x}]) \in \varepsilon(h_1, h_2)$. This, applied to $\bar{x} = (e_1, \ldots, e_n)$, means that for all $(h_1, h_2) \in \kappa$ from $g_1 \in [h_1, h_2]_{(\zeta,\chi)}$ it follows that $(g_1, g_2) \in \varepsilon(h_1, h_2)$. If $g_1 \neq 0$, then according to the 0-reflexivity of κ, we have $(g_1, g_1) \in \kappa$. Consequently, putting $h_1 = h_2 = g_1$, we get

$$(g_1, g_2) \in \overline{\zeta^{-1} \circ \zeta \cap \chi\langle g_1\rangle \times \chi\langle g_1\rangle},$$

whence, as it is known (see the proof of Theorem 4.3.3), we obtain $g_1 \leqslant g_2$. The same argumentation take place when $g_1 = 0$ and $(0,0) \in \kappa$. If $g_1 = 0$ and $(0,0) \notin \kappa$, then, according to (4.6.1), we have $g_1 = 0 \leqslant g_2$. This proves the inclusion $\zeta_P \subset \zeta$, and consequently, the equality $\zeta = \zeta_P$. The proof of the equality $\chi = \chi_P$ is similar. Since ζ is an order, from the above, it follows that the representation P is faithful.

Now let $(g_1, g_2) \in \kappa$. Since $\{g_1, g_2\} \subset [g_1, g_2]_{(\zeta,\chi)}$ and $(g_1, g_2) \in \theta(g_1, g_2)$, we have also $(g_1, g_2) \in \varepsilon(g_1, g_2)$, i.e., $(g_1[e_1, \ldots, e_n], g_2[e_1, \ldots, e_n]) \in \varepsilon(g_1, g_2)$. So, there exist such $(h_1, h_2) \in \kappa$ and $\bar{x} \in B$ such that $g_1[\bar{x}] \in [h_1, h_2]_{(\zeta,\chi)}$ and $(g_1[\bar{x}], g_2[\bar{x}]) \in \varepsilon(g_1, g_2)$. Thus $(g_1, g_2) \in \kappa_P$. Conversely, if $(g_1, g_2) \in \kappa_P$, then $g_1[\bar{x}] \in [h_1, h_2]_{(\zeta,\chi)}$ and $g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon(h_1, h_2))$ for some $\bar{x} \in B$ and $(h_1, h_2) \in \kappa$. Whence, according to (4.6.2), we get $(g_1[\bar{x}], g_2[\bar{x}]) \in \kappa$. This, according to the l-cancellativity of κ, implies $(g_1, g_2) \in \kappa$. Therefore $\kappa = \kappa_P$, which completes the proof. □

Based on the definition of the relation $\varepsilon(h_1, h_2)$, it is not difficult to see that condition (4.6.2) is equivalent to the system of implications $(A_m)_{m\in\mathbb{N}}$, where

$$A_m : \ (h_1, h_2) \in \kappa \wedge A'_m \longrightarrow (x_0, x_m) \in \kappa \,,$$

and

$$A'_m : \bigwedge_{i=0}^{m-1} \begin{pmatrix} (\, x_i \leqslant z_i \wedge x_{i+1} \leqslant z_i \,) \vee \\ (\, x_i = t_i(h_1) \wedge x_{i+1} = t_i(h_2) \,) \end{pmatrix} \wedge \ x_0, \ldots, x_m \in [h_1, h_2]_{(\zeta,\chi)}$$

for all $h_1, h_2, x_i, z_i \in G,\ t_i \in T_n(G),\ i = 0, 1, \ldots, m.$

Thus the following theorem is true.

Theorem 4.6.2. *For a relation $\kappa \subset G \times G$ to be a connectivity relation on a f.o.p. Menger algebra (G, o, ζ, χ) of rank n it is necessary and sufficient that κ is 0-reflexive, l-cancellative and condition (4.6.1) together with the system of elementary formulas $(A_m)_{m\in\mathbb{N}}$ are satisfied.*

Now let (G, o, ζ) be a f.o. Menger algebra of rank n. As it is known,[9] the system $(G, o, \zeta, \delta \circ \zeta)$ is a f.o.p. Menger algebra, so, as a simple consequence of our Theorem 4.6.2, we obtain the following corollary.

Corollary 4.6.3. *For a relation $\kappa \subset G \times G$ to be a connectivity relation on a f.o. Menger algebra (G, o, ζ) it is necessary and sufficient to be 0-reflexive, l-cancellative and to satisfy condition (4.6.1) together with the system of formulas $(A_m)_{m\in\mathbb{N}}$, in which χ is replaced by $\delta \circ \zeta$.*

[9] See the proof of Theorem 3.2.3.

Consider now a p.q-o. Menger algebra (G, o, χ) of rank n and a binary relation $\kappa \subset G \times G$. Let

$$\zeta_0 = \begin{cases} \triangle_G & \text{if } (0,0) \in \kappa, \\ \triangle_G \cup \{0\} \times G & \text{if } (0,0) \notin \kappa. \end{cases}$$

Then, as it is not difficult to see, ζ_0 is a stable order on (G, o).

Theorem 4.6.4. *For κ to be a connectivity relation on a p.q-o. Menger algebra (G, o, χ) of rank n it is necessary and sufficient to be 0-reflexive, l-cancellative and to satisfy for all $h_1, h_2, g, g_1, g_2 \in G$ the following conditions:*

$$(0,0) \notin \kappa \;\wedge\; g \sqsubset 0 \longrightarrow g = 0, \tag{4.6.3}$$

$$\left.\begin{array}{r} (h_1, h_2) \in \kappa \\ (g_1, g_2) \in \varepsilon^0(h_1, h_2) \\ g_1 \in [h_1, h_2]_{(\zeta_0, \chi)} \end{array}\right\} \longrightarrow (g_1, g_2) \in \kappa, \tag{4.6.4}$$

where

$$\varepsilon^0(h_1, h_2) = \varepsilon(\zeta_0^{-1} \circ \zeta_0 \cup \theta(h_1, h_2), [h_1, h_2]_{(\zeta_0, \chi)}).$$

Proof. Let P be a faithful representation of a p.q-o. Menger algebra (G, o, χ) by n-place functions such that $\chi = \chi_P$, $\kappa = \kappa_P$. If $(0,0) \notin \kappa$ and $g \sqsubset 0$, then, evidently, $P(0) = \varnothing$ and $\mathrm{pr}_1\, P(g) = \varnothing$. So, $P(g) = P(0)$, whence $g = 0$, which proves (4.6.3). Furthermore, from $\zeta_0 \subset \zeta_P$ we obtain $[h_1, h_2]_{(\zeta_0, \chi)} \subset [h_1, h_2]_{(\zeta_P, \chi_P)}$, and consequently, $\varepsilon^0(h_1, h_2) \subset \varepsilon(h_1, h_2)$. Thus, if the premise of (4.6.4) is satisfied, then the premise of (4.6.2) is also satisfied on a f.o.p. Menger algebra (G, o, ζ_P, χ_P), because κ_P is its connectivity relation. Hence $(g_1, g_2) \in \kappa_P$, i.e., $(g_1, g_2) \in \kappa$. The necessity is proved.

In order to prove the sufficiency observe that any system (G, o, ζ_0, χ) satisfying (4.6.3) is a f.o.p. Menger algebra of rank n, in which κ satisfies all the conditions of Theorem 4.6.1. Therefore κ is a connectivity relation on (G, o, ζ_0, χ), and consequently, on (G, o, χ). □

It is not difficult to see that if (G, o, χ) is a p.q-o. Menger algebra of rank n, then for every $h_1, h_2 \in G$ the subset $X \subset G$ is $(\rho_0(h_1, h_2), \chi)$-closed, where $\rho_0(h_1, h_2) = \zeta_0 \cup \theta(h_1, h_2)$, if and only if

$$t(a) \sqsubset g \;\wedge\; t(b) \in X \longrightarrow g \in X$$

for all $t \in T_n(G)$, $a, b \in \{h_1, h_2\}$ if $a \neq b$, or $a \in G$ if $a = b$. For every $X \subset G$ we define the subset $E_0(X)$ of G containing all these $g \in G$ for which $t(a) \sqsubset g$ and $t(b) \in X$ for some $t \in T_n(G)$, $a, b \in \{h_1, h_2\}$ if $a \neq b$, or $a \in G$ if $a = b$. Of course,

$$[X]_{(\zeta_0, \chi)} = \bigcup_{n=0}^{\infty} \overset{n}{E_0}(X),$$

where $\overset{n}{E_0}(X) = X$ for $n = 0$, and $\overset{n}{E_0}(X) = E_0(\overset{n-1}{E_0}(X))$ for $n > 0$. By induction, one can prove that for every $m \in \mathbb{N}$ condition, $g \in \overset{m}{E_0}(X)$ means that there exist $t_i \in T_n(G)$, $a_i, b_i \in \{h_1, h_2\}$ for $a_i \neq b_i$ or $a_i \in G$ for $a_i = b_i$, $i = 0, 1, \ldots, m$, such that the following formula

$$t_0(a_0) \sqsubset g \wedge \bigwedge_{i=0}^{m-1} \Big(t_{i+1}(a_{i+1}) \sqsubset t_i(b_i)\Big) \wedge t_m(b_m) \in X \qquad (4.6.5)$$

is true.

Let $(h_1, h_2) \in \kappa$. If $(0,0) \notin \kappa$, then, as it is not difficult to see, by (4.6.4), we get $0 \notin [h_1, h_2]_{(\zeta_0,\chi)}$. This means that for all $(g_1, g_2) \in \varepsilon^0(h_1, h_2)$, one can find $x_0, x_1, \ldots, x_m \in [h_1, h_2]_{(\zeta_0,\chi)}$ such that $g_1 = x_0$, $g_2 = x_m$ and $(x_i, x_{i+1}) \in \triangle_G \cup \theta(h_1, h_2)$ for all $i = 0, 1, \ldots, m-1$. Therefore for all $i = 0, 1, \ldots, m-1$, we have $x_i = t_i(a_i) \in [h_1, h_2]_{(\zeta_0,\chi)}$ and $x_{i+1} = t_i(b_i) \in [h_1, h_2]_{(\zeta_0,\chi)}$, where $t_i \in T_n(G)$, $a_i, b_i \in \{h_1, h_2\}$ if $a_i \neq b_i$, and $a_i \in G \cup \{0\}$ if $a_i = b_i$. Thus, condition (4.6.4) is equivalent to the system of formulas $(C_m)_{m\in\mathbb{N}}$, where

$$C_m : (h_1, h_2) \in \kappa \wedge C'_m \longrightarrow (t_0(a_0), t_m(b_m)) \in \kappa$$

and

$$C'_m : \begin{cases} t_0(a_0) \in [h_1, h_2]_{(\zeta_0,\chi)}\,, \\ \bigwedge_{i=0}^{m-1} \Big(t_i(b_i) = t_{i+1}(a_{i+1}) \in [h_1, h_2]_{(\zeta_0,\chi)}\Big), \\ t_m(b_m) \in [h_1, h_2]_{(\zeta_0,\chi)} \end{cases}$$

for all $t_i \in T_n(G)$, $a_i, b_i \in \{h_1, h_2\}$ when $a_i \neq b_i$, and $a_i \in G \setminus \{0\}$ when $a_i = b_i$, $i = 0, 1, \ldots, m$.

The above considerations lead to the following theorem.

Theorem 4.6.5. *For $\kappa \subset G \times G$ to be a connectivity relation on a p.q-o. Menger algebra (G, o, χ) of rank n it is necessary and sufficient to be 0-reflexive, l-cancellative and to satisfy condition* (4.6.3) *together with the system of formulas $(C_m)_{m\in\mathbb{N}}$.*

From the above theorem, it follows that κ is a connectivity relation on a p.q-o. Menger algebra (G, o, δ), where

$$\delta = \{(g_1, g_2) \,|\, (\exists t \in T_n(G))\ g_1 = t(g_2)\}.$$

We have therefore the following characterization connectivity relations in Menger algebras.

Corollary 4.6.6. *For $\kappa \subset G \times G$ to be a connectivity relation on a Menger algebra (G, o) of rank n it is necessary and sufficient to be 0-reflexive, l-cancellative and that condition* (4.6.3) *together with the system of formulas $(C_m)_{m\in\mathbb{N}}$, in which χ is replaced by δ, is satisfied.*

Let (G, o) be a Menger algebra of rank n and ζ be its order. The subset $H \subset G$ is called ζ-*strong*, if for any $g, g_1, g_2 \in G,\ t_1, t_2 \in T_n(G)$ the implication

$$t_2(g_1) \leqslant g \ \wedge\ t_1(g_1), t_1(g_2) \in H \longrightarrow g \in H$$

is valid. The least ζ-strong subset containing $X \subset G$ will be denoted by $\langle X \rangle$. In the case when $X = \{g_1, \ldots, g_n\}$ instead of $\langle \{g_1, \ldots, g_n\} \rangle$ we will write $\langle g_1, \ldots, g_n \rangle$. One can prove that

$$\langle X \rangle = \bigcup_{n=0}^{\infty} K_n(X),$$

where $K_0(X) = X,\ K_{n+1}(X) = K(K_n(X))$ and $g \in K(X)$ if and only if $t_2(g_1) \leqslant g,\ \{t_1(g_1), t_1(g_2), t_2(g_2)\} \subset X$ for some $g_1, g_2 \in G$, $t_1, t_2 \in T_n(G)$.

Theorem 4.6.7. *For $\kappa \subset G \times G$ to be a reversive connectivity relation on a strong f.o. Menger algebra (G, o, ζ) of rank $n \geqslant 2$ it is necessary and sufficient to be* 0*-reflexive, l-cancellative and that for all $g, g_1, g_2, h_1, h_2 \in G$ the following two conditions:*

$$g \in \langle g_1 \rangle \longrightarrow g_1 \leqslant g, \tag{4.6.6}$$

$$(h_1, h_2) \in \kappa \ \wedge\ t(g_1), t(g_2) \in \langle h_1, h_2 \rangle \longrightarrow (g_1, g_2) \in \kappa \tag{4.6.7}$$

are satisfied.

Proof. Necessity. Let $(\mathbf{\Phi}, O, \zeta_{\mathbf{\Phi}})$ be a f.o. Menger algebra of reversive n-place functions and $\kappa_{\mathbf{\Phi}}$ be its connectivity relation. Then for any subset $H \subset \mathbf{\Phi}$ the following implication

$$\varphi \in \langle H \rangle \longrightarrow \bigcap H \subset \varphi, \tag{4.6.8}$$

where $\bigcap H$ is the intersection of all $\psi \in H$, is true. Indeed, if $\varphi \in K(H)$, then $\{t_1(\psi_1), t_1(\psi_2), t_2(\psi_2)\} \subset H$ and $t_2(\psi_1) \subset \varphi$ for some $\psi_1, \psi_2 \in \mathbf{\Phi}$, $t_1, t_2 \in T_n(\mathbf{\Phi})$. Thus $\bigcap H \subset t_1(\psi_1) \cap t_1(\psi_1)$ and $\bigcap H \subset t_2(\psi_2)$, whence $t_1(\psi_1) \circ \triangle_{\mathrm{pr}_1 \bigcap H} = t_1(\psi_2) \circ \triangle_{\mathrm{pr}_1 \bigcap H}$. So, $\psi_1 \circ \triangle_{\mathrm{pr}_1 \bigcap H} = \psi_2 \circ \triangle_{\mathrm{pr}_1 \bigcap H}$, because our functions are reversive. Hence $\bigcap H \subset t_2(\psi_2) \circ \triangle_{\mathrm{pr}_1 \bigcap H} = t_2(\psi_2 \circ \triangle_{\mathrm{pr}_1 \bigcap H}) = t_2(\psi_1 \circ \triangle_{\mathrm{pr}_1 \bigcap H}) \subset t_2(\psi_1) \subset \varphi$. This means that $\varphi \in K(H)$ implies $\bigcap H \subset \varphi$. Therefore

$$\bigcap H \subset \bigcap K(H) \subset \bigcap K_2(H) \subset \cdots \subset \bigcap K_n(H) \subset \cdots .$$

So, from $\varphi \in \langle H \rangle$, i.e., from $\varphi \in K_m(H)$ for some $m \in \mathbb{N}$, it follows that $\bigcap H \subset \bigcap K_m(H) \subset \varphi$, which was to be proved.

If $\psi \in \langle \varphi \rangle$, then, according to (4.6.6), we have $\varphi = \bigcap\{\varphi\} \subset \psi$, which proves (4.6.6). Furthermore, if $\varphi_1 \cap \varphi_2 \neq \varnothing$ and $t(\psi_1), t(\psi_2) \in \langle \varphi_1, \varphi_2 \rangle$, then, by

(4.6.8), we get $\varnothing \neq \varphi_1 \cap \varphi_2 \subset t(\psi_1) \cap t(\psi_2) = t(\psi_1 \cap \psi_2)$, because all the functions from Φ are reversive. So, $t(\psi_1 \cap \psi_2) \neq \varnothing$, whence $\psi_1 \cap \psi_2 \neq \varnothing$. This proves (4.6.7).

Sufficiency. Let (G, o, ζ) be a strong f.o. Menger algebra of rank n and κ a relation satisfying all the conditions of the theorem. Consider the sum P of the family of these simplest representations, which correspond to the determining pairs $\big((\varepsilon^*_{\langle h_1,h_2\rangle}, W_{\langle h_1,h_2\rangle})\big)_{(h_1,h_2)\in\kappa}$, where

$$\varepsilon^*_{\langle h_1,h_2\rangle} = \varepsilon_{\langle h_1,h_2\rangle} \cup \{(e_1,e_1),\ldots,(e_n,e_n)\},$$

$e_1,\ldots,e_n$ – the selectors in (G^*, o^*) and $\varepsilon_{\langle h_1,h_2\rangle} = \varepsilon_v(\langle h_1,h_2\rangle)$, $W_{\langle h_1,h_2\rangle} = W_v(\langle h_1,h_2\rangle)$ (see p. 31). Because for any $h_1, h_2 \in G$ the subset $\langle h_1,h_2\rangle$ is strong, then, evidently, P is the representation of (G, o) by reversive n-place functions.

Let us show that $\zeta = \zeta_P$ and $\kappa = \kappa_P$. If $g_1 \leqslant g_2$ and $g_1[\bar{x}] \notin W_{\langle h_1,h_2\rangle}$ for some $\bar{x} \in B = G^n \cup \{(e_1,\ldots,e_n)\}$, $(h_1,h_2) \in \kappa$, then one can find a $t_1 \in T_n(G)$ such that $t_1(g_1[\bar{x}]) \in \langle h_1,h_2\rangle$. Let $t(g_1[\bar{x}]) \in \langle h_1,h_2\rangle$ for some $t \in T_n(G)$. From $g_1 \leqslant g_2$ and the stability of ζ, it follows that $t(g_1[\bar{x}]) \leqslant t(g_2[\bar{x}])$, whence $t(g_2[\bar{x}]) \in \langle h_1,h_2\rangle$. Analogously $t_1(g_2[\bar{x}]) \in \langle h_1,h_2\rangle$. This means that $\{t_1(g_1[\bar{x}]), t_1(g_2[\bar{x}]), t(g_2[\bar{x}])\} \subset \langle h_1,h_2\rangle$ implies $t(g_1[\bar{x}]) \in \langle h_1,h_2\rangle$. So, $(g_1,g_2) \in \zeta_{(\varepsilon^*_{\langle h_1,h_2\rangle}, W_{\langle h_1,h_2\rangle})}$, i.e., $(g_1,g_2) \in \zeta_P$. Thus $\zeta \subset \zeta_P$. Conversely, let $(g_1,g_2) \in \zeta_P$. If $g_1 = 0$, then $g_1 \leqslant g_1[g_2g_1\cdots g_1]$, $g_1 \leqslant g_1[g_1g_1\cdots g_1]$ and $g_1 \leqslant g_1$, whence $g_1 \leqslant g_2$, because ζ is the steady relation. If $g_1 \neq 0$, then $(g_1,g_1) \in \kappa$. So, for all $\bar{x} \in B$, from $g_1[\bar{x}] \notin W_{\langle g_1\rangle}$ we get $g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon_{\langle g_1\rangle})$, which for $\bar{x} = (e_1,\ldots,e_n)$ gives $g_1 \equiv g_2(\varepsilon_{\langle g_1\rangle})$.[10] Therefore

$$(\forall\, t \in T_n(G))\ (\, t(g_1) \in \langle g_1\rangle \longleftrightarrow t(g_2) \in \langle g_1\rangle\,).$$

Putting $t = \triangle_G$ in the last formula, we obtain $g_1 \in \langle g_1\rangle$, whence, according to (4.6.6), we deduce $g_1 \leqslant g_2$. So, $\zeta_P \subset \zeta$. This completes the proof of the equality $\zeta = \zeta_P$. The equality $\kappa = \kappa_P$ can be proved in the same way as in the previous theorems.

From the antisymmetry of ζ, it follows that P is one-to-one. □

It can be showed also that each of the conditions (4.6.6) and (4.6.7) is equivalent to the system of elementary formulas (for details see [226]).

4.7 Projection equivalence relation

A binary relation π on a Menger algebra (G, o) of rank n is called a (*reversive*) *projection equivalence* if there exists a faithful representation P of (G, o) by (reversive) n-place functions such that $\pi = \pi_P$, where

$$\pi_P = \{(g_1,g_2) \,|\, \mathrm{pr}_1 P(g_1) = \mathrm{pr}_1 P(g_2)\}$$

[10] Note that $\varepsilon_{\langle g_1\rangle} = \varepsilon_{\langle g_1,g_1\rangle}$ and $W_{\langle g_1\rangle} = W_{\langle g_1,g_1\rangle}$.

(see Section 2.7). It is easy to see that *a binary relation* $\pi \subset G \times G$ *is a (reversive) projection equivalence on a (strong) f.o.p. Menger algebra* (G, o, ζ, χ) *if and only if* $\pi = \chi \cap \chi^{-1}$. A similar result is true for a (strong) p.q-o. Menger algebra (G, o, χ).

Now let ζ be a stable order on a Menger algebra (G, o) and π its l-regular equivalence. Consider on (G, o) the binary relation

$$\chi(\zeta, \pi) = \{(g_1, g_2) \mid \overline{\overline{g_2}} \subset \overline{\overline{g_1}}\},$$

where $\overline{\overline{g}}$ means the $(\zeta, \delta \circ \pi \circ \zeta)$-closure of the subset $\{g\} \subset G$ and

$$\delta = \{(g_1, g_2) \mid (\exists t \in T_n(G))\ g_1 = t(g_2)\}.$$

Lemma 4.7.1. *The relation* $\chi(\zeta, \pi)$ *is an* l*-regular* v*-negative quasi-order containing* $\zeta \cup \pi$ *and connected with* ζ *by condition* (3.2.2).

Proof. Indeed, the fact that $\chi(\zeta, \pi)$ is a quasi-order follows immediately from the definition of this relation. Let $g_1 \leqslant g_2$, $g \sqsubset g_1$ and $g \sqsubset u[\bar{w}|_i\, g_2]$, where the symbol $\sqsubset$ denotes the relation $\chi(\zeta, \pi)$. Directly from the definition we have $g_1 \leqslant g_2$, $g_1 \in \overline{\overline{g}}$ and $u[\bar{w}|_i\, g_2] \in \overline{\overline{g}}$. Since the relations ζ, π and δ are reflexive, then

$$(u[\bar{w}\,|_i\, g_1], u[\bar{w}\,|_i\, g_1]) \in \delta \circ \pi \circ \zeta.$$

This, together with $g_1, u[\bar{w}\,|_i\, g_2] \in \overline{\overline{g}}$ and the fact that $\overline{\overline{g}}$ is a $(\zeta, \delta \circ \pi \circ \zeta)$-closed subset, implies $u[\bar{w}|_i\, g_1] \in \overline{\overline{g}}$, i.e., $g \sqsubset u[\bar{w}|_i\, g_1]$. So, $\chi(\zeta, \pi)$ and ζ are connected by condition (3.2.2).

Let $(a, g) \in \delta$. Then $(a, g) \in \delta \circ \pi \circ \zeta$, because $\delta \subset \delta \circ \pi \circ \zeta$. But $a \in \overline{\overline{a}}$ and $\overline{\overline{a}}$ is a $(\delta \circ \pi \circ \zeta)$–saturated subset, so, $g \in \overline{\overline{a}}$. Thus $a \sqsubset g$, i.e., $\delta \subset \chi(\zeta, \pi)$. Therefore $\chi(\zeta, \pi)$ is v-negative.

In a similar way, we can verify that $\zeta \subset \chi(\zeta, \pi)$ and $\pi \subset \chi(\zeta, \pi)$.

Now let $(g_1, g_2) \in \chi(\zeta, \pi)$, i.e., $\overline{\overline{g_2}} \subset \overline{\overline{g_1}}$. Since $g_2 \in \overline{\overline{g_2}}$ implies $g_2 \in \overline{\overline{g_1}}$, we have also $g_2[\bar{x}] \in \overline{\overline{g_1}}[\bar{x}]$ for all $\bar{x} \in G^n$. But, as it is not difficult to see, $\overline{\overline{g_1}}\,[\bar{x}] \subset \overline{\overline{g_1[\bar{x}]}}$, so, $g_2[\bar{x}] \in \overline{\overline{g_1[\bar{x}]}}$, whence $g_1[\bar{x}] \sqsubset g_2[\bar{x}]$, i.e., $\chi(\zeta, \pi)$ is l-regular, which was to be proved. □

Theorem 4.7.2. *For a binary relation* π *on a f.o. Menger algebra* (G, o, ζ) *of rank* n *to be its projection equivalence it is necessary and sufficient to be an* l*-regular equivalence satisfying the following two conditions:*

$$\zeta^{-1} \circ \zeta \cap \chi(\zeta, \pi) \subset \zeta, \tag{4.7.1}$$

$$\chi(\zeta, \pi) \cap \chi(\zeta, \pi)^{-1} \subset \pi. \tag{4.7.2}$$

Proof. Let π be a projection equivalence on a f.o. Menger algebra (G, o, ζ) of rank n. On this algebra, there is a faithful representation P of (G, o) by

n-place functions such that $\zeta = \zeta_P$ and $\pi = \pi_P$. Consider a f.o.p. Menger algebra (G, o, ζ_P, χ_P), i.e., (G, o, ζ, χ_P). If $(g_1, g_2) \in \chi(\zeta, \pi)$, then $g_2 \in \overline{\overline{g_1}}$. As $\zeta \subset \chi_P$ and $\pi = \pi_P \subset \chi_P$, then, according to condition (3.2.2) of Theorem 3.2.1, for any $g \in G$ the subset $\chi_P\langle g\rangle$ is $(\zeta, \delta \circ \pi \circ \zeta)$-closed. Thus, $g_1 \in \chi_P\langle g_1\rangle$ implies $\overline{\overline{g_1}} \subset \chi_P\langle g_1\rangle$, whence $g_2 \in \chi_P\langle g_1\rangle$, i.e., $(g_1, g_2) \in \chi_P$. So, $\chi(\zeta, \pi) \subset \chi_P$. Since $\zeta^{-1} \circ \zeta \cap \chi_P \subset \zeta$, by Theorem 3.2.1, we have

$$\zeta^{-1} \circ \zeta \cap \chi(\zeta, \pi) \subset \zeta^{-1} \circ \zeta \cap \chi_P \subset \zeta,$$

which proves (4.7.1). As $\chi_P \cap \chi_P^{-1} = \pi_P = \pi$, then, clearly

$$\chi(\zeta, \pi) \cap \chi(\zeta, \pi)^{-1} \subset \chi_P \cap \chi_P^{-1} = \pi$$

proves (4.7.2). The l-regularity of π is a consequence of the l-regularity of χ_P.

Conversely, if π satisfies all the conditions of the theorem, the system $(G, o, \zeta, \chi(\zeta, \pi))$ is a f.o.p. Menger algebra. Since $\pi \subset \chi(\zeta, \pi)$, from (4.7.2), we obtain $\pi = \chi(\zeta, \pi) \cap \chi(\zeta, \pi)^{-1}$, which means that π is a projection equivalence on $(G, o, \zeta, \chi(\zeta, \pi))$. □

It is not difficult to see that conditions (4.7.1) and (4.7.2) are equivalent to a system of the two conditions:

$$g_1 \leqslant g \wedge g_2 \leqslant g \wedge g_2 \in \overline{\overline{g_1}} \longrightarrow g_1 \leqslant g_2, \tag{4.7.3}$$

$$g_1 \in \overline{\overline{g_2}} \wedge g_2 \in \overline{\overline{g_1}} \longrightarrow (g_1, g_2) \in \pi \tag{4.7.4}$$

where $g, g_1, g_2 \in G$. Based on (2.1.12), we can show that (4.7.3) is equivalent to the system $(A_m)_{m\in\mathbb{N}}$, and (4.7.4) — to the system $(B_m)_{m\in\mathbb{N}}$, where

$$A_m : g_1 \leqslant g \wedge g_2 \leqslant g \wedge g_2 \in \overset{m}{E}(\{g_1\}) \longrightarrow g_1 \leqslant g_2,$$
$$B_m : g_1 \in \overset{m}{E}(\{g_2\}) \wedge g_2 \in \overset{m}{E}(\{g_1\}) \longrightarrow (g_1, g_2) \in \pi$$

and $\rho = \zeta,\ \sigma = \delta \circ \pi \circ \zeta$ (see p. 34). Moreover, by Proposition 2.1.15, A_m and B_m can be written in an elementary form.

Theorem 4.7.3. *For a binary relation π to be a projection equivalence on a Menger algebra (G, o) of rank n it is necessary and sufficient to be an l-regular equivalence satisfying for every $m \in \mathbb{N}$ and all $x_1, \ldots, x_m \in G,\ t_1, \ldots, t_m \in T_n(G)$ the condition*

$$C_m : \bigwedge_{i=1}^{m} (x_i, t_i(x_{i+1})) \in \pi \longrightarrow \bigwedge_{i=1}^{m-1} (x_i, x_{i+1}) \in \pi\,,$$

where $x_{m+1} = x_1$.

Proof. Let a Menger algebra (G, o) be isomorphic to a Menger algebra (Φ, O) of n-place functions. If $(\varphi_1, t_1(\varphi_2)) \in \pi_\Phi$, $(\varphi_2, t_2(\varphi_3)) \in \pi_\Phi, \ldots,$ $(\varphi_{m-1}, t_{m-1}(\varphi_m)) \in \pi_\Phi$, $(\varphi_m, t_m(\varphi_1)) \in \pi_\Phi$ for some $\varphi_1, \ldots, \varphi_m \in \Phi$ and $t_1, \ldots, t_m \in T_n(\Phi)$, then $\mathrm{pr}_1\, \varphi_i \subset \mathrm{pr}_1\, \varphi_{i+1}$ for every $i = 1, \ldots, m-1$ and $\mathrm{pr}_1\, \varphi_m \subset \mathrm{pr}_1\, \varphi_1$. Thus $\mathrm{pr}_1\, \varphi_1 \subset \mathrm{pr}_1\, \varphi_2 \subset \cdots \subset \mathrm{pr}_1\, \varphi_m \subset \mathrm{pr}_1\, \varphi_1$, whence $\mathrm{pr}_1\, \varphi_1 = \mathrm{pr}_1\, \varphi_2 = \cdots = \mathrm{pr}_1\, \varphi_m$, i.e., $(\varphi_1, \varphi_2) \in \pi_\Phi$, $(\varphi_2, \varphi_3) \in \pi_\Phi, \ldots,$ $(\varphi_{m-1}, \varphi_m) \in \pi_\Phi$. The necessity is proved.

To prove the sufficiency, consider a f.o.p. Menger algebra of the form $(G, o, \triangle_G, \chi(\triangle_G, \pi))$. From Theorem 4.7.2, it follows that π is a projection equivalence on this algebra if and only if π is an l-regular equivalence such that

$$\chi(\triangle_G, \pi) \cap \chi(\triangle_G, \pi)^{-1} \subset \pi. \tag{4.7.5}$$

The last condition is equivalent to condition C_m. Indeed, it is clear that a subset of a Menger algebra (G, o) is $(\triangle_G, \delta \circ \pi \circ \triangle_G)$-closed if and only if it is $(\delta \circ \pi)$-saturated. Whence, the $(\triangle_G, \delta \circ \pi \circ \triangle_G)$-closure of $\{g\}$ coincides with

$$\bigcup_{m=0}^{\infty} (\delta \circ \pi)^m \langle g \rangle.$$

Thus for $(g_1, g_2) \in \chi(\triangle_G, \pi)$ there exist $m \in \mathbb{N}$, $x_2, \ldots, x_m \in G$ such that $(g_1, x_2), (x_2, x_3), \ldots, (x_m, g_2) \in \delta \circ \pi$. As $(x_i, x_{i+1}) \in \delta \circ \pi$ is equivalent to $(x_i, t_i(x_{i+1})) \in \pi$ for some $t_i \in T_n(G)$, then $(g_1, g_2) \in \chi(\triangle_G, \pi)$ is equivalent to the fact that

$$(g_1, t_1(x_2)),\ (x_2, t_2(x_3)),\ \ldots, (x_{m-1}, t_{m-1}(x_m)),\ (x_m, t_m(g_2)) \in \pi$$

for some $m \in \mathbb{N}$, $x_2, \ldots, x_m \in G$, $t_1, \ldots, t_m \in T_n(G)$. Thus, (4.7.5) is equivalent to the system of conditions $(C_{p,m})_{p,m \in \mathbb{N}}$, where $p \leqslant m$,

$$C_{p,m}\ :\ \bigwedge_{i=1}^{m} (x_i, t_i(x_{i+1})) \in \pi \longrightarrow (x_1, x_{p+1}) \in \pi$$

and $x_{m+1} = x_1$. It is evident that the system of such conditions is equivalent to condition C_m. □

Corollary 4.7.4. *For a binary relation π to be a projection equivalence on a Menger algebra (G, o) of rank n it is necessary and sufficient to be an l-regular equivalence satisfying for every $m \in \mathbb{N}$ and all $x_1, \ldots, x_m \in G$, $t_1, \ldots, t_m \in T_n(G)$ the condition*

$$C_m^0 :\ \bigwedge_{i=1}^{m} (x_i, t_i(x_{i+1})) \in \pi \longrightarrow (x_1, x_2) \in \pi,$$

where $x_{m+1} = x_1$.

Proof. Let us show that the system of conditions $(C_m)_{m\in\mathbb{N}}$ of Theorem 4.7.3 is equivalent to the system $(C_m^0)_{m\in\mathbb{N}}$. It is clear that for any $m\in\mathbb{N}$ condition C_m implies C_m^0. Thus $(C_m^0)_{m\in\mathbb{N}}$ is a consequence of $(C_m)_{m\in\mathbb{N}}$.

Assume now that the system $(C_m^0)_{m\in\mathbb{N}}$ is satisfied. If the premise of C_m is satisfied, then $(x_1,x_2)\in\pi$, by C_m^0. But $\bigwedge\limits_{i=2}^{m}(x_i,t_i(x_{i+1}))\in\pi$, where $x_{m+1}=x_1$, and $(x_1,x_2)\in\pi$ imply $(x_1,t_3(x_3))\in\pi,\ldots,(x_m,t_m(x_1))\in\pi$. Therefore, from the above, by C_{m-1}^0, we conclude $(x_1,x_3)\in\pi$. Repeating this procedure, after a finite number of steps, from $(x_1,t_{m-1}(x_m))\in\pi$ and $(x_m,t_m(x_1))\in\pi$, by C_2^0, we obtain $(x_1,x_m)\in\pi$. So, $(x_i,x_{i+1})\in\pi$ for every $i=1,\ldots,m-1$, which proves that condition C_m holds. Thus $(C_m)_{m\in\mathbb{N}}$ is a consequence of $(C_m^0)_{m\in\mathbb{N}}$. □

Now let ζ be an l-regular order on a Menger algebra (G,o) of rank n. For an l-regular equivalence π on (G,o), we define a new binary relation $\widetilde{\chi}\langle\zeta,\pi\rangle$ putting

$$\widetilde{\chi}\langle\zeta,\pi\rangle=\{(g_1,g_2)\,|\,\widetilde{g_2}\subset\widetilde{g_1}\},$$

where $\widetilde{g}$ is the $\langle\zeta^{-1}\circ\zeta,\delta\circ\pi\circ\zeta\rangle$-closure of $\{g\}$. In the same way as in the case of the relation $\chi\langle\zeta,\pi\rangle$, one can prove that $\widetilde{\chi}\langle\zeta,\pi\rangle$ is a v-negative quasi-order containing $\zeta\cup\pi$ and connected with ζ by condition (3.3.3). It is also l-regular. Indeed, if $(g_1,g_2)\in\widetilde{\chi}\langle\zeta,\pi\rangle$, i.e., $g_2\in\widetilde{g_1}$, then $g_2[\bar{x}]\in\widetilde{g_1}[\bar{x}]$ for all $\bar{x}\in G^n$. According to the formula (2.1.14), where $\rho=\zeta^{-1}\circ\zeta$ and $\sigma=\delta\circ\pi\circ\zeta$, the following equality

$$\widetilde{g_1}=\bigcup_{m=0}^{\infty}F_m(\{g_1\})$$

is valid.[11] Therefore $\widetilde{g_1}[\bar{x}]=\bigcup\limits_{m=0}^{\infty}F_m(\{g_1\})[\bar{x}]$. So, we must only prove that

$$F_m(\{g_1\})[\bar{x}]\subset F_m(\{g_1[\bar{x}]\}) \tag{4.7.6}$$

for all $m\in\mathbb{N}$. We prove it by induction. For $m=0$, this inclusion is valid because $F_0(\{g_1\})=\{g_1\}$. Assume that it is valid for some $m\in\mathbb{N}$. If $g\in F_{m+1}(\{g_1\})[\bar{x}]$, then $g=z[\bar{x}]$ for some $z\in F_{m+1}(\{g_1\})$. This means that

$$(a,b)\in\varepsilon_m\langle\zeta^{-1}\circ\zeta,F_m(\{g_1\})\rangle\,,\qquad (u[\bar{y}|_i a],z)\in\delta\circ\pi\circ\zeta$$

and $u[\bar{y}|_i b],z)\in F_m(\{g_1\})$ for some $a,b\in G$, $u\in G\cup\{e_1,\ldots,e_n\}$, $\bar{y}\in G^n$, $i=1,\ldots,n$. Since relations δ,π and ζ are l-regular, $(u[\bar{y}|_i a],z)\in\delta\circ\pi\circ\zeta$ implies

$$(u[\bar{y}*\bar{x}|_i\,a[\bar{x}]\,],z[\bar{x}])\in\delta\circ\pi\circ\zeta,$$

[11] For the definition of $F_m(\{g_1\})$ see p. 35.

where $\bar{y} * \bar{x} = (y_1[\bar{x}], \ldots, y_n[\bar{x}])$. From $u[\bar{y}|_i b] \in F_m(\{g_1\})$, according to our hypothesis, follows

$$u[\bar{y}*\bar{x}|_i\, b[\bar{x}]\,] \in F_m(\{g_1\})[\bar{x}] \subset F_m(\{g_1[\bar{x}]\}).$$

Further, using the l-regularity of $\delta \circ \pi \circ \zeta$ and ζ, the results of Section 2.1 and our hypothesis, from $(a, b) \in \varepsilon_m\langle \zeta^{-1} \circ \zeta, F_m(\{g_1\})\rangle$ we conclude that

$$(a[\bar{x}], b[\bar{x}]) \in \varepsilon_m\langle \zeta^{-1} \circ \zeta, F_m(\{g_1[\bar{x}]\})\rangle.$$

Hence $g \in F_{m+1}(\{g_1[\bar{x}]\})$, i.e., $F_{m+1}(\{g_1\})[\bar{x}] \subset F_{m+1}(\{g_1[\bar{x}]\})$. So, (4.7.6) is valid for every $m \in \mathbb{N}$.

Theorem 4.7.5. *For a binary relation π on a strong p.q-o. Menger algebra (G, o, ζ) of rank $n \geqslant 2$ to be reversive projection equivalence it is necessary and sufficient to be an l-regular equivalence satisfying for all $m \in \mathbb{N}$, $g_1, g_2 \in G$ the following two conditions:*

$$(g_1, g_2) \in \varepsilon_m\langle \zeta^{-1} \circ \zeta, \widetilde{\chi}\langle \zeta, \pi\rangle\langle g_1\rangle\rangle \longrightarrow g_1 \leqslant g_2, \tag{4.7.7}$$

$$\widetilde{\chi}\langle \zeta, \pi\rangle \cap \widetilde{\chi}\langle \zeta, \pi\rangle^{-1} \subset \pi. \tag{4.7.8}$$

Proof. Let π be a reversive projection equivalence on a strong p.q-o. Menger algebra (G, o, ζ) of rank $n \geqslant 2$. Then there exists a faithful representation P of (G, o) by reverse n-place functions such that $\zeta = \zeta_P$ and $\pi = \pi_P$. Let (G, o, ζ, χ_P) be a strong f.o.p. Menger algebra. It is known that this algebra satisfies (3.3.2) and (3.3.3). Moreover, for all $m \in \mathbb{N}$, $g \in G$ we have

$$\varepsilon_m\langle \zeta^{-1} \circ \zeta, \chi\langle \zeta, \pi\rangle\langle g\rangle\rangle \subset \varepsilon_m\langle \zeta^{-1} \circ \zeta, \chi_P\langle g\rangle\rangle,$$

because $\widetilde{\chi}\langle \zeta, \pi\rangle \subset \chi_P$. Thus, from the premise of (4.7.7), it follows that $(g_1, g_2) \in \varepsilon_m\langle \zeta^{-1} \circ \zeta, \chi_P\langle g_1\rangle\rangle$, whence $g_1 \leqslant g_2$, by (3.3.2). So, the implication (4.7.7) is true. The condition (4.7.8) is a consequence of the inclusion $\widetilde{\chi}\langle \zeta, \pi\rangle \subset \chi_P$ and the equality $\chi_P \cap \chi_P^{-1} = \pi_P$.

Conversely, if π satisfies all the conditions of the theorem, then, obviously, $(G, o, \zeta, \widetilde{\chi}\langle \zeta, \pi\rangle)$ is a strong f.o.p. Menger algebra of rank $n \geqslant 2$. Therefore, $\pi \subset \widetilde{\chi}\langle \zeta, \pi\rangle$ and (4.7.8) imply $\pi = \widetilde{\chi}\langle \zeta, \pi\rangle \cap \widetilde{\chi}\langle \zeta, \pi\rangle^{-1}$, which means that π is a reversive projection equivalence. □

Starting from (2.1.13) and (2.1.14), one can show that (4.7.7) is equivalent to the system of conditions $(A_m)_{m\in\mathbb{N}}$, where

$$A_m \; : \quad (g_1, g_2) \in \varepsilon_m\langle \zeta^{-1} \circ \zeta, F_m(\{g_2\})\rangle \longrightarrow g_1 \leqslant g_2,$$

and (4.7.8) to the system $(B_m)_{m\in\mathbb{N}}$, where

$$B_m \; : \quad g_1 \in F_m(\{g_2\}) \wedge g_2 \in F_m(\{g_1\}) \longrightarrow (g_1, g_2) \in \pi$$

and $\rho = \zeta^{-1} \circ \zeta$, $\sigma = \delta \circ \pi \circ \zeta$. Each of these conditions, for every concrete $m \in \mathbb{N}$, can be written in an elementary form in a finite number of steps.

Theorem 4.7.6. *For a binary relation π on a strong Menger algebra (G, o) of rank $n \geqslant 2$ to be its reversive projection equivalence it is necessary and sufficient to be an l-regular equivalence such that*

$$\widetilde{\chi}\langle\triangle_G,\pi\rangle \cap \widetilde{\chi}\langle\triangle_G,\pi\rangle^{-1} \subset \pi$$

and

$$(g_1,g_2) \in \varepsilon_m\langle\triangle_G,\widetilde{\chi}\langle\triangle_G,\pi\rangle\langle g_1\rangle\rangle \wedge (g_2,g_1) \in \widetilde{\chi}\langle\triangle_G,\pi\rangle \longrightarrow g_1 = g_2$$

for all $m \in \mathbb{N}$, $g_1, g_2 \in G$.

Proof. Indeed, if P is a faithful representation of a strong Menger algebra (G, o) by reversive n-place functions such that $\pi = \pi_P$, then π is a reversive projection equivalence on a strong p.q-o. Menger algebra (G, o, χ_P). But $\widetilde{\chi}\langle\triangle_G,\pi\rangle \subset \chi_P$, so, $(g_1,g_2) \in \varepsilon_m\langle\triangle_G,\widetilde{\chi}\langle\triangle_G,\pi\rangle\langle g_1\rangle\rangle$ and $(g_2,g_1) \in \widetilde{\chi}\langle\triangle_G,\pi\rangle$ imply $(g_1,g_2) \in \varepsilon_m\langle\triangle_G,\chi_P\langle g_1\rangle\rangle$ and $(g_2,g_1) \in \chi_P$, whence, according to (3.3.4), we get $g_1 = g_2$. So, the necessity of the second condition is proved. The necessity of the first condition can be proved similarly.

From the fact that $(G, o, \widetilde{\chi}\langle\triangle_G,\pi\rangle)$ is, by Theorem 3.3.3, a strong p.q-o. Menger algebra, for which $\pi = \widetilde{\chi}\langle\triangle_G,\pi\rangle \cap \widetilde{\chi}\langle\triangle_G,\pi\rangle^{-1}$ follows the sufficiency of these conditions. □

4.8 Semiadjacency relation

One of the important relations on the set of partial functions is a *semiadjacency relation*

$$\delta_\Phi = \{(f,g) \,|\, \mathrm{pr}_2\, f \subset \mathrm{pr}_1\, g\},$$

i.e., the inclusion of the image of the first function in the domain of the second. The investigation of such relation was initiated in [141] and [48]. V.V. Vagner studied in [253] the *semidirect product of the second type* of the family binary relations $\rho_i \subset A \times B_i$, where $i \in I$, defined as a binary relation $\underset{i}{\rtimes}\rho_i \subset A \times \Big(\overset{n}{\underset{i=1}{\times}} B_i\Big)$ such that

$$\underset{i}{\rtimes}\rho_i = \{(a,(b_i)_{i\in I}) \,|\, (\forall i)(a,b_i) \in \rho_i\}.$$

An analogous construction can be considered for $(n+1)$-ary relations.

As is well known the composition of $(n+1)$-ary relations $\rho, \rho_1, \ldots, \rho_n$ on the set A is defined as an $(n+1)$-ary relation $\rho[\rho_1 \cdots \rho_n]$ such that

$$(\bar{a},c) \in \rho[\rho_1\cdots\rho_n] \longleftrightarrow (\exists \bar{b})\Big((\bar{a},b_1) \in \rho_1 \wedge \cdots \wedge (\bar{a},b_n) \in \rho_n \wedge (\bar{b},c) \in \rho\Big),$$

where $\bar{a}, \bar{b} \in A^n$, $c \in A$. It is easy to see that

$$\rho[\rho_1 \cdots \rho_n] = \rho \circ \overset{n}{\underset{i=1}{\rtimes}} \rho_i \quad \text{and} \quad \mathrm{pr}_1 \overset{n}{\underset{i=1}{\rtimes}} \rho_i = \bigcap_{i=1}^{n} \mathrm{pr}_1 \, \rho_i \, .$$

On a Menger algebra (Φ, O) of n-place functions, we can consider two $(n+1)$-ary relations

$$\overset{n}{\chi}_{\Phi} = \Big\{ ((\varphi_1, \ldots, \varphi_n), \psi) \,|\, \mathrm{pr}_1 \overset{n}{\underset{i=1}{\rtimes}} \varphi_i \subset \mathrm{pr}_1 \, \psi \Big\},$$

$$\overset{n}{\delta}_{\Phi} = \Big\{ ((\varphi_1, \ldots, \varphi_n), \psi) \,|\, \mathrm{pr}_2 \overset{n}{\underset{i=1}{\rtimes}} \varphi_i \subset \mathrm{pr}_1 \, \psi \Big\}.$$

The first relation is called the relation of *inclusion of domains*, the second — the *semiadjacency relation*. (For $n = 1$, these two relations coincide with the corresponding binary relations.) It is easily to show that

$$((\psi_1, \ldots, \psi_n), \varphi) \in \overset{n}{\delta}_{\Phi} \longleftrightarrow ((\psi_1, \ldots, \psi_n), \varphi[\psi_1 \cdots \psi_n]) \in \overset{n}{\chi}_{\Phi}, \tag{4.8.1}$$

$$((\varphi_1, \ldots, \varphi_n), \psi) \in \overset{n}{\delta}_{\Phi} \longleftrightarrow ((\varphi_1[\chi_1 \cdots \chi_n], \ldots, \varphi_n[\chi_1 \ldots \chi_n]), \psi) \in \overset{n}{\delta}_{\Phi} \tag{4.8.2}$$

for all $\chi_1, \ldots, \chi_n \in \Phi$.

We say that an $(n+1)$-ary relation ρ defined on the set A is *projective* if $(\bar{a}, a_i) \in \rho$ for all $\bar{a} = (a_1, \ldots, a_n)$ and $i = 1, \ldots, n$. An $(n+1)$-ary relation ρ defined on A is *transitive*, if

$$(\bar{a}, \bar{b}) \in \overset{n}{\rtimes} \rho \wedge (\bar{b}, c) \in \rho \longrightarrow (\bar{a}, c) \in \rho,$$

for $\bar{a}, \bar{b} \in A^n$, $c \in A$. By $\overline{\rho}$ we denote the transitive closure of ρ, by $F(\rho)$ the relation:

$$F(\rho) = \{ (\bar{a}, c) \,|\, (\exists \bar{b})(\bar{a}, b_1), \ldots, (\bar{a}, b_n), (\bar{b}, c) \in \rho \vee (\bar{a}, c) \in \rho \}, \tag{4.8.3}$$

where $\bar{a}, \bar{b} \in A^n$, $c \in A$. It is clear that $\rho \subset F(\rho)$. If ρ is transitive, then $F(\rho) = \rho$.

Proposition 4.8.1. $\overline{\rho} = \bigcup_{n=1}^{\infty} \overset{n}{F}(\rho)$, *where* $\overset{n}{F}(\rho) = F(\overset{n-1}{F}(\rho))$ *for* $n \geqslant 2$.

Proof. For any $\bar{a}, \bar{b} \in A^n$, $c \in A$ and $(\bar{a}, b_1), \ldots, (\bar{a}, b_n)$, $(\bar{b}, c) \in \bigcup_{n=1}^{\infty} \overset{n}{F}(\rho)$ one can find $m_0, m_1, \ldots, m_n$ such that $(\bar{a}, b_i) \in \overset{m_i}{F}(\rho)$ and $(\bar{b}, c) \in \overset{m_0}{F}(\rho)$. So, $(\bar{a}, b_1), \ldots, (\bar{a}, b_n), (\bar{b}, c) \in \overset{m}{F}(\rho)$ because for $m = \max\{m_0, m_1, \ldots, m_n\}$ we have

$\overset{m_i}{F}(\rho) \subset \overset{m}{F}(\rho)$. From this, according to (4.8.3), we get $(\bar{a}, c) \in \overset{m+1}{F}(\rho)$, which implies $(\bar{a}, c) \in \bigcup\limits_{n=1}^{\infty} \overset{n}{F}(\rho)$. Thus the relation $\bigcup\limits_{n=1}^{\infty} \overset{n}{F}(\rho)$ is transitive. If $\rho_1 \subset \rho_2$, where ρ_1, ρ_2 are $(n+1)$-ary relations, then $F(\rho_1) \subset F(\rho_2)$ by (4.8.3). Consequently, $\rho \subset \overline{\rho}$ implies $F(\rho) \subset F(\overline{\rho}) = \overline{\rho}$. Analogously, $\overset{n}{F}(\rho) \subset \overline{\rho}$ for every n. Hence, $\bigcup\limits_{n=1}^{\infty} \overset{n}{F}(\rho) \subset \overline{\rho}$. But $\overline{\rho}$ is the smallest $(n+1)$-ary relation containing ρ, thus $\bigcup\limits_{n=1}^{\infty} \overset{n}{F}(\rho) = \overline{\rho}$. □

Proposition 4.8.2. *For projective $(n+1)$-ary relations ρ_1, ρ_2 we have:*
(a) $\rho_1 \subset \rho_2[\rho_1 \cdots \rho_1]$,
(b) $\rho_2 \subset \rho_2[\rho_1 \cdots \rho_1]$,
(c) $\overline{\rho_1 \cup \rho_2} = \overline{\rho_2[\rho_1 \cdots \rho_1]}$.

Proof. The first two inclusions are obvious. So, $\rho_1 \cup \rho_2 \subset \rho_2[\rho_1 \cdots \rho_1]$. Thus $\overline{\rho_1 \cup \rho_2} \subset \overline{\rho_2[\rho_1 \cdots \rho_1]}$. Since $\rho[\rho \cdots \rho] \subset \overline{\rho}$, we have $\rho_2[\rho_1 \cdots \rho_1] \subset (\rho_1 \cup \rho_2)[\rho_1 \cup \rho_2 \cdots \rho_1 \cup \rho_2] \subset \overline{\rho_1 \cup \rho_2}$, which implies $\overline{\rho_2[\rho_1 \cdots \rho_1]} \subset \overline{\overline{\rho_1 \cup \rho_2}} = \overline{\rho_1 \cup \rho_2}$. This proves (c). □

Let $(G^n, *)$ be a binary comitant of a Menger algebra (G, o). We say that an $(n+1)$-ary relation ρ defined on a Menger algebra (G, o) is *l-regular*, *v-negative* and an *l-ideal*, if it satisfies following conditions:

$$(\bar{y}, \bar{z}) \in \overset{n}{\bowtie}\rho \longrightarrow (\bar{y} * \bar{x}, \bar{z} * \bar{x}) \in \overset{n}{\bowtie}\rho,$$
$$(\bar{y} * \bar{x}, \bar{x}) \in \overset{n}{\bowtie}\rho,$$
$$(\bar{y}, \bar{z}) \in \overset{n}{\bowtie}\rho \longrightarrow (\bar{y} * \bar{x}, \bar{z}) \in \overset{n}{\bowtie}\rho$$

for all $\bar{x}, \bar{y}, \bar{z} \in G^n$.

The following proposition is obvious.

Proposition 4.8.3. *A projective and transitive $(n+1)$-ary relation ρ defined on a Menger algebra (G, o) is v-negative if and only if $(\bar{x}|_i(x_i[\bar{y}]), y_j) \in \rho$ for all $i, j = 1, \ldots, n$ and $\bar{x}, \bar{y} \in G^n$.*

Let (G^*, o^*) be an unitary Menger algebra of rank n obtained from a Menger algebra (G, o) by external adjoining the full set of selectors (see p. 31). On G we define an $(n+1)$-ary relation σ by putting

$$(\bar{g}, g_0) \in \sigma \longleftrightarrow \begin{cases} (\forall i \in \{1, \ldots, n\})(\exists u_i \in G^*)(\bar{w} \in G^n)(s_i \in \{1, \ldots, n\}) \\ g_1 = u_i[\bar{w}|_{s_i} g_0]. \end{cases}$$

A relation σ defined in such way is projective, l-regular and v-negative.

Any $(n+1)$-ary relation ρ defined on G induces on $D = G \cup \{e_1, \ldots, e_n\}$, where $e_1, \ldots, e_n$ are selectors in (G^*, o^*), the following three relations:

$$\overset{*}{\rho} = \rho \cup G^n \times \{e_1, \ldots, e_n\},$$

$$\pi_\rho = \big\{(\bar{x}, z) \,|\, (\exists y \in G^*)(\bar{x}, y) \in \overset{*}{\rho} \wedge z = y[\bar{x}]\big\},$$

$$\overset{n}{\chi}(\rho) = \overline{\pi_\rho \cup \sigma}.$$

The relation π_ρ is projective. So, π_ρ is l-regular if ρ is an l-ideal. In this case $\overset{n}{\chi}(\rho)$ is projective, transitive, l-regular and v-negative. Thus, $\overset{n}{\chi}(\rho) = \overline{\sigma[\pi_\rho \cdots \pi_\rho]}$ in view of Proposition 4.8.2.

Any representation P of (G, o) by n-place functions induces on G two $(n+1)$-ary relations $\overset{n}{\chi}_P$ and $\overset{n}{\delta}_P$ defined by

$$((g_1, \ldots, g_n), g_0) \in \overset{n}{\chi}_P \longleftrightarrow ((P(g_1), \ldots, P(g_n)), P(g_0)) \in \overset{n}{\chi}_{\check{P}(G)},$$

$$((g_1, \ldots, g_n), g_0) \in \overset{n}{\delta}_P \longleftrightarrow ((P(g_1), \ldots, P(g_n)), P(g_0)) \in \overset{n}{\delta}_{\check{P}(G)}\ .$$

It is not difficult to see that $\overset{n}{\chi}_P$ is a projective, transitive, l-regular, and v-negative $(n+1)$-ary relation.

Theorem 4.8.4. *A Menger algebra $(G, o, \overset{n}{\chi})$ is isomorphic to some Menger algebra $(\Phi, O, \overset{n}{\chi}_\Phi)$ of n-place functions if and only if $\overset{n}{\chi}$ is a projective, transitive, l-regular and v-negative $(n+1)$-ary relation.*

Proof. The necessity of the above conditions is obvious. To prove their sufficiency, consider the sum P of the family of representations $\big(P_{\left(\varepsilon_{\overset{n}{\chi}\langle\bar{z}\rangle}, W_{\overset{n}{\chi}\langle\bar{z}\rangle}\right)}\big)_{\bar{z} \in G^n}$ defined by the determining pairs $\big(\varepsilon_{\overset{n}{\chi}\langle\bar{z}\rangle}, W_{\overset{n}{\chi}\langle\bar{z}\rangle}\big)_{\bar{z} \in G^n}$. Then P is a homomorphism. It is not difficult to verify that for each determining pair (ε, W) we have

$$(\bar{g}, g_0) \in \overset{n}{\chi}_{(\varepsilon, W)} \longleftrightarrow (\forall \bar{x} \in B)\Big(\bar{g} * \bar{x} \in (W')^n \longrightarrow g_0[\bar{x}] \in W'\Big),$$

where $\bar{g} \in G^n$, $B = G^n \cup \{(e_1, \ldots, e_n)\}$, $W' = G \setminus W$ and $\overset{n}{\chi}_{(\varepsilon, W)} = \overset{n}{\chi}_{P_{(\varepsilon, W)}}$.

Using this equivalence we obtain $\overset{n}{\chi}_P \subset \overset{n}{\chi}$. Indeed,

$$\begin{aligned}
(\bar{g}, g_0) \in \overset{n}{\chi}_P &\longleftrightarrow (\forall \bar{z} \in G^n)\Big((\bar{g}, g_0) \in \overset{n}{\chi}_{\left(\varepsilon_{\overset{n}{\chi}\langle\bar{z}\rangle}, W_{\overset{n}{\chi}\langle\bar{z}\rangle}\right)}\Big) \\
&\longleftrightarrow (\forall \bar{z} \in G^n)(\forall \bar{x} \in B)\Big(\bar{g} * \bar{x} \in \Big(W'_{\overset{n}{\chi}\langle\bar{z}\rangle}\Big)^n \longrightarrow g_0[\bar{x}] \in W'_{\overset{n}{\chi}\langle\bar{z}\rangle}\Big) \\
&\longrightarrow (\forall \bar{z} \in G^n)\Big(\bar{g} \in \Big(W'_{\overset{n}{\chi}\langle\bar{z}\rangle}\Big)^n \longrightarrow g_0 \in W'_{\overset{n}{\chi}\langle\bar{z}\rangle}\Big) \\
&\longrightarrow (\bar{g}, g_0) \in \overset{n}{\chi}\ .
\end{aligned}$$

Similarly we can prove the converse inclusion. Hence, $\overset{n}{\chi}=\overset{n}{\chi}_P$.

Now let Λ^* be a representation of (G, o) transforming each element g of G into a full n-place function $\Lambda^*(g)$ defined on G^* such that $\Lambda^*(g)(\bar{x}) = g[\bar{x}]$ for every $\bar{x} \in (G^*)^n$. Then $\overset{n}{\chi}_{\Lambda^*}= G^n \times G$. Moreover, as it is not difficult to see, $P_0 = P + \Lambda^*$ is an isomorphism. For this isomorphism, we have $\overset{n}{\chi}_{P_0}=\overset{n}{\chi}_P \cap \overset{n}{\chi}_{\Lambda^*}=\overset{n}{\chi}_P \cap G^n \times G = \overset{n}{\chi}_P=\overset{n}{\chi}$. Hence, P_0 is an isomorphism of $(G, o, \overset{n}{\chi})$ onto $(P_0(G), O, \overset{n}{\chi}_{P_0(G)})$. This completes the proof. □

Corollary 4.8.5. *A Menger algebra $(G, o, \overset{n}{\chi}, \overset{n}{\delta})$ is isomorphic to some Menger algebra $(\Phi, O, \overset{n}{\chi}_\Phi, \overset{n}{\delta}_\Phi)$ of n-place functions if and only if the algebra $(G, o, \overset{n}{\chi})$ satisfies the conditions of Theorem* 4.8.4 *and*

$$(\bar{x}, y) \in \overset{n}{\delta} \longleftrightarrow (\bar{x}, y[\bar{x}]) \in \overset{n}{\chi} \,.$$

Proof. The necessity follows from (4.8.1). To prove the sufficiency suppose that $(G, o, \overset{n}{\chi}, \overset{n}{\delta})$ satisfies all the conditions of our corollary. Then, by Theorem 4.8.4, $(G, o, \overset{n}{\chi})$ is isomorphic to $(P_0(G), O, \overset{n}{\chi}_{P_0(G)})$. Therefore

$$(\bar{x}, y) \in \overset{n}{\delta} \longleftrightarrow (\bar{x}, y[\bar{x}]) \in \overset{n}{\chi} \longleftrightarrow (\bar{x}, y[\bar{x}]) \in \overset{n}{\chi}_{P_0} \longleftrightarrow (\bar{x}, y) \in \overset{n}{\delta}_{P_0} \,.$$

This means that algebras $(G, o, \overset{n}{\chi}, \overset{n}{\delta})$ and $(P_0(G), O, \overset{n}{\chi}_{P_0(G)}, \overset{n}{\delta}_{P_0(G)})$ are isomorphic. □

Theorem 4.8.6. *A Menger algebra $(G, o, \overset{n}{\delta})$ is isomorphic to a Menger algebra $(\Phi, O, \overset{n}{\delta}_\Phi)$ of n-place functions if and only if the relation $\overset{n}{\delta}$ is an l-ideal such that*

$$(\bar{x}, y[\bar{x}]) \in \overset{n}{\chi}(\overset{n}{\delta}) \longrightarrow (\bar{x}, y) \in \overset{n}{\delta} \,. \tag{4.8.4}$$

Proof. Necessity. Let P be the corresponding isomorphism. Then $\overset{n}{\delta}=\overset{n}{\delta}_P$. Thus, according to (4.8.1), we have $\pi_{\overset{n}{\delta}_P} \subset \overset{n}{\chi}_P$. Consequently, $\pi_{\overset{n}{\delta}} \subset \overset{n}{\chi}_P$. Also $\pi_{\overset{n}{\delta}} \cup \sigma \subset \overset{n}{\chi}_P$ because $\sigma \subset \overset{n}{\chi}_P$. So, $\overset{n}{\chi}(\overset{n}{\delta}) = \overline{\pi_{\overset{n}{\delta}} \cup \sigma} \subset \overline{\overset{n}{\chi}_P} = \overset{n}{\chi}_P$. Hence $(\bar{x}, y[\bar{x}]) \in \overset{n}{\chi}(\overset{n}{\delta})$ implies $(\bar{x}, y[\bar{x}]) \in \overset{n}{\chi}_P$, which, by (4.8.1), gives $(\bar{x}, y) \in \overset{n}{\delta}_P=\overset{n}{\delta}$.

Sufficiency. If an algebra $(G, o, \overset{n}{\chi}(\overset{n}{\delta}), \overset{n}{\delta})$ satisfies the above implication, then by Theorem 4.8.4, it is isomorphic to some algebra $(\Phi, O, \overset{n}{\chi}_\Phi, \overset{n}{\delta}_\Phi)$ of n-place functions. So, $(G, o, \overset{n}{\delta})$ and $(\Phi, O, \overset{n}{\delta}_\Phi)$ are isomorphic. □

The implication (4.8.4) is equivalent to an infinite system of elementary axioms (for details see [221]).

4.9 Notes on Chapter 4

The first abstract characterization of stabilizers of points and stationary subsets of semigroups of partial functions was given by B. M. Schein [176].

The relation of semi-compatibility was introduced by V. V. Vagner [249] on semigroups of partial transformations. He considered this relation together with the relation of inclusion of domains. Semigroups of partial transformations ordered by these two relations are called transformative semigroups. The first abstract characterization of transformative semigroups in the language of second order predicates was given by V. V. Vagner [249]. Later on, an elementary characterization was obtained independently by B. M. Schein and V. N. Salii. Schein's characterization is based on his method of the determining pairs. In [171], he proved that transformative semigroups are characterized by infinite system of axioms, which is not equivalent to any finite system. By using other methods, V. N. Salii found the abstract characterization of transformative semigroups [155] and transformative semigroups with a third relation added — the relation of inclusion of images of functions [156]. The system of elementary axioms for transformative semigroups of one-to-one partial functions was found by V. S. Trokhimenko [235].

The relation of co-definability was introduced by B. M. Schein [178]. He proved that on semigroups of partial mappings this relation can be characterized by a finite system of elementary axioms. The relation of connectivity can be characterized by an infinite system of axioms [47].

The investigation of the projection equivalence relation defined as a binary relation which for two given relations (functions) identifies their domains and their images was initiated by B. M. Schein, which described this relation on a semigroup of binary relations [167] and on a semigroup of functions [185]. On a semigroup of one-to-one partial mappings this relation was characterized by V. S. Garvackii [45]. A projection equivalence on these two semigroups is characterized by an infinite system of axioms. This system is not equivalent to any finite system of elementary axioms [45].

The first abstract characterization of stabilizers of points and stationary subsets on Menger algebras was given in [228] and [232]. For restrictive Menger $\mathcal{P}$-algebras of multiplace functions, these sets are characterized in [239]. The investigation on the co-definability relation on Menger algebras was initiated in [223]; on Menger $\mathcal{P}$-algebras – in [225]. The first results on the connectivity relation for multiplace functions were obtained by V. S. Trokhimenko in [226] and [229]. The investigation on a projection equivalence relation on f.o.p. Menger algebras was initiated in [214]. The last relation together with the relation of inclusions of domains was also studied in other types of Menger algebras (cf. Section 5.3).

Chapter 5

$(2,n)$-semigroups of functions

In this chapter, we will consider other naturally defined binary operations on the set $\mathcal{F}(A^n, A)$ of all partial n-place functions on A. Such compositions (called now *Mann superpositions*) were studied for the first time by Mann in his paper [109]. Afterwards they were studied by many authors, for example in [5, 217, 264].

In Section 5.1 we describe the representations of sets of n-place functions closed with respect to all Mann superpositions, in Section 5.2, the sets of n-place functions closed with respect to Mann and Menger superpositions, in Section 5.3, the sets of n-place functions with some relations defined on their domains.

5.1 $(2,n)$-semigroups and their representations

On the set $\mathcal{F}(A^n, A)$ of all n-place functions defined on the set A one consider n binary operations $\underset{1}{\oplus}, \ldots, \underset{n}{\oplus}$, called *Mann superpositions*, such that for all $f, g \in \mathcal{F}(A^n, A)$ and $a_1, \ldots, a_n \in A$ we have

$$(f \underset{i}{\oplus} g)(a_1, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, g(a_1, \ldots, a_n), a_{i+1}, \ldots, a_n),$$

or, in an abbreviated notation, as

$$(f \underset{i}{\oplus} g)(a_1^n) = f(a_1^{i-1}, g(a_1^n), a_{i+1}^n), \tag{5.1.1}$$

where $i = 1, \ldots, n$. Since all Menger superpositions are associative, the algebra $(\Phi, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$, where $\Phi \subset \mathcal{F}(A^n, A)$, is called a $(2,n)$*-semigroup of n-place functions*. In the case when $\Phi \subset \mathcal{T}(A^n, A)$, i.e., when Φ is the set of n-ary operations, this algebra is called a $(2,n)$*-semigroup of n-ary operations*.

An abstract algebra $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$, where $\underset{1}{\oplus}, \ldots, \underset{n}{\oplus}$ are associative binary operations defined on G, is called a $(2,n)$*-semigroup*.[1] Each of its homomorphisms into some $(2,n)$-semigroup of n-place functions (n-ary operations) is called a *representation by n-place functions* (respectively, *by n-ary operations*). A representation, which is one-to-one, is called *faithful*. A $(2,n)$-semigroup for which there exists a faithful representation is called *representable*.

[1] For $n = 1$ it is an arbitrary semigroup.

Definition 5.1.1. We say that a $(2,n)$-semigroup $(G,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ is *unitary*, if it contains elements $e_1,\ldots,e_n$, called *selectors*, such that

$$g\underset{i}{\oplus}e_i=e_i\underset{i}{\oplus}g=g, \tag{5.1.2}$$

$$e_k\underset{i}{\oplus}g=e_k \tag{5.1.3}$$

for all $g\in G,\ i,k=1,\ldots,n$, where $i\neq k$.

Further expressions of the form $(\cdots((x\underset{i_1}{\oplus}y_1)\underset{i_2}{\oplus}y_2)\cdots)\underset{i_k}{\oplus}y_k$ are denoted by $x\underset{i_1}{\oplus}y_1\underset{i_2}{\oplus}\cdots\underset{i_k}{\oplus}y_k$ or, in abbreviated form, by $x\overset{i_k}{\underset{i_1}{\oplus}}y_1^k$. The symbol $\mu_i(\overset{i_s}{\underset{i_1}{\oplus}}x_1^s)$ will be reserved for the expression $x_{i_k}\overset{i_s}{\underset{i_{k+1}}{\oplus}}x_{k+1}^s$, if $i\neq i_1,\ \ldots,\ i\neq i_{k-1},\ i=i_k$ for some $k\in\{1,\ldots,s\}$. In any other case, this symbol is empty. For example, $\mu_1(\underset{2}{\oplus}x\underset{1}{\oplus}y\underset{3}{\oplus}z)=y\underset{3}{\oplus}z$, $\mu_2(\underset{2}{\oplus}x\underset{1}{\oplus}y\underset{3}{\oplus}z)=x\underset{1}{\oplus}y\underset{3}{\oplus}z$, $\mu_3(\underset{2}{\oplus}x\underset{1}{\oplus}y\underset{3}{\oplus}z)=z$. The symbol $\mu_4(\underset{2}{\oplus}x\underset{1}{\oplus}y\underset{3}{\oplus}z)$ is empty.

As it is known, a Menger superposition of n-place functions $f,g_1,\ldots,g_n$ defined on A is denoted by $f[g_1\ldots g_n]$ or shortly by $f[g_1^n]$, i.e.,

$$f[g_1\cdots g_n](a_1^n)=f\left(g_1(a_1^n),\ldots,g_n(a_1^n)\right) \tag{5.1.4}$$

for any $a_1,\ldots,a_n\in A$, where the left and right hand side of (5.1.4) are simultaneously defined or not defined. As earlier, $I_1^n,\ldots,I_n^n$ denotes the n-place projectors, i.e., full n-place functions such that

$$I_i^n(a_1,\ldots,a_n)=a_i \tag{5.1.5}$$

for all $a_1,\ldots,a_n\in A$ and $i=1,\ldots,n$. It is not difficult to verify that for all n-place functions defined on A we have

$$f[I_1^n\cdots I_n^n]=f, \tag{5.1.6}$$

$$I_i^n[g_1\cdots g_n]=g_i\circ\triangle_{\mathrm{pr}_1 g_1\cap\cdots\cap\mathrm{pr}_1 g_n}, \tag{5.1.7}$$

$$f\underset{i}{\oplus}g=f[I_1^n\cdots I_{i-1}^n g I_{i+1}^n\cdots I_n^n], \tag{5.1.8}$$

$$f[g_1^n][h_1^n]=f[g_1[h_1^n]\cdots g_n[h_1^n]], \tag{5.1.9}$$

$$(f\underset{i}{\oplus}g)[h_1^n]=f[h_1^{i-1}g[h_1^n]\,h_{i+1}^n], \tag{5.1.10}$$

$$f[g_1^n]\underset{i}{\oplus}h=f[(g_1\underset{i}{\oplus}h)\cdots(g_n\underset{i}{\oplus}h)], \tag{5.1.11}$$

$$g\underset{i}{\oplus}I_i^n=I_i^n\underset{i}{\oplus}g=g, \tag{5.1.12}$$

$$I_k^n\underset{i}{\oplus}g=I_k^n\circ\triangle_{\mathrm{pr}_1 g} \tag{5.1.13}$$

for all $i,k\in\{1,\ldots,n\}$, $k\neq i$.

Proposition 5.1.2. *For all n-place functions* $f, g_1, \ldots, g_n \in \mathcal{F}(A^n, A)$ *and* $\underset{i_1}{\oplus}, \ldots, \underset{i_s}{\oplus}$ *we have*

$$f \overset{i_s}{\underset{i_1}{\oplus}} g_1^s = f[\,(I_1^n \overset{i_s}{\underset{i_1}{\oplus}} g_1^s) \cdots (I_n^n \overset{i_s}{\underset{i_1}{\oplus}} g_1^s)\,]. \tag{5.1.14}$$

Proof. We prove (5.1.14) by induction. For $s = 1$, by (5.1.8) and (5.1.13), we have

$$\begin{aligned} f \underset{i_1}{\oplus} g_1 &= f[I_1^n \cdots I_{i_1-1}^n g_1 I_{i_1+1}^n \cdots I_n^n] \\ &= f[(I_1^n \circ \triangle_{\mathrm{pr}_1 g_1}) \cdots (I_{i_1-1}^n \circ \triangle_{\mathrm{pr}_1 g_1})\, g_1\, (I_{i_1+1}^n \circ \triangle_{\mathrm{pr}_1 g_1}) \cdots (I_n^n \circ \triangle_{\mathrm{pr}_1 g_1})] \\ &= f[(I_1^n \underset{i_1}{\oplus} g_1) \cdots (I_{i_1-1}^n \underset{i_1}{\oplus} g_1)\,(I_{i_1}^n \underset{i_1}{\oplus} g_1)\,(I_{i_1+1}^n \underset{i_1}{\oplus} g_1) \cdots (I_n^n \underset{i_1}{\oplus} g_1)] \\ &= f[(I_1^n \underset{i_1}{\oplus} g_1) \cdots (I_n^n \underset{i_1}{\oplus} g_1)]. \end{aligned}$$

Thus, for $s = 1$, condition (5.1.14) is satisfied.

Assume that it is valid for $s = k$, i.e.,

$$f \overset{i_k}{\underset{i_1}{\oplus}} g_1^k = f[(I_1^n \overset{i_k}{\underset{i_1}{\oplus}} g_1^k) \cdots (I_n^n \overset{i_k}{\underset{i_1}{\oplus}} g_1^k)].$$

Then, according to this assumption and (5.1.11), we obtain

$$\begin{aligned} f \overset{i_{k+1}}{\underset{i_1}{\oplus}} g_1^{k+1} &= (f \overset{i_k}{\underset{i_1}{\oplus}} g_1^k) \underset{i_{k+1}}{\oplus} g_{k+1} = f[(I_1^n \overset{i_k}{\underset{i_1}{\oplus}} g_1^k) \cdots (I_n^n \overset{i_k}{\underset{i_1}{\oplus}} g_1^k)] \underset{i_{k+1}}{\oplus} g_{k+1} \\ &= f[\,((I_1^n \overset{i_k}{\underset{i_1}{\oplus}} g_1^k) \underset{i_{k+1}}{\oplus} g_{k+1}) \cdots ((I_n^n \overset{i_k}{\underset{i_1}{\oplus}} g_1^k) \underset{i_{k+1}}{\oplus} g_{k+1})\,] \\ &= f[\,(I_1^n \overset{i_{k+1}}{\underset{i_1}{\oplus}} g_1^{k+1}) \cdots (I_n^n \overset{i_{k+1}}{\underset{i_1}{\oplus}} g_1^{k+1})\,], \end{aligned}$$

which proves (5.1.14) for $s = k + 1$. So, by induction, it is true for all natural numbers. □

Proposition 5.1.3. *The implication*

$$\bigwedge_{i=1}^{n} \Big(\mu_i(\overset{i_s}{\underset{i_1}{\oplus}} g_1^s) = \mu_i(\overset{j_k}{\underset{j_1}{\oplus}} h_1^k) \Big) \longrightarrow f \overset{i_s}{\underset{i_1}{\oplus}} g_1^s = f \overset{j_k}{\underset{j_1}{\oplus}} h_1^k \tag{5.1.15}$$

is valid for all n-place functions $f, g_1, \ldots, g_s, h_1, \ldots, h_k \in \mathcal{F}(A^n, A)$ *and all* $i_1, \ldots, i_s, j_1, \ldots, j_k \in \{1, \ldots, n\}$.

Proof. Assume that the premise of (5.1.15) is satisfied. Then $\{i_1, \ldots, i_s\} = \{j_1, \ldots, j_k\}$. Indeed, for $i \notin \{i_1, \ldots, i_s\}$, the symbol $\mu_i(\overset{i_s}{\underset{i_1}{\oplus}} g_1^s)$ is empty. So, $\mu_i(\overset{j_k}{\underset{j_1}{\oplus}} h_1^k)$ also is empty. Thus $i \notin \{j_1, \ldots, j_k\}$.

If $(a_1,\ldots,a_n)\in \mathrm{pr}_1\,(f\overset{i_s}{\underset{i_1}{\oplus}}g_1^s)$, then, according to (5.1.14), we have

$$\begin{aligned}(f\overset{i_s}{\underset{i_1}{\oplus}}g_1^s)(a_1^n) &= f[(I_1^n\overset{i_s}{\underset{i_1}{\oplus}}g_1^s)\cdots(I_n^n\overset{i_s}{\underset{i_1}{\oplus}}g_1^s)](a_1^n)\\ &= f((I_1^n\overset{i_s}{\underset{i_1}{\oplus}}g_1^s)(a_1^n),\ldots,(I_n^n\overset{i_s}{\underset{i_1}{\oplus}}g_1^s)(a_1^n))\\ &= f(\mu_1'(\overset{i_s}{\underset{i_1}{\oplus}}g_1^s)(a_1^n),\ldots,\mu_n'(\overset{i_s}{\underset{i_1}{\oplus}}g_1^s)(a_1^n)),\end{aligned}$$

where $\mu_i'(\overset{i_s}{\underset{i_1}{\oplus}}g_1^s)(a_1^n)$ is equal to a_i for $i\notin\{i_1,\ldots,i_s\}$, and $\mu_i(\overset{i_s}{\underset{i_1}{\oplus}}g_1^s)(a_1^n)$ for $i\in\{i_1,\ldots,i_s\}$. As $\{i_1,\ldots,i_s\}=\{j_1,\ldots,j_k\}$ and the premise of (5.1.15) is satisfied, then

$$\mu_i'(\overset{i_s}{\underset{i_1}{\oplus}}g_1^s)(a_1^n)=\mu_i'(\overset{j_k}{\underset{j_1}{\oplus}}h_1^k)(a_1^n),$$

for all $i=1,\ldots,n$. Hence

$$\begin{aligned}f\overset{i_s}{\underset{i_1}{\oplus}}g_1^s(a_1^n) &= f(\mu_1'(\overset{i_s}{\underset{i_1}{\oplus}}g_1^s)(a_1^n),\ldots,\mu_n'(\overset{i_s}{\underset{i_1}{\oplus}}g_1^s)(a_1^n))\\ &= f(\mu_1'(\overset{j_k}{\underset{j_1}{\oplus}}h_1^k)(a_1^n),\ldots,\mu_n'(\overset{j_k}{\underset{j_1}{\oplus}}h_1^k)(a_1^n))=(f\overset{j_k}{\underset{j_1}{\oplus}}h_1^k)(a_1^n),\end{aligned}$$

which proves the inclusion $f\overset{i_s}{\underset{i_1}{\oplus}}g_1^s\subset f\overset{j_k}{\underset{j_1}{\oplus}}h_1^k$. The converse inclusion can be proved similarly. So, implication (5.1.15) is valid. □

Theorem 5.1.4. *A unitary $(2,n)$-semigroup $(G,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ with selectors $e_1,\ldots,e_n$ can be isomorphically represented by n-place functions if and only if*

$$\bigwedge_{i=1}^{n}\left(e_i\overset{i_s}{\underset{i_1}{\oplus}}x_1^s=e_i\overset{j_k}{\underset{j_1}{\oplus}}y_1^k\right)\longrightarrow g\overset{i_s}{\underset{i_1}{\oplus}}x_1^s=g\overset{j_k}{\underset{j_1}{\oplus}}y_1^k \tag{5.1.16}$$

for all $g,x_1,\ldots,x_s,y_1,\ldots,y_k\in G$ and $i_1,\ldots,i_s,j_1,\ldots,j_k\in\{1,\ldots,n\}$.

Proof. It is clear that $\{i_1,\ldots,i_s\}=\{j_1,\ldots,j_k\}$. Let

$$\{r_1,\ldots,r_p\}=\{1,\ldots,n\}\setminus\{i_1,\ldots,i_s\}.$$

Then, according to (5.1.2), we have

$$e_i\overset{i_s}{\underset{i_1}{\oplus}}x_1^s=e_i\overset{i_s}{\underset{i_1}{\oplus}}x_1^s\underset{r_1}{\oplus}e_{r_1}\underset{r_2}{\oplus}\cdots\underset{r_p}{\oplus}e_{r_p}=e_i\overset{i_s}{\underset{i_1}{\oplus}}x_1^s\overset{r_p}{\underset{r_1}{\oplus}}e_{r_1}^{r_p},$$

$$g\overset{i_s}{\underset{i_1}{\oplus}}x_1^s=g\overset{i_s}{\underset{i_1}{\oplus}}x_1^s\overset{r_p}{\underset{r_1}{\oplus}}e_{r_1}^{r_p},\qquad g\overset{j_k}{\underset{j_1}{\oplus}}y_1^k=g\overset{j_k}{\underset{j_1}{\oplus}}y_1^k\overset{r_p}{\underset{r_1}{\oplus}}e_{r_1}^{r_p},$$

$$e_i\overset{i_k}{\underset{j_1}{\oplus}}y_1^k=e_i\overset{j_k}{\underset{j_1}{\oplus}}y_1^k\overset{r_p}{\underset{r_1}{\oplus}}e_{r_1}^{r_p}.$$

So, (5.1.16) can be rewritten in the form

$$\bigwedge_{i=1}^{n}\left(e_i \overset{i_s}{\underset{i_1}{\oplus}} x_1^s \overset{r_p}{\underset{r_1}{\oplus}} e_{r_1}^{r_p} = e_i \overset{j_k}{\underset{j_1}{\oplus}} y_1^k \overset{r_p}{\underset{r_1}{\oplus}} e_{r_1}^{r_p}\right) \longrightarrow g \overset{i_s}{\underset{i_1}{\oplus}} x_1^s \overset{r_p}{\underset{r_1}{\oplus}} e_{r_1}^{r_p} = g \overset{j_k}{\underset{j_1}{\oplus}} y_1^k \overset{r_p}{\underset{r_1}{\oplus}} e_{r_1}^{r_p}\,,$$

i.e., in the form

$$\bigwedge_{i=1}^{n}\left(\mu_i(\overset{i_s}{\underset{i_1}{\oplus}} x_1^s \overset{r_p}{\underset{r_1}{\oplus}} e_{r_1}^{r_p}) = \mu_i(\overset{j_k}{\underset{j_1}{\oplus}} y_1^k \overset{r_p}{\underset{r_1}{\oplus}} e_{r_1}^{r_p})\right) \longrightarrow g \overset{i_s}{\underset{i_1}{\oplus}} x_1^s \overset{r_p}{\underset{r_1}{\oplus}} e_{r_1}^{r_p} = g \overset{j_k}{\underset{j_1}{\oplus}} y_1^k \overset{r_p}{\underset{r_1}{\oplus}} e_{r_1}^{r_p},$$

and the necessity of the condition follows from Proposition 5.1.3.

To prove the sufficiency of condition (5.1.16), consider the set $\mathfrak{A}$ of all those elements $(x_1,\ldots,x_n)\in G^n$ for which there are $y_1,\ldots,y_s\in G$ and $i_1,\ldots,i_s\in\{1,\ldots,n\}$ such that $x_i = e_i \overset{i_s}{\underset{i_1}{\oplus}} y_1^s,\ i=1,\ldots,n$. Next, for every $g\in G$, consider the n-place function $\lambda_g:\mathfrak{A}\to G$ defined by the equality

$$\lambda_g(x_1^n) = g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s\,. \tag{5.1.17}$$

Note that the value of $\lambda_g(x_1^n)$ does not depends on the choice of $y_1,\ldots,y_s$. Indeed, if for some $z_1,\ldots,z_k\in G$ and $j_1,\ldots,j_k\in\{1,\ldots,n\}$ we have also $x_i = e_i \overset{j_k}{\underset{j_1}{\oplus}} z_1^k$, then $e_i \overset{i_s}{\underset{i_1}{\oplus}} y_1^s = e_i \overset{j_k}{\underset{j_1}{\oplus}} z_1^k$ for every $i=1,\ldots,n$, whence, according to (5.1.16), we conclude $g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s = g \overset{j_k}{\underset{j_1}{\oplus}} z_1^k$.

Let $g_1,g_2\in G$ and $(x_1,\ldots,x_n)\in\mathfrak{A}$. Then

$$\lambda_{g_1 \underset{i}{\oplus} g_2}(x_1^n) = (g_1 \underset{i}{\oplus} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, \tag{5.1.18}$$

where $1\leqslant i\leqslant n,\ x_k = e_k \overset{i_s}{\underset{i_1}{\oplus}} y_1^s,\ k=1,\ldots,n$. Further we have

$$e_i \underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s = g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s = \lambda_{g_2}(x_1^n)$$

and

$$e_k \underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s = e_k \overset{i_s}{\underset{i_1}{\oplus}} y_1^s = x_k$$

for $k\neq i,\ k=1,\ldots,n$.

Thus, $(x_1^{i-1},\lambda_{g_2}(x_1^n),x_{i+1}^n)\in\mathfrak{A}$ and

$$\lambda_{g_1}(x_1^{i-1},\lambda_{g_2}(x_1^n),x_{i+1}^n) = g_1 \underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s,$$

which together with (5.1.18) gives

$$\lambda_{g_1 \underset{i}{\oplus} g_2}(x_1^n) = \lambda_{g_1}(x_1^{i-1}, \lambda_{g_2}(x_1^n), x_{i+1}^n),$$

i.e., $\lambda_{g_1 \underset{i}{\oplus} g_2}(x_1^n) = \lambda_{g_1} \underset{i}{\oplus} \lambda_{g_2}(x_1^n)$. So, $\lambda_{g_1 \underset{i}{\oplus} g_2} = \lambda_{g_1} \underset{i}{\oplus} \lambda_{g_2}$.

As $e_i \overset{n}{\underset{1}{\oplus}} e_1^n = e_i$ for every $i = 1, \ldots, n$, then $e_1^n = (e_1, \ldots, e_n) \in \mathfrak{A}$. Consequently

$$\lambda_g(e_1^n) = g \overset{n}{\underset{1}{\oplus}} e_1^n = g \overset{n}{\underset{2}{\oplus}} e_2^n = \cdots = g \underset{n}{\oplus} e_n = g$$

for every $g \in G$.

Now let $\lambda_{g_1} = \lambda_{g_2}$. Then $\lambda_{g_1}(e_1^n) = \lambda_{g_2}(e_1^n)$, whence $g_1 = g_2$.

So, we have proved that the mapping $g \mapsto \lambda_g$ is a faithful representation of a unitary $(2, n)$-semigroup $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ by n-place functions. □

Theorem 5.1.5. *A $(2, n)$-semigroup $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ has a faithful representation by n-place functions if and only if for all $g, x_1, \ldots, x_s, y_1, \ldots, y_k \in G$ and $i_1, \ldots, i_s, j_1, \ldots, j_k \in \{1, \ldots, n\}$ the implication*

$$\bigwedge_{i=1}^{n} \Big(\mu_i(\overset{i_s}{\underset{i_1}{\oplus}} x_1^s) = \mu_i(\overset{j_k}{\underset{j_1}{\oplus}} y_1^k) \Big) \longrightarrow g \overset{i_s}{\underset{i_1}{\oplus}} x_1^s = g \overset{j_k}{\underset{j_1}{\oplus}} y_1^k \tag{5.1.19}$$

is satisfied.

Proof. The necessity of condition (5.1.19) follows from Proposition 5.1.3. To prove the sufficiency, consider a $(2, n)$-semigroup $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ and the set $G^* = G \cup \{e_1, \ldots, e_n\}$, where the elements $e_1, \ldots, e_n \notin G$ are fixed. For $x_1, \ldots, x_s \in G$, $i_1, \ldots, i_s \in \{1, \ldots, n\}$ and $i = 1, \ldots, n$, by $\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} x_1^s)$, we denote the element of G^* such that

$$\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} x_1^s) = \begin{cases} \mu_i(\overset{i_s}{\underset{i_1}{\oplus}} x_1^s) & \text{if } \ i \in \{i_1, \ldots, i_s\}, \\ e_i & \text{if } \ i \notin \{i_1, \ldots, i_s\}. \end{cases}$$

Let $\mathfrak{A}_0$ be the collection of all n-tuples $(x_1, \ldots, x_n) \in (G^*)^n$ for which there exist $y_1, \ldots, y_s \in G$ and $i_1, \ldots, i_s \in \{1, \ldots, n\}$ such that $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$. Denote by $\mathfrak{A}^*$ the set $\mathfrak{A}_0 \cup \{(e_1, \ldots, e_n)\}$.

Next, for every $g \in G$ we define the n-place function $\lambda_g^* : (G^*)^n \to G^*$ putting

$$\lambda_g^*(x_1^n) = \begin{cases} g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s & \text{if } \ (x_1, \ldots, x_n) \in \mathfrak{A}_0, \\ g & \text{if } \ (x_1, \ldots, x_n) = (e_1, \ldots, e_n). \end{cases}$$

In the other cases this function is not defined.

The mapping $P : g \mapsto \lambda^*_g$ is a faithful representation of $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ by the so defined n-place functions. Indeed, if $(x_1, \ldots, x_n) \in \mathfrak{A}_0$, then

$$\lambda^*_{g_1 \underset{i}{\oplus} g_2}(x_1^n) = (g_1 \underset{i}{\oplus} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s.$$

But for $i \in \{i, i_1, \ldots, i_s\}$, we have

$$\mu^*_i(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = \mu_i(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s = \lambda^*_{g_2}(x_1^n)$$

and $\mu^*_k(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = \mu_k(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = x_k$ for all $k \neq i$. This means that

$$(x_1^{i-1}, \lambda^*_{g_2}(x_1^n), x_{i+1}^n) \in \mathfrak{A}_0$$

and as a consequence

$$g_1 \underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s = \lambda^*_{g_1}(x_1^{i-1}, \lambda^*_{g_2}(x_1^n), x_{i+1}^n) = \lambda^*_{g_1} \underset{i}{\oplus} \lambda^*_{g_2}(x_1^n).$$

Thus

$$\lambda^*_{g_1 \underset{i}{\oplus} g_2}(x_1^n) = \lambda^*_{g_1} \underset{i}{\oplus} \lambda^*_{g_2}(x_1^n)$$

for all $(x_1, \ldots, x_n) \in \mathfrak{A}_0$.

In the case $(x_1, \ldots, x_n) = (e_1, \ldots, e_n)$ we have $\lambda^*_{g_1 \underset{i}{\oplus} g_2}(e_1^n) = g_1 \underset{i}{\oplus} g_2$ and

$$\lambda^*_{g_2}(e_1^n) = g_2 = \mu_i(\underset{i}{\oplus} g_2) = \mu^*_i(\underset{i}{\oplus} g_2).$$

Since $\mu^*_k(\underset{i}{\oplus} g_2) = e_k$ for $k \neq i$, the element $(e_1^{i-1}, \lambda^*_{g_2}(e_1^n), e_{i+1}^n)$ lies in $\mathfrak{A}_0$ and

$$\lambda^*_{g_1}(e_1^{i-1}, \lambda^*_{g_2}(e_1^n), e_{i+1}^n) = g_1 \underset{i}{\oplus} g_2.$$

So, $\lambda^*_{g_1 \underset{i}{\oplus} g_2}(e_1^n) = \lambda^*_{g_1} \underset{i}{\oplus} \lambda^*_{g_2}(e_1^n)$.

Thus, we have

$$P(g_1 \underset{i}{\oplus} g_2) = P(g_1) \underset{i}{\oplus} P(g_2),$$

which completes the proof that P is a homomorphism. Hence P is a representation of $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ by partial n-place functions.

This representation is faithful because $P(g_1) = P(g_2)$, i.e., $\lambda^*_{g_1} = \lambda^*_{g_2}$, implies $\lambda^*_{g_1}(e_1^n) = \lambda^*_{g_2}(e_1^n)$, whence $g_1 = g_2$. □

Corollary 5.1.6. *Every* $(2,n)$*-semigroup satisfying condition* (5.1.19) *has a faithful representation by full* n*-place functions, i.e., by* n*-ary operations.*

Proof. Let $(G,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ be some $(2,n)$-semigroup. By Theorem 5.1.5 it is isomorphic to some $(2,n)$-semigroup $(\Phi,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ of (partial) n-place functions, where $\Phi\subset\mathcal{F}(A^n,A)$. Consider now the set $A_0=A\cup\{c\}$, where $c\notin A$, and the extension f^0 of $f\in\Phi$ defined in the following way:

$$f^0(x_1^n)=\begin{cases} f(x_1^n) & \text{if} \quad (x_1,\ldots,x_n)\in \operatorname{pr}_1 f,\\ c & \text{if} \quad (x_1,\ldots,x_n)\notin \operatorname{pr}_1 f\end{cases}$$

for all $x_1,\ldots,x_n\in A_0$. It is clear that f^0 is a full n-place function on A_0.

Let us show that the mapping $f\mapsto f^0$ is an isomorphism of $(\Phi,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ onto $(\Phi_0,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$, where $\Phi_0=\{f^0\,|\,f\in\Phi\}$. Indeed, if $f,g\in\Phi$, then in the case when $(x_1,\ldots,x_n)\in\operatorname{pr}_1(f\underset{i}{\oplus}g)$ we have $(f\underset{i}{\oplus}g)^0(x_1^n)=f\underset{i}{\oplus}g(x_1^n)$ and $(x_1,\ldots,x_n)\in\operatorname{pr}_1 g$, $(x_1^{i-1},g(x_1^n),x_{i+1}^n)\in\operatorname{pr}_1 f$. Hence $g^0(x_1^n)=g(x_1^n)$, and consequently

$$\begin{aligned} f^0\underset{i}{\oplus}g^0(x_1^n)&=f^0(x_1^{i-1},g^0(x_1^n),x_{i+1}^n)=f^0(x_1^{i-1},g(x_1^n),x_{i+1}^n)\\ &=f(x_1^{i-1},g(x_1^n),x_{i+1}^n)=f\underset{i}{\oplus}g(x_1^n). \end{aligned}$$

Thus $(f\underset{i}{\oplus}g)^0(x_1^n)=f^0\underset{i}{\oplus}g^0(x_1^n)$.

In the case when $(x_1,\ldots,x_n)\notin\operatorname{pr}_1(f\underset{i}{\oplus}g)$ we have $(f\underset{i}{\oplus}g)^0(x_1^n)=c$ and $(x_1,\ldots,x_n)\notin\operatorname{pr}_1 g$, or $(x_1,\ldots,x_n)\in\operatorname{pr}_1 g$ and $(x_1^{i-1},g(x_1^n),x_{i+1}^n)\notin\operatorname{pr}_1 f$. If $(x_1,\ldots,x_n)\notin\operatorname{pr}_1 g$, then $g^0(x_1^n)=c$ and

$$f^0\underset{i}{\oplus}g^0(x_1^n)=f^0(x_1^{i-1},g^0(x_1^n),x_{i+1}^n)=f^0(x_1^{i-1},c,x_{i+1}^n)=c.$$

If $(x_1,\ldots,x_n)\in\operatorname{pr}_1 g$ and $(x_1^{i-1},g(x_1^n),x_{i+1}^n)\notin\operatorname{pr}_1 f$, then $g^0(x_1^n)=g(x_0^n)$ and

$$f^0\underset{i}{\oplus}g^0(x_1^n)=f^0(x_1^{i-1},g^0(x_1^n),x_{i+1}^n)=f^0(x_1^{i-1},g(x_1^n),x_{i+1}^n)=c.$$

So, in all cases $(f\underset{i}{\oplus}g)^0(x_1^n)=f^0\underset{i}{\oplus}g^0(x_1^n)$ for all $(x_1,\ldots,x_n)\in(A_0)^n$. Thus, the mapping $f\mapsto f^0$ is a homomorphism.

If $f^0=g^0$ for some $f,g\in\Phi$, then $\operatorname{pr}_1 f=\operatorname{pr}_1 g$ and $f(x_1^n)=g(x_1^n)$ for all $x_1^n\in\operatorname{pr}_1 f$, i.e., $f=g$. So, $f\mapsto f^0$ is an isomorphism. □

Let $(G,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ be a $(2,n)$-semigroup. We say that a unitary $(2,n)$-semigroup $(G^*,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ with selectors $e_1,\ldots,e_n$ is a *unitary extension* of the $(2,n)$-semigroup $(G,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$, if

(a) $G \subset G^*$,

(b) $G \cap \{e_1, \dots, e_n\} = \varnothing$,

(c) $G \cup \{e_1, \dots, e_n\}$ is a generating set of the $(2,n)$-semigroup $(G^*, \underset{1}{\oplus}, \dots, \underset{n}{\oplus})$.

Theorem 5.1.7. *Every representable $(2,n)$-semigroup can be isomorphically embedded into a unitary extension of some $(2,n)$-semigroup.*

Proof. In view of our Theorem 5.1.5 every representable $(2,n)$-semigroup $(G, \underset{1}{\oplus}, \dots, \underset{n}{\oplus})$ satisfies condition (5.1.19). By Corollary 5.1.6, it is isomorphic to some $(2,n)$-semigroup $(\Phi_0, \underset{1}{\oplus}, \dots, \underset{n}{\oplus})$ of full n-place functions on some set A_0.

Consider now the set $\{I_1^n, \dots, I_n^n\}$ of n-place projectors on A_0 and the family of subsets $(F_k(\Phi_0))_{k\in\mathbb{N}}$ satisfying the following two conditions:

(1) $F_0(\Phi_0) = \Phi_0 \cup \{I_1^n, \dots, I_n^n\}$,

(2) $f, g \in F_k(\Phi_0) \longrightarrow f \underset{i}{\oplus} g \in F_{k+1}(\Phi_0)$.

Obviously $\{I_1^n, \dots, I_n^n\} \cap \Phi_0 = \varnothing$ and $\{I_1^n, \dots, I_n^n\} \subset F_k(\Phi_0)$ for every $k \in \mathbb{N}$. Moreover, if $f \in F_k(\Phi_0)$, then $f = f \underset{i}{\oplus} I_i^n \in F_{k+1}(\Phi_0)$. So, $\Phi_0 \subset F_k(\Phi_0) \subset F_{k+1}(\Phi_0)$ for every $k \in \mathbb{N}$.

Now let $\Phi^* = \bigcup\limits_{k=0}^{\infty} F_k(\Phi_0)$. Then $\Phi_0 \cup \{I_1^n, \dots, I_n^n\} = F_0(\Phi_0) \subset \Phi^*$ and $\{I_1^n, \dots, I_n^n\} \cap \Phi_0 = \varnothing$. If $f \in F_n(\Phi_0)$, $g \in F_m(\Phi_0)$ for some $n, m \in \mathbb{N}$, then $f, g \in F_k(\Phi_0)$, where $k = \max\{n, m\}$. Therefore $f \underset{i}{\oplus} g \in F_{k+1}(\Phi_0) \subset \Phi^*$. This means that the set Φ^* is closed with respect to the operations $\underset{1}{\oplus}, \dots, \underset{n}{\oplus}$ and contains the projectors $I_1^n, \dots, I_n^n$. Hence $(\Phi^*, \underset{1}{\oplus}, \dots, \underset{n}{\oplus})$ is a unitary extension of a $(2,n)$-semigroup $(\Phi_0, \underset{1}{\oplus}, \dots, \underset{n}{\oplus})$. □

Let $(P_i)_{i\in I}$ be the family of representations of a given $(2,n)$-semigroup $(G, \underset{1}{\oplus}, \dots, \underset{n}{\oplus})$ by n-place functions defined on sets $(A_i)_{i\in I}$, respectively. By the *union* of this family we mean the mapping $P : g \mapsto P(g)$, where $g \in G$, and $P(g)$ is an n-place function on $A = \bigcup\limits_{i\in I} A_i$ defined by

$$P(g) = \bigcup_{i\in I} P_i(g)\,.$$

If $A_i \cap A_j = \varnothing$ for all $i, j \in I$, $i \neq j$, then P is called the *sum* of $(P_i)_{i\in I}$ and is denoted by $P = \sum_{i\in I} P_i$. It is not difficult to see that the sum of representations is a representation, but the union of representations need not necessarily be a representation.

Let $(G, \underset{1}{\oplus}, \dots, \underset{n}{\oplus})$ be a $(2,n)$-semigroup. A binary relation $\rho \subset G \times G$ is called

- *v-regular*, if
$$\bigwedge_{i=1}^{n}(x_i,y_i)\in\rho\longrightarrow(g\overset{i_s}{\underset{i_1}{\oplus}}u_1^s\,,\,g\overset{j_k}{\underset{j_1}{\oplus}}v_1^k)\in\rho$$
for all $g\in G$ and $x_i=\mu_i(\overset{i_s}{\underset{i_1}{\oplus}}u_1^s)$, $y_i=\mu_i(\overset{j_k}{\underset{j_1}{\oplus}}v_1^k)$, $i=1,\ldots,n$, where $u_1,\ldots,u_s\in G$, $v_1,\ldots,v_k\in G$,
- *l-regular*, if
$$(x,y)\in\rho\longrightarrow(x\underset{i}{\oplus}z\,,\,y\underset{i}{\oplus}z)\in\rho$$
for all $x,y,z\in G$, $i=1,\ldots,n$,
- *v-negative*, if
$$\Big(x\overset{i_s}{\underset{i_1}{\oplus}}y_1^n\,,\,\mu_i(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)\Big)\in\rho$$
for all $i\in\{i_1,\ldots,i_s\}$ and $x,y_1,\ldots,y_s\in G$.

A nonempty subset W of G is called an *l-ideal*, if the implication
$$g\overset{i_s}{\underset{i_1}{\oplus}}x_1^s\notin W\longrightarrow\mu_i(\overset{i_s}{\underset{i_1}{\oplus}}x_1^s)\notin W$$
is true for all $g,x_1,\ldots,x_s\in G$ and $i\in\{i_1,\ldots,i_s\}\subset\{1,\ldots,n\}$.

By *determining pair* of a $(2,n)$-semigroup $(G,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ we mean an ordered pair $(\mathcal{E},W)$, where $\mathcal{E}$ is a symmetric and transitive binary relation defined on the unitary extension $(G^*,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ of the $(2,n)$-semigroup $(G,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ and W is the subset of G^* such that

(1) $G\cup\{e_1,\ldots,e_n\}\subset\operatorname{pr}_1\mathcal{E}$,

(2) $\{e_1,\ldots,e_n\}\cap W=\varnothing$,

(3) $\bigwedge\limits_{i=1}^{n}e_i\equiv x_i(\mathcal{E})\longrightarrow g\equiv g\overset{i_s}{\underset{i_1}{\oplus}}y_1^s(\mathcal{E})$, where $x_i=\mu_i(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)$, $i=1,\ldots,n$, $y_1,\ldots,y_s\in G^*$,

(4) $\bigwedge\limits_{i=1}^{n}x_i\equiv y_i(\mathcal{E})\longrightarrow g\overset{i_s}{\underset{i_1}{\oplus}}u_1^s\equiv g\overset{j_k}{\underset{j_1}{\oplus}}v_1^k(\mathcal{E})$ for all $g\in G$, $x_i=\mu_i(\overset{i_s}{\underset{i_1}{\oplus}}u_1^s)$, $y_i=\mu_i(\overset{j_k}{\underset{j_1}{\oplus}}v_1^k)\in G$, $i=1,\ldots,n$, where $u_1,\ldots,u_s,v_1,\ldots,v_k\in G^*$,

(5) if $W\neq\varnothing$, then W is an $\mathcal{E}$-class and $W\cap G$ is an l-ideal of G.

Let $(H_a)_{a\in A}$ be a collection of $\mathcal{E}$-classes (uniquely indexed by elements of A) such that $H_a\neq W$ and $H_a\cap(G\cup\{e_1,\ldots,e_n\})\neq\varnothing$ for all $a\in A$. Consider the set $\mathfrak{A}$ of elements $(a_1,\ldots,a_n)\in A^n$ satisfying one of the conditions:

(a) $H_{a_i} = \mathcal{E}\langle \mu^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)\rangle$ for all $i = 1,\ldots,n$ and some $y_1,\ldots,y_s \in G$,

(b) $H_{a_i} = \mathcal{E}\langle e_i\rangle$ for all $i = 1,\ldots,n$,

where $\mathcal{E}\langle x\rangle$ denotes the $\mathcal{E}$-class of x.

Next, for each $g \in G$, we define an n-place function $P_{(\mathcal{E},W)}(g)$ on A putting

$$(a_1^n, b) \in P_{(\mathcal{E},W)}(g) \longleftrightarrow (a_1,\ldots,a_n) \in \mathfrak{A} \wedge \begin{cases} g \overset{i_s}{\underset{i_2}{\oplus}} y_1^s \in H_b & \text{if holds (a)}, \\ \quad g \in H_b & \text{if holds (b)}. \end{cases}$$

Proposition 5.1.8. *If a $(2,n)$-semigroup $(G, \underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ is representable, then a mapping $g \mapsto P_{(\mathcal{E},W)}(g)$, where $(\mathcal{E},W)$ is a determining pair, is a representation of the $(2,n)$-semigroup by n-place functions.*

Proof. We must show that

$$P_{(\mathcal{E},W)}(g_1 \underset{i}{\oplus} g_2) = P_{(\mathcal{E},W)}(g_1) \underset{i}{\oplus} P_{(\mathcal{E},W)}(g_2) \tag{5.1.20}$$

for all $g_1, g_2 \in G$ and $i = 1,\ldots,n$.

Let $(a_1^n, b) \in P_{(\mathcal{E},W)}(g_1 \underset{i}{\oplus} g_2)$ for some $i = 1,\ldots,n$. Then, according to the definition of $P_{(\mathcal{E},W)}$, we have $(a_1,\ldots,a_n) \in \mathfrak{A}$. In the case (a), we have also $(g_1 \underset{i}{\oplus} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \in H_b$. But $H_b \cap W = \varnothing$, so, $(g_1 \underset{i}{\oplus} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \notin W$. Therefore $\mu_i^*(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \notin W$, i.e., $g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \notin W$, because W is an l-ideal. Assume that $g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \in H_c$. Since

$$\mu_k^*(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = \mu_k^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \notin W \quad \text{for} \quad k \neq i,$$

we have $(a_1^{i-1}, c, a_{i+1}^n) \in \mathfrak{A}$, which, together with $(a_1^n, c) \in P_{(\mathcal{E},W)}(g_2)$, implies $(a_1^n, b) \in P_{(\mathcal{E},W)}(g_1) \underset{i}{\oplus} P_{(\mathcal{E},W)}(g_2)$.

In the case (b) we obtain $g_1 \underset{i}{\oplus} g_2 \in H_b$. Thus $g_2 \notin W$, and consequently, $g_2 \in H_c$ for some $c \in A$. Therefore $(a_1^{i-1}, c, a_{i+1}^n) \in \mathfrak{A}$, whence

$$(a_1^{i-1}\, c\, a_{i+1}^n, b) \in P_{(\mathcal{E},W)}(g_1) \qquad \text{and} \qquad (a_1^n, c) \in P_{(\mathcal{E},W)}(g_2).$$

Consequently $(a_1^n, b) \in P_{(\mathcal{E},W)}(g_1) \underset{i}{\oplus} P_{(\mathcal{E},W)}(g_2)$, which shows that the inclusion

$$P_{(\mathcal{E},W)}(g_1 \underset{i}{\oplus} g_2) \subset P_{(\mathcal{E},W)}(g_1) \underset{i}{\oplus} P_{(\mathcal{E},W)}(g_2)$$

is valid in any case.

Now let $(a_1^n, b) \in P_{(\mathcal{E},W)}(g_1) \underset{i}{\oplus} P_{(\mathcal{E},W)}(g_2)$. Then there exists $c \in A$ such that $(a_1^n, c) \in P_{(\mathcal{E},W)}(g_2)$ and $(a_1^{i-1} c\, a_{i+1}^n, b) \in P_{(\mathcal{E},W)}(g_1)$.

If $H_{a_i} = \mathcal{E}\langle e_i \rangle$ for all $i = 1, \ldots, n$, then $g_2 \in H_c$. Thus $(a_1^{i-1}, c, a_{i+1}^n) \in \mathfrak{A}$ and $g_1 \underset{i}{\oplus} g_2 \in H_b$, whence $(a_1^n, b) \in P_{(\mathcal{E},W)}(g_1 \underset{i}{\oplus} g_2)$.

If $H_{a_i} = \mathcal{E}\langle \mu^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \rangle$ for all $i = 1, \ldots, n$ and some $y_1, \ldots, y_s \in G$, then $H_b = \mathcal{E}\langle g_1 \underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \rangle$, because $H_c = \mathcal{E}\langle g_2 \overset{i_2}{\underset{i_1}{\oplus}} y_1^s \rangle = \mathcal{E}\langle \mu_i^*(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \rangle$ and

$$H_k = \mathcal{E}\langle \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \rangle = \mathcal{E}\langle \mu_i^*(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \rangle, \quad k \neq i.$$

Therefore $(a_1^n, b) \in P_{(\mathcal{E},W)}(g_1 \underset{i}{\oplus} g_2)$. So,

$$P_{(\mathcal{E},W)}(g_1) \underset{i}{\oplus} P_{(\mathcal{E},W)}(g_2) \subset P_{(\mathcal{E},W)}(g_1 \underset{i}{\oplus} g_2),$$

which, together with the previous inclusion, proves (5.1.20). □

The representation $P_{(\mathcal{E},W)}$, uniquely determined by the pair $(\mathcal{E}, W)$, will be called *simplest*.

Theorem 5.1.9. *Any representation of a $(2,n)$-semigroup by n-place functions is a union of some family of its simplest representations.*

Proof. Let P be a representation of the $(2,n)$-semigroup $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ by n-place functions defined on A, and let $c \notin A$ be some fixed element. For every $g \in G$ we define on $A_0 = A \cup \{c\}$ an n-ary operation $P^*(g)$ putting

$$P^*(g) = \begin{cases} P(g)(a_1^n) & \text{if} \quad (a_1, \ldots, a_n) \in \mathrm{pr}_1\, P(g), \\ c & \text{if} \quad (a_1, \ldots, a_n) \notin \mathrm{pr}_1\, P(g). \end{cases}$$

It is not difficult to see that P^* is a representation of $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ by n-ary operations defined on A_0, and $P(g) \mapsto P^*(g)$, where $g \in G$, is an isomorphism of $(P(G), \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ onto $(P^*(G), \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$. Because $G \cup \{e_1, \ldots, e_n\}$ is a generating set of the unitary extension $(G^*, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ with selectors $e_1, \ldots, e_n$, so putting $P^*(e_i) = I_i^n$, $i = 1, \ldots, n$, where I_i^n is the ith n-place projector of A_0, we obtain a unique extension of P^* from G to G^*.

For any $(a_1, \ldots, a_n) \in A^n$ we define on G^* an equivalence $\boldsymbol{\Theta}_{a_1^n}$ such that

$$x \equiv y(\theta_{a_1^n}) \longleftrightarrow P^*(x)(a_1^n) = P^*(y)(a_1^n).$$

This equivalence is v-regular. Indeed, if $x_i \equiv y_i(\theta_{a_1^n})$, i.e., $P^*(x_i)(a_1^n) = P^*(y_i)(a_1^n)$ for $i = 1, \ldots, n$, then for $g \in G$ and $x_i = \mu_i(\overset{i_s}{\underset{i_1}{\oplus}} u_1^s)$, $y_i = \mu_i(\overset{j_k}{\underset{j_1}{\oplus}} v_1^k)$, $i = 1, \ldots, n$, by Proposition 5.1.2, we have

$$\begin{aligned}
P^*(g \overset{i_s}{\underset{v_1}{\oplus}} u_1^s)(a_1^n) &= P^*(g) \underset{i_1}{\oplus} P^*(u_1) \underset{i_2}{\oplus} \cdots \underset{i_s}{\oplus} P^*(u_s)(a_1^n) \\
&= P^*(g)\Big[\Big(I_1^n \underset{i_1}{\oplus} P^*(u_1) \underset{i_2}{\oplus} \cdots \underset{i_s}{\oplus} P^*(u_s)\Big) \cdots \\
&\quad \cdots \Big(I_n^n \underset{i_1}{\oplus} P^*(u_1) \underset{i_2}{\oplus} \cdots \underset{i_s}{\oplus} P^*(u_s)\Big)\Big](a_1^n) \\
&= P^*(g)\Big(P^*(\mu_1(\overset{i_s}{\underset{i_1}{\oplus}} u_1^s))(a_1^n), \ldots, P^*(\mu_n(\overset{i_s}{\underset{i_1}{\oplus}} u_1^s))(a_1^n)\Big) \\
&= P^*(g)\Big(P^*(x_1)(a_1^n), \ldots, P^*(x_n)(a_1^n)\Big) \\
&= P^*(g)\Big(P^*(y_1)(a_1^n), \ldots, P^*(y_n)(a_1^n)\Big) \\
&= \cdots = P^*(g \overset{j_k}{\underset{j_1}{\oplus}} v_1^k)(a_1^n).
\end{aligned}$$

So, $g \overset{i_s}{\underset{i_1}{\oplus}} u_1^s \equiv g \overset{j_k}{\underset{j_1}{\oplus}} v_1^k(\theta_{a_1^n})$. This proves the v-regularity of the equivalence $\theta_{a_1^n}$. All subsets of the form

$$H_b^{a_1^n} = \{x \in G^* \mid (a_1^n, b) \in P^*(x)\}$$

are, of course, equivalence classes of this relation. Moreover, the pair $(\mathcal{E}_{a_1^n}, W_{a_1^n})$, where

$$\mathcal{E}_{a_1^n} = \theta_{a_1^n} \cap \theta_{a_1^n}\Big(G \cup \{e_1, \ldots, e_n\}\Big) \times \theta_{a_1^n}\Big(G \cup \{e_1, \ldots, e_n\}\Big),$$

$$W_{a_1^n} = \{x \in G^* \mid P^*(x)(a_1^n) = c\},$$

is a determining pair of a $(2, n)$-semigroup $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$.

We prove that a representation P is the union of a family of the simplest representations $P_{a_1^n}$ of $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ induced by a determining pair $(\mathcal{E}_{a_1^n}, W_{a_1^n})$, i.e., that

$$P(g) = \bigcup_{a_1^n \in A^n} P_{a_1^n}(g) \tag{5.1.21}$$

for every $g \in G$. Indeed, if $(b_1^n, d) \in P(g)$, then $g \in H_d^{b_1^n}$, where $b_1^n \in A^n$, $d \in A$. But $e_i \in H_{b_i}^{b_1^n}$, $i = 1, \ldots, n$, and $g \in H_d^{b_1^n}$, imply, according to the definition, $(b_1^n, d) \in P_{b_1^n}(g)$. Therefore $(b_1^n, d) \in \bigcup_{a_1^n \in A^n} P_{a_1^n}(g)$. So,

$$P(g) \subset \bigcup_{a_1^n \in A^n} P_{a_1^n}(g).$$

Conversely, if $(b_1^n, d) \in P_{a_1^n}(g)$ for some $a_1^n \in A^n$, then $g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \in H_d^{a_1^n}$ (for $H_{b_i}^{a_1^n} = \mathcal{E}_{a_1^n}\langle \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)\rangle$, $i = 1, \ldots, n$), or $g \in H_d^{a_1^n}$ (for $H_{b_i}^{a_1^n} = \mathcal{E}_{a_1^n}\langle e_i\rangle$, $i = 1, \ldots, n$, where $b_1^n \in A^n$).

For $\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \in H_{b_i}^{a_1^n}$ we have $b_i = P\Big(\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)\Big)(a_1^n)$, $i = 1, \ldots, n$. From $g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \in H_d^{a_1^n}$ we obtain $d = P\Big(g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s\Big)(a_1^n)$. But P is a homomorphism, so,

$$d = P(g)\left(P\Big(\mu_1^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)\Big)(a_1^n), \ldots, P\Big(\mu_n^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)\Big)(a_1^n)\right) = P(g)(b_1^n).$$

Hence $(b_1^n, c) \in P(g)$.

For $e_i \in H_{b_i}^{a_1^n}$, $i = 1, \ldots, n$, we get $(a_1^n, b_i) \in P^*(e_i) = I_i^n$, $i = 1, \ldots, n$, whence $a_i = b_i$ for all $i = 1, \ldots, n$. So, $g \in H_d^{b_1^n}$, i.e., $(b_1^n, d) \in P(g)$.

Thus, in both these cases we have

$$\bigcup_{a_1^n \in A^n} P_{a_1^n}(g) \subset P(g)$$

which, together with the previous inclusion, proves (5.1.21). □

Let P be a representation of a $(2, n)$-semigroup $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ by n-place functions. Define on G two binary relations χ_P and ε_P putting

$$(g_1, g_2) \in \chi_P \longleftrightarrow \mathrm{pr}_1\, P(g_1) \subset \mathrm{pr}_1\, P(g_2),$$
$$(g_1, g_2) \in \varepsilon_P \longleftrightarrow P(g_1) = P(g_2).$$

It is not difficult to see that the relation χ_P is reflexive and transitive, i.e., it is a quasi-order. The relation ε_P is an equivalence on G. If a representation P is faithful, then $\varepsilon_P = \triangle_G = \{(g, g) \,|\, g \in G\}$. In the case when P is a representation by full n-place functions, we have $\chi_P = G \times G$. Moreover, if P is the sum of a family $(P_i)_{i \in I}$ of representations P_i, then

$$\chi_P = \bigcap_{i \in I} \chi_{P_i} \quad \text{and} \quad \varepsilon_P = \bigcap_{i \in I} \varepsilon_{P_i}. \tag{5.1.22}$$

An algebraic system $(\Phi, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \chi_\Phi)$ is called a *projection quasi-ordered* $(2, n)$*-semigroup of* n*-place functions* if $(\Phi, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ is a $(2, n)$-semigroup of n-place functions on a set A and

$$\chi_\Phi = \{(f, g) \in \Phi \times \Phi \,|\, \mathrm{pr}_1 f \subset \mathrm{pr}_1 g\}.$$

It is characterized by the following theorem.

Theorem 5.1.10. *An algebraic system* $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \chi)$, *where* $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ *is a* $(2,n)$*-semigroup and* χ *is a binary relation on* G, *is isomorphic to a projection quasi-ordered* $(2,n)$*-semigroup of* n*-place functions if an only if it satisfies the condition* (5.1.19) *and* χ *is an* l*-regular,* v*-negative quasi-order.*

Proof. Necessity. Let $(\Phi, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \chi_\Phi)$ be a projection quasi-ordered $(2,n)$-semigroup of n-place functions. It is clear that the relation χ_Φ is reflexive and transitive, i.e., it is a quasi-order. By Theorem 5.1.5 condition (5.1.19) is satisfied.

Assume that for some $f, g \in \Phi$ we have $(f,g) \in \chi_\Phi$, i.e., $\mathrm{pr}_1 f \subset \mathrm{pr}_1 g$. If $(a_1, \ldots, a_n) \in \mathrm{pr}_1 (f \underset{i}{\oplus} h)$, where $h \in \Phi$, then there exists $c \in A$ such that $(a_1^n, c) \in f \underset{i}{\oplus} h$, whence $(a_1^n, b) \in h$ and $(a_1^{i-1}\, b\, a_{i+1}^n, c) \in f$ for some $b \in A$. So, $(a_1^{i-1}\, b\, a_{i+1}^n) \in \mathrm{pr}_1 f$. Therefore $(a_1^{i-1}\, b\, a_{i+1}^n) \in \mathrm{pr}_1 g$, whence $(a_1^{i-1}\, b\, a_{i+1}^n, d) \in g$ for some $d \in A$. Thus, $(a_1^n, b) \in h$ and $(a_1^{i-1}\, b\, a_{i+1}^n, d) \in g$. Hence $(a_1^n, d) \in g \underset{i}{\oplus} h$, i.e., $(a_1, \ldots, a_n) \in \mathrm{pr}_1 (g \underset{i}{\oplus} h)$, which proves the inclusion $\mathrm{pr}_1 (f \underset{i}{\oplus} h) \subset \mathrm{pr}_1 (g \underset{i}{\oplus} h)$. So, the relation χ_Φ is l-regular.

Now let $(a_1, \ldots, a_n) \in \mathrm{pr}_1 (f \overset{i_s}{\underset{i_1}{\oplus}} g_1^s)$ for some $f, g_1, \ldots, g_s \in \Phi$. This means that $f \overset{i_s}{\underset{i_1}{\oplus}} g_1^s \langle a_1^n \rangle \neq \varnothing$, where the expression $f\langle a_1^n\rangle$ denotes the set $\{f(a_1^n)\}$, if $(a_1, \ldots, a_n) \in \mathbf{pr}_1 f$, and the empty set if $(a_1, \ldots, a_n) \notin \mathbf{pr}_1 f$. Thus, according to (5.1.14), we obtain

$$\begin{aligned} \varnothing \neq f \overset{i_s}{\underset{i_1}{\oplus}} g_1^s \langle a_1^n \rangle &= f[(I_1^n \overset{i_s}{\underset{i_1}{\oplus}} g_1^s) \cdots (I_n^n \overset{i_s}{\underset{i_1}{\oplus}} g_1^s)]\langle a_1^n \rangle \\ &= f\Big(I_1^n \overset{i_s}{\underset{i_1}{\oplus}} g_1^s \langle a_1^n \rangle, \ldots, I_n^n \overset{i_s}{\underset{i_1}{\oplus}} g_1^s \langle a_1^n \rangle\Big), \end{aligned}$$

where $I_1^n, \ldots, I_n^n$ are the n-place projectors on A. Hence $I_i^n \overset{i_s}{\underset{i_1}{\oplus}} g_1^s \langle a_1^n \rangle \neq \varnothing$ for $i \in \{i_1, \ldots, i_s\}$. Thus $\mu_i(\overset{i_s}{\underset{i_1}{\oplus}} g_1^s)\langle a_1^n \rangle \neq \varnothing$, i.e., $(a_1, \ldots, a_n) \in \mathrm{pr}_1\, \mu_i(\overset{i_s}{\underset{i_1}{\oplus}} g_1^s)$. This shows that the relation χ_Φ is v-negative.

Sufficiency. Let all the conditions of the theorem be satisfied by an algebraic system $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \chi)$ and let $G^* = G \cup \{e_1, \ldots, e_n\}$, where $e_1, \ldots, e_n \notin G$. Consider the set $\mathfrak{A}_0$ defined in the following way:

$$(x_1, \ldots, x_n) \in \mathfrak{A}_0 \longleftrightarrow (\forall i = 1, \ldots, n)\ x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$$

for some $y_1, \ldots, y_s \in G$, $i_1, \ldots, i_s \in \{1, \ldots, n\}$, where $\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$ denotes an element from G^* defined in the proof of Theorem 5.1.4.

Let $a \in G$ be fixed. For every $g \in G$ we define an n-place function $P_a(g)$ from $\mathfrak{A}^* = \mathfrak{A}_0 \cup \{(e_1, \dots, e_n)\}$ to G by putting

$$P_a(g)(x_1^n) = \begin{cases} g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s & \text{if } x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s),\ i = 1, \dots, n, \text{ and } \big(a, g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s\big) \in \chi \\ & \text{for some } y_1, \dots, y_s \in G, \\ g & \text{if } (x_1, \dots, x_n) = (e_1, \dots, e_n) \text{ and } (a, g) \in \chi, \end{cases}$$

where $(x_1, \dots, x_n) \in \mathfrak{A}^*$. It is clear that $P_a(g)$ is a partial n-place function on G^*.

Let us show that $P_a : g \mapsto P_a(g)$ is a homomorphism, i.e., we verify the identity

$$P_a(g_1 \underset{i}{\oplus} g_2) = P_a(g_1) \underset{i}{\oplus} P_a(g_2). \tag{5.1.23}$$

For this, consider arbitrary $g_1, g_2 \in G$, $(x_1, \dots, x_n) \in \mathfrak{A}^*$ and $(x_1^n, y) \in P_a(g_1 \underset{i}{\oplus} g_2)$, where $y \in G$.

(1) If $(x_1, \dots, x_n) \in \mathfrak{A}_0$, i.e., $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$, $i = 1, \dots, n$, for some $y_1, \dots, y_s \in G$ and $i_1, \dots, i_s \in \{1, \dots, n\}$, then obviously $(a, y) \in \chi$ for $y = (g_1 \underset{i}{\oplus} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s$. But $i \in \{i, i_1, \dots, i_s\}$, so

$$\mu_i^*(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = \mu_i(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s\,. \tag{5.1.24}$$

Therefore, by the v-negativity of χ, from $(a, y) \in \chi$ we deduce

$$\big(a, (g_1 \underset{i}{\oplus} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s\big) \in \chi \quad \text{and} \quad \big((g_1 \underset{i}{\oplus} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s), \mu_i(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s\big) \in \chi.$$

Hence $\big(a, g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s\big) \in \chi$, i.e., $P_a(g_2)(x_1^n) = g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s$. For $k \neq i$ we have obviously

$$\mu_k^*(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = \mu_k^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = x_k, \quad k = 1, \dots, n,$$

which together with (5.1.24) implies $(x_1^{i-1}, g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, x_{i+1}^n) \in \mathfrak{A}_0$. Thus, from $(a, y) \in \chi$, we obtain

$$P_a(g_1)(x_1^{i-1}, g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, x_{i+1}^n) = g_1 \underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s = y,$$

i.e., $P_a(g_1)\Big(x_1^{i-1}, P_a(g_2)(x_1^n), x_{i+1}^n\Big) = y$. So, $P_a(g_1) \underset{i}{\oplus} P_a(g_2)(x_1^n) = y$, and consequently, $(x_1^n, y) \in P_a(g_1) \underset{i}{\oplus} P_a(g_2)$.

(2) If $(x_1,\dots,x_n)=(e_1,\dots,e_n)$, then $(a,y)\in\chi$ for $y=g_1\underset{i}{\oplus}g_2$. We have also $(g_1\underset{i}{\oplus}g_2,g_2)\in\chi$, $\mu_i^*(\underset{i}{\oplus}g_2)=g_2$ and $\mu_k^*(\underset{i}{\oplus}g_2)=e_k$ for $k\neq i$, $k=1,\dots,n$. From the above we obtain $P_a(g_2)(e_1,\dots,e_n)=g_2$, $(e_1^{i-1},g_2,e_{i+1}^n)\in\mathfrak{A}_0$ and $P_a(g_1)(e_1^{i-1},g_2,e_{i+1}^n)=g_1\underset{i}{\oplus}g_2$. Thus

$$P_a(g_1)\Big(e_1^{i-1},P_a(g_2)(e_1^n),e_{i+1}^n\Big)=y,$$

whence $(e_1^n,y)\in P_a(g_1)\underset{i}{\oplus}P_a(g_2)$.

We have shown in this way that in both cases $(x_1^n,y)\in P_a(g_1)\underset{i}{\oplus}P_a(g_2)$. This proves the inclusion

$$P_a(g_1\underset{i}{\oplus}g_2)\subset P_a(g_1)\underset{i}{\oplus}P_a(g_2).$$

To prove the converse inclusion, let $(x_1^n,y)\in P_a(g_1)\underset{i}{\oplus}P_a(g_2)$. Then there exists $z\in G$ such that

$$(x_1^n,z)\in P_a(g_2), \tag{5.1.25}$$

$$(x_1^{i-1}\,z\,x_{i+1}^n,y)\in P_a(g_1). \tag{5.1.26}$$

(1) If $(x_1,\dots,x_n)\in\mathfrak{A}_0$, then $x_i=\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)$, $i=1,\dots,n$, for some $y_1,\dots,y_s\in G$, $i_1,\dots,i_s\in\{1,\dots,n\}$. So, from (5.1.25) we get $(a,z)\in\chi$ for $z=g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s$. Since

$$\mu_i^*(\underset{i}{\oplus}g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)=g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s=z$$

and

$$\mu_k^*(\underset{i}{\oplus}g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)=\mu_k^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)=x_k$$

for $k\neq i$, $k=1,\dots,n$, condition (5.1.26) can be written as $(a,y)\in\chi$, where $y=g_1\underset{i}{\oplus}g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s$, which is equivalent to $(x_1^n,y)\in P_a(g_1\underset{i}{\oplus}g_2)$.

(2) If $(x_1,\dots,x_n)=(e_1,\dots,e_n)$, then (5.1.25) gives $(a,z)\in\chi$ and $z=g_2$. Similarly, (5.1.26) implies $(e_1^{i-1},g_2,e_{i+1}^n)\in P_a(g_1)$. But $(e_1^{i-1},g_2,e_{i+1}^n)\in\mathfrak{A}_0$, therefore $P_a(g_1)(e_1^{i-1},g_2,e_{i+1}^n)=g_1\underset{i}{\oplus}g_2=y$, i.e., $(a,y)\in\chi$ for $y=g_1\underset{i}{\oplus}g_2$, which means that $(e_1^n,y)\in P_a(g_1\underset{i}{\oplus}g_2)$.

So, in both cases we have $P_a(g_1)\underset{i}{\oplus}P_a(g_2)\subset P_a(g_1\underset{i}{\oplus}g_2)$, which together with the previous inclusion proves (5.1.23). Thus, P_a is a representation of the $(2,n)$-semigroup $(G,\underset{1}{\oplus},\dots,\underset{n}{\oplus})$ by n-place functions.

Let P_0 be the sum of a family of representations $(P_a)_{a\in G}$, i.e.,

$$P_0 = \sum_{a\in G} P_a\,. \tag{5.1.27}$$

Then, of course, P_0 also is a representation of this $(2,n)$-semigroup by n-place functions.

Now we show that $\chi = \chi_{P_0}$. Indeed, if $(g_1,g_2)\in\chi_{P_0}$, then, by (5.1.22), we have $(g_1,g_2)\in \bigcap\limits_{a\in G}\chi_{P_a}$, i.e., $\mathrm{pr}_1\,P_a(g_1)\subset \mathrm{pr}_1\,P_a(g_2)$ for every $a\in G$, which means that

$$(\forall a\in G)(\forall x_1^n\in\mathfrak{A}^*)\Big((x_1,\ldots,x_n)\in \mathrm{pr}_1 P_a(g_1)\longrightarrow (x_1,\ldots,x_n)\in \mathrm{pr}_1 P_a(g_2)\Big).$$

This, applied to $(x_1,\ldots,x_n)=(e_1,\ldots,e_n)$, gives

$$(\forall a\in G)\Big((\exists y\in G)(e_1^n,y)\in P_a(g_1)\longrightarrow(\exists z\in G)(e_1^n,z)\in P_a(g_2)\Big),$$

which, in particular, implies

$$(\forall a\in G)(\forall g_1\in G)\Big((a,g_1)\in\chi\longrightarrow(a,g_2)\in\chi\Big).$$

So, $(g_1,g_2)\in\chi$ by the reflexivity of χ, whence $\chi_{P_0}\subset\chi$.

Conversely, let $(g_1,g_2)\in\chi$, $a\in G$ and $(x_1,\ldots,x_n)\in\mathrm{pr}_1\,P_a(g_1)$. If $(x_1,\ldots,x_n)\in\mathfrak{A}_0$, then we have $(a,g_1\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)\in\chi$, where $x_i=\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)$, $i=1,\ldots,n$. Since χ is l-regular, $(g_1,g_2)\in\chi$ implies $(g_1\overset{i_s}{\underset{i_1}{\oplus}}y_1^s, g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)\in\chi$. Thus $(a,g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)\in\chi$, i.e., $(x_1,\ldots,x_n)\in\mathrm{pr}_1\,P_a(g_2)$. If $(x_1,\ldots,x_n)=(e_1,\ldots,e_n)$, then $(a,g_1)\in\chi$. Therefore $(a,g_2)\in\chi$, which means that $(e_1,\ldots,e_n)\in\mathrm{pr}_1\,P_a(g_2)$. Thus, for any $(x_1,\ldots,x_n)\in\mathfrak{A}^*$ from $(x_1,\ldots,x_n)\in\mathrm{pr}_1\,P_a(g_1)$, it follows that $(x_1,\ldots,x_n)\in\mathrm{pr}_1\,P_a(g_2)$. From this, according to (5.1.22) and (5.1.27), we conclude $\mathrm{pr}_1\,P_0(g_1)\subset\mathrm{pr}_1\,P_0(g_2)$, i.e., $(g_1,g_2)\in\chi_{P_0}$. So, $\chi\subset\chi_{P_0}$. Hence $\chi=\chi_{P_0}$.

Since the $(2,n)$-semigroup $(G,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ satisfies condition (5.1.19), by Corollary 5.1.6, there exists an isomorphic representation Λ of this $(2,n)$-semigroup by n-ary operations. Hence $\chi_\Lambda = G\times G$ and $\varepsilon_\Lambda=\triangle_G$. Consider now the representation P of the given $(2,n)$-semigroup, which is defined by the equality $P=\Lambda+P_0$. We have $\chi_P=\chi_\Lambda\cap\chi_{P_0}=G\times G\cap\chi=\chi$ and $\varepsilon_P=\varepsilon_\Lambda\cap\varepsilon_{P_0}=\triangle_G\cap\varepsilon_{P_0}=\triangle_G$. This means that P is a faithful representation. So, $(G,\underset{1}{\oplus},\ldots,\underset{n}{\oplus},\chi)$ is isomorphic to some projection quasi-ordered $(2,n)$-semigroup of n-place functions. □

5.2 Menger $(2, n)$-semigroups

On the set $\mathcal{F}(A^n, A)$ of all n-place functions on A, in addition to the binary compositions $\underset{1}{\oplus}, \ldots, \underset{n}{\oplus}$ defined according to the formula (5.1.1) we can consider the $(n+1)$-ary Menger composition of n-place functions, which, as previously, will be denoted by O. The algebra $(\Phi, O, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$, where $\Phi \subset \mathcal{F}(A^n, A)$ will be called a *Menger* $(2, n)$*-semigroup of* n*-place functions.* In the case when all elements of the set Φ are n-ary operations we will say that this algebra is a *Menger* $(2, n)$*-semigroup of* n*-ary operations* or a *Menger* $(2, n)$*-semigroup of full* n*-place functions.*

The algebra $\mathcal{G} = (G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ of type $(n+1, 2, \ldots, 2)$, where (G, o) is a Menger algebra of rank n, and $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ is a $(2, n)$-semigroup, will be called a *Menger* $(2, n)$*-semigroup.* Any its homomorphism into some Menger $(2, n)$-semigroup of n-place functions will be called a *representation* of this Menger $(2, n)$-semigroup by n-place functions. A representation, which is an isomorphism, is called *faithful.* Any Menger $(2, n)$-semigroup, for which there exists a faithful representation is called *representable.*

Definition 5.2.1. A Menger $(2, n)$-semigroup $\mathcal{G}$ is called *unitary* if it contains *selectors*, i.e., elements $e_1, \ldots, e_n \in G$ such that

$$x[e_1^n] = x, \tag{5.2.1}$$

$$e_i[x_1^n] = x_i, \quad i = 1, \ldots, n, \tag{5.2.2}$$

$$x[e_1^{i-1}\, y\, e_{i+1}^n] = x \underset{i}{\oplus} y, \quad i = 1, \ldots, n, \tag{5.2.3}$$

for all $x, x_1, \ldots, x_n, y \in G$.

Theorem 5.2.2. *Every unitary Menger* $(2, n)$*-semigroup is faithfully representable by full* n*-place functions (i.e.,* n*-ary operations) on some set in such a way that its selectors correspond to* n*-place projectors* $I_1^n, \ldots, I_n^n$ *on this set.*

Proof. Let $\mathcal{G}$ be a unitary Menger $(2, n)$-semigroup with selectors $e_1, \ldots, e_n$. Consider the set Φ of n-place projectors $I_1^n, \ldots, I_n^n$ and functions λ_g defined on G by the formula

$$\lambda_g(x_1^n) = g[x_1^n]\,. \tag{5.2.4}$$

We prove that $P : G \to \Phi$ such that $P(g) = \lambda_g$ is an isomorphism.

Indeed, for all $g, g_1, \ldots, g_n \in G$ and $x_1, \ldots, x_n \in G$, according to (5.1.1), (5.1.4), and (5.2.4), we have

$$\begin{aligned}\lambda_{g[g_1^n]}(x_1^n) &= g[g_1^n][x_1^n] = g[g_1[x_1^n]\,\cdots\,g_n[x_1^n]] = \lambda_g\Big(g_1[x_1^n],\cdots,g_n[x_1^n]\Big)\\ &= \lambda_g\Big(\lambda_{g_1}(x_1^n),\ldots,\lambda_{g_n}(x_1^n)\Big) = \lambda_g[\lambda_{g_1}\,\cdots\,\lambda_{g_n}](x_1^n),\end{aligned}$$

which implies $\lambda_{g[g_1^n]} = \lambda_g[\lambda_{g_1}\,\cdots\,\lambda_{g_n}]$. This proves that

$$P(g[g_1^n]) = P(g)[P(g_1)\,\cdots\,P(g_n)].$$

Similarly, by (5.1.1), (5.2.2), and (5.2.4), for all $g_1, g_2, x_1, \ldots, x_n \in G$, $i = 1, \ldots, n$, we obtain

$$\begin{aligned}\lambda_{g_1 \underset{i}{\oplus} g_2}(x_1^n) &= (g_1 \underset{i}{\oplus} g_2)[x_1^n] = g_1[e_1^{i-1}\, g_2\, e_{i+1}^n][x_1^n]\\ &= g_1[e_1[x_1^n]\,\cdots\,e_{i-1}[x_1^n]\; g_2[x_1^n]\; e_{i+1}[x_1^n]\,\cdots\,e_n[x_1^n]]\\ &= g_1[x_1^{i-1}\, g_2[x_1^n]\, x_{i+1}^n] = \lambda_{g_1}\Big(x_1^{i-1}, \lambda_{g_2}(x_1^n), x_{i+1}^n\Big)\\ &= \lambda_{g_1} \underset{i}{\oplus} \lambda_{g_2}(x_1^n).\end{aligned}$$

Thus $\lambda_{g_1 \underset{i}{\oplus} g_2} = \lambda_{g_1} \underset{i}{\oplus} \lambda_{g_2}$, i.e.,

$$P(g_1 \underset{i}{\oplus} g_2) = P(g_1) \underset{i}{\oplus} P(g_2).$$

Moreover, $P(e_i) = I_i^n$ for $i = 1, \ldots, n$, where $e_1, \ldots, e_n$ are selectors on G. Indeed,

$$\lambda_{e_i}(x_1^n) = e_i[x_1^n] = x_i = I_i^n(x_1^n)$$

for all $x_1, \ldots, x_n \in G$. Hence $P(e_i) = \lambda_{e_i} = I_i^n$, i.e., λ_{e_i} is the ith projector in G.

Finally, if $\lambda_{g_1} = \lambda_{g_2}$, then $\lambda_{g_1}(e_1^n) = \lambda_{g_1}(e_1^n)$, and consequently, $g_1[e_1^n] = g_2[e_1^n]$, which (by (5.2.1)) gives $g_1 = g_2$. Hence $P : G \to \mathbf{\Phi}$ is an isomorphism. □

From this point on, projectors of a Menger $(2,n)$-semigroup of n-ary operations will be identified with the selectors. For a Menger $(2,n)$-semigroup of partial n-place functions[2] such identification is impossible because

$$I_i^n[g_1\,\cdots\,g_n] = g_i \circ \Delta_{\mathrm{pr}_1 g_1 \cap \cdots \cap \mathrm{pr}_1 g_n}.$$

Theorem 5.2.3. *A Menger $(2,n)$-semigroup $\mathcal{G}$ is isomorphic to a Menger $(2,n)$-semigroup of n-place functions if and only if it satisfies the following three identities:*

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[2] I.e., partial n-place functions.

$$(x \underset{i}{\oplus} y)[z_1^n] = x[z_1^{i-1}\ y[z_1^n]\ z_{i+1}^n]\,, \tag{5.2.5}$$

$$x[y_1^n] \underset{i}{\oplus} z = x[(y_1 \underset{i}{\oplus} z) \cdots (y_n \underset{i}{\oplus} z)]\,, \tag{5.2.6}$$

$$x \overset{i_s}{\underset{i_1}{\oplus}} y_1^s = x[\mu_1(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)\ \cdots\ \mu_n(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)]\,, \tag{5.2.7}$$

where $\{i_1, \ldots, i_s\} = \{1, \ldots, n\}$, *and the implication* (5.1.19).

Proof. We limit ourselves to the proof of sufficiency because the necessity is a consequence of (5.1.10), (5.1.11), Proposition 5.1.2, and Proposition 5.1.3.

Let all the conditions of the theorem be satisfied by $\mathcal{G}$. Consider the set $G^* = G \cup \{e_1, \ldots, e_n\}$, where $e_1, \ldots, e_n$ are different elements not belonging to G. For every $g \in G$, we define on G^* an n-place function λ_g^* letting

$$\lambda_g^*(x_1^n) = \begin{cases} g[x_1^n] & \text{if } x_1, \ldots, x_n \in G, \\ g & \text{if } (x_1, \ldots, x_n) = (e_1, \ldots, e_n), \\ g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s & \text{if } x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s),\ i = 1, \ldots, n, \\ & \quad \text{where } y_1, \ldots, y_s \in G \end{cases}$$

and $\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$ is defined as in the proof of Theorem 5.1.5. In other cases, λ_g^* is not defined.

Let us show that the mapping $P : g \mapsto \lambda_g^*$ is a faithful representation of $\mathcal{G}$ by n-place functions.

Let $g, g_1, \ldots, g_n \in G$. Then for any $x_1, \ldots, x_n \in G$, according to (5.1.1), (5.1.4), and (5.2.4), we have

$$\begin{aligned} \lambda^*_{g[g_1^n]}(x_1^n) &= g[g_1^n][x_1^n] = g[g_1[x_1^n] \cdots g_n[x_1^n]] \\ &= \lambda_g^*\left(g_1[x_1^n], \ldots, g_n[x_1^n]\right) = \lambda_g^*\left(\lambda_{g_1}^*(x_1^n), \ldots, \lambda_{g_n}^*(x_1^n)\right) \\ &= \lambda_g^*[\lambda_{g_1}^* \cdots \lambda_{g_n}^*](x_1^n). \end{aligned}$$

If $(x_1, \ldots, x_n) = (e_1, \ldots, e_n)$, then obviously $\lambda^*_{g[g_1^n]}(e_1^n) = g[g_1^n]$ and

$$\lambda_g^*[\lambda_{g_1}^* \cdots \lambda_{g_n}^*](e_1^n) = \lambda_g^*\left(\lambda_{g_1}^*(e_1^n) \cdots \lambda_{g_n}^*(e_1^n)\right) = \lambda_g^*(g_1, \ldots, g_n) = g[g_1^n],$$

whence $\lambda^*_{g[g_1^n]}(e_1^n) = \lambda_g^*[\lambda_{g_1}^* \cdots \lambda_{g_n}^*](e_1^n)$.

Now let $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$, $i = 1, \ldots, n$, where $y_1, \ldots, y_s \in G$. Then

$$\begin{aligned} \lambda^*_{g[g_1^n]}(x_1^n) &= g[g_1^n] \overset{i_s}{\underset{i_1}{\oplus}} y_1^s = g[g_1 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \cdots g_n \overset{i_s}{\underset{i_1}{\oplus}} y_1^s] \\ &= \lambda_g^*\left(\lambda_{g_1}^*(x_1^n), \ldots, \lambda_{g_n}^*(x_1^n)\right) = \lambda_g^*[\lambda_{g_1}^* \cdots \lambda_{g_n}^*](x_1^n). \end{aligned}$$

So, $\lambda^*_{g[g_1^n]} = \lambda^*_g[\lambda^*_{g_1}\cdots\lambda^*_{g_n}]$.

In this way we have proved that

$$P(g[g_1^n]) = \lambda^*_g[\lambda^*_{g_1}\cdots\lambda^*_{g_n}] = O(P(g),P(g_1),\ldots,P(g_n))$$

for all $g, g_1, \ldots, g_n \in G$.

Now we consider the operation $\underset{i}{\oplus}$, where $i = 1,\ldots,n$.

Let $g_1, g_2 \in G$. If $x_1,\ldots,x_n \in G$, then, according to (5.2.5), we get

$$\begin{aligned}\lambda^*_{g_1\underset{i}{\oplus}g_2}(x_1^n) &= (g_1\underset{i}{\oplus}g_2)[x_1^n] = g_1[x_1^{i-1}\,g_2[x_1^n]\,x_{i+1}^n]\\ &= \lambda^*_{g_1}\left(x_1^{i-1},\lambda^*_{g_2}(x_1^n),x_{i+1}^n\right) = \lambda^*_{g_1}\underset{i}{\oplus}\lambda^*_{g_2}(x_1^n).\end{aligned}$$

If $(x_1,\ldots,x_n) = (e_1,\ldots,e_n)$, then $\lambda^*_{g_1\underset{i}{\oplus}g_2}(e_1^n) = g_1\underset{i}{\oplus}g_2$ and

$$\lambda^*_{g_1}\underset{i}{\oplus}\lambda^*_{g_2}(e_1^n) = \lambda^*_{g_1}\left(e_1^{i-1},\lambda^*_{g_2}(e_1^n),e_{i+1}^n\right) = \lambda^*_{g_1}\left(e_1^{i-1},g_2,e_{i+1}^n\right).$$

Since $\mu_i^*(\underset{i}{\oplus}g_2) = g_2$ and $\mu_k^*(\underset{i}{\oplus}g_2) = e_k$ for $k \neq i$, $k = 1,\ldots,n$, then

$$\lambda^*_{g_1}\left(e_1^{i-1},g_2,e_{i+1}^n\right) = g_1\underset{i}{\oplus}g_2.$$

Thus, $\lambda^*_{g_1\underset{i}{\oplus}g_2}(e_1^n) = \lambda^*_{g_1}\underset{i}{\oplus}\lambda^*_{g_2}(e_1^n)$.

Finally, let $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)$, $i = 1,\ldots,n$, where $y_1,\ldots,y_s \in G$, then

$$\lambda^*_{g_1\underset{i}{\oplus}g_2}(x_1^n) = g_1\underset{i}{\oplus}g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s.$$

On the other side, $\mu_i^*(\underset{i}{\oplus}g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s) = g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s = \lambda^*_{g_2}(x_1^n)$ and $\mu_k^*(\underset{i}{\oplus}g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s) = \mu_k^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s) = x_k$ for all $k = 1,\ldots,n$, $k \neq i$. Hence

$$\lambda^*_{g_1}\underset{i}{\oplus}\lambda^*_{g_2}(x_1^n) = \lambda^*_{g_1}\left(x_1^{i-1},\lambda^*_{g_2}(x_1^n),x_{i+1}^n\right) = g_1\underset{i}{\oplus}g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s.$$

Thus $\lambda^*_{g_1\underset{i}{\oplus}g_2}(x_1^n) = \lambda^*_{g_1}\underset{i}{\oplus}\lambda^*_{g_2}(x_1^n)$.[3]

So, in all cases we have $\lambda^*_{g_1\underset{i}{\oplus}g_2} = \lambda^*_{g_1}\underset{i}{\oplus}\lambda^*_{g_2}$, i.e.,

$$P(g_1\underset{i}{\oplus}g_2) = P(g_1)\underset{i}{\oplus}P(g_2)$$

[3] Observe that for $\{i_1,\ldots,i_s\} = \{1,\ldots,n\}$ this case, according to (5.2.7), is reduced to the case considered earlier, i.e., to the case when $x_1,\ldots,x_n \in G$.

for every $i = 1, \dots, n$.

This completes the proof that P is a representation of $\mathcal{G}$ by n-place functions.

This representation is faithful because $P(g_1) = P(g_2)$, i.e., $\lambda^*_{g_1} = \lambda^*_{g_2}$, implies $\lambda^*_{g_1}(e_1^n) = \lambda^*_{g_2}(e_1^n)$, whence $g_1 = g_2$. □

Corollary 5.2.4. *In any unitary Menger $(2,n)$-semigroup $\mathcal{G}$ we have*

$$x \overset{i_s}{\underset{i_1}{\oplus}} y_1^s = x[\mu_1^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \cdots \mu_n^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)] \tag{5.2.8}$$

for all $x, y_1, \dots, y_s \in G$ and $i_1, \dots, i_s \in \{1, \dots, n\}$.

Proof. Let $\{j_1, \dots, j_k\} = \{1, \dots, n\} \setminus \{i_1, \dots, i_s\}$. According to the definition 5.2.1, in a Menger $(2,n)$-semigroup containing selectors $e_1, \dots, e_n$, we have $x \underset{i}{\oplus} e_i = x$ for all $x \in G$ and $i = 1, \dots, n$. Thus

$$\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = \mu_i^*((\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \underset{j_1}{\oplus} e_{j_1} \underset{j_2}{\oplus} \cdots \underset{j_k}{\oplus} e_{j_k}),$$

for $i = 1, \dots, n$, which implies (5.2.7), and consequently, (5.2.8). □

Theorem 5.2.5. *Any Menger $(2,n)$-semigroup of n-place functions is isomorphic to some Menger $(2,n)$-semigroup of n-ary operations.*

Proof. Let $\mathbf{F} = (\Phi, O, \underset{1}{\oplus}, \dots, \underset{n}{\oplus})$, where $\Phi \subset \mathcal{F}(A^n, A)$, be a Menger $(2,n)$-semigroup of n-place functions. For every $f \in \Phi$, we define on $A_0 = A \cup \{c\}$, where $c \notin A$, an n-ary operation f^0 putting

$$f^0(x_1^n) = \begin{cases} f(x_1^n) & \text{if } \ (x_1, \dots, x_n) \in \mathrm{pr}_1 f, \\ \quad c & \text{if } \ (x_1, \dots, x_n) \notin \mathrm{pr}_1 f. \end{cases} \tag{5.2.9}$$

Let $f, g_1, \dots, g_n \in \Phi$. If $(x_1, \dots, x_n) \in \mathrm{pr}_1 f[g_1^n]$, then $(x_1, \dots, x_n) \in \mathrm{pr}_1 g_i$ for all $i = 1, \dots, n$ and, consequently, $(g_1(x_1^n), \dots, g_n(x_1^n)) \in \mathrm{pr}_1 f$. Hence

$$\begin{aligned} f[g_1^n]^0(x_1^n) &= f[g_1^n](x_1^n) = f(g_1(x_1^n), \dots, g_n(x_1^n)) \\ &= f^0(g_1(x_1^n), \dots, g_n(x_1^n)) = f^0(g_1^0(x_1^n), \dots, g_n^0(x_1^n)) \\ &= f^0[g_1^0 \cdots g_n^0](x_1^n), \end{aligned}$$

which gives

$$f[g_1^n]^0(x_1^n) = f^0[g_1^0 \cdots g_n^0](x_1^n)$$

for all $(x_1, \dots, x_n) \in \mathrm{pr}_1 f[g_1^n]$.

For $(x_1, \dots, x_n) \notin \mathrm{pr}_1 f[g_1^n]$, we have $f[g_1^n]^0(x_1^n) = c$. Moreover, from $(x_1, \dots, x_n) \notin \mathrm{pr}_1 f[g_1^n]$ it follows either $(x_1, \dots, x_n) \notin \mathrm{pr}_1 g_i$ (for some i)

or $(g_1(x_1^n),\ldots,g_n(x_1^n)) \notin \mathrm{pr}_1 f$. In the first case, $g_i^0(x_1^n) = c$, which gives $f^0(g_1^0(x_1^n),\ldots,g_n^0(x_1^n)) = c$. In the second (according to the definition of the operation f^0), we have $f^0(g_1(x_1^n),\ldots,g_n(x_1^n)) = c$, which implies

$$f^0(g_1^0(x_1^n),\ldots,g_n^0(x_1^n)) = c,$$

i.e., $f^0[g_1^0 \cdots g_n^0](x_1^n) = c$. This means that $f[g_1^n]^0(x_1^n) = f^0[g_1^0 \cdots g_n^0](x_1^n)$ holds also for $(x_1,\ldots,x_n) \notin \mathrm{pr}_1 f[g_1^n]$. Hence

$$f[g_1^n]^0 = f^0[g_1^0 \cdots g_n^0]$$

is satisfied for all $f, g_1,\ldots,g_n \in \mathbf{\Phi}$.

A similar argumentation proves that

$$(f \underset{i}{\oplus} g)^0 = f^0 \underset{i}{\oplus} g^0$$

for all $f, g \in \mathbf{\Phi}$ and $i = 1,\ldots,n$.

This means that the mapping $P : f \mapsto f^0$ is a homomorphism of $\mathbf{F}$ onto $(\mathbf{\Phi}_0, O, \underset{1}{\oplus},\ldots,\underset{n}{\oplus})$, where $\mathbf{\Phi}_0 = \{f^0 \mid f \in \mathbf{\Phi}\}$. In fact, as it is not difficult to see, P is an isomorphism. This completes the proof. □

As a simple consequence of Theorems 5.2.3 and 5.2.5 we obtain the following result.

Corollary 5.2.6. *Any Menger $(2,n)$-semigroup satisfying all the conditions of Theorem* 5.2.3 *is isomorphic to some Menger $(2,n)$-semigroup of n-ary operations.*

A *unitary extension* $\mathcal{G}^*$ of a Menger $(2,n)$-semigroup $\mathcal{G}$ is defined in the same way as a unitary extension of the $(2,n)$-semigroup $(G, \underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ (see p. 237).

In a unitary extension of a Menger $(2,n)$-semigroup of n-ary operations selectors are replaced by projectors.

Theorem 5.2.7. *Any Menger $(2,n)$-semigroup satisfying the assumptions of Theorem* 5.2.3 *can be isomorphically embedded into a unitary extension of some Menger $(2,n)$-semigroup of n-ary operations.*

Proof. If a Menger $(2,n)$-semigroup $\mathcal{G}$ is isomorphic to a Menger $(2,n)$-semigroup $(\mathbf{\Phi}, O, \underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ of n-place functions defined on A, then, by Theorem 5.2.5, it is isomorphic to a Menger $(2,n)$-semigroup $(\mathbf{\Phi}_0, O, \underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ of n-ary operations defined on A_0 (see the proof of Theorem 5.2.5).

Let $I_1^n,\ldots,I_n^n$ be n-ary projectors defined on A_0. Consider the family $(F_k(\mathbf{\Phi}_0))_{k=0,1,\ldots}$ of sets $F_k(\mathbf{\Phi}_0)$ such that:

(1) $F_0(\Phi_0) = \Phi_0 \cup \{I_1^n, \ldots, I_n^n\}$,

(2) $f, g_1, \ldots, g_n \in F_k(\Phi_0) \longrightarrow f[g_1^n] \in F_{k+1}(\Phi_0)$,

(3) $f, g \in F_k(\Phi_0) \longrightarrow f \underset{i}{\oplus} g \in F_{k+1}(\Phi_0)$.

It is clear that $\{I_1^n, \ldots, I_n^n\} \cap \Phi_0 = \varnothing$ and $\{I_1^n, \ldots, I_n^n\} \subset F_k(\Phi_0)$ for every $k = 0, 1, \ldots$ Moreover, if $f \in F_k(\Phi_0)$, then $f = f[I_1^n \ldots I_n^n] \in F_{k+1}(\Phi_0)$ (by (2)). Thus $F_k(\Phi_0) \subset F_{k+1}(\Phi_0)$ for every $k = 0, 1, \ldots$

Let $\Phi^* = \bigcup\limits_{k=0}^{\infty} F_k(\Phi_0)$. For all $f, g_1, \ldots, g_n \in \Phi^*$, there are $m_0, \ldots, m_n$ such that $f \in F_{m_0}(\Phi_0)$, $g_i \in F_{m_i}(\Phi_0)$. Thus $f, g_1, \ldots, g_n \in F_k(\Phi_0)$ for $k = \max\{m_0, m_1, \ldots, m_n\}$, and, consequently, $f[g_1^n] \in F_{k+1}(\Phi_0) \subset \Phi^*$. This proves that Φ^* is closed with respect to the operation O.

Analogously we can verify that Φ^* is closed with respect to the operations $\underset{1}{\oplus}, \ldots, \underset{n}{\oplus}$. Hence $(\Phi^*, O, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ is a unitary Menger $(2,n)$-semigroup generated by $\Phi_0 \cup \{I_1^n, \ldots, I_n^n\}$. By (5.2.9), we also have $\Phi_0 \cap \{I_1^n, \ldots, I_n^n\} = \varnothing$. Therefore $(\Phi^*, O, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ is a unitary extension of $(\Phi_0, O, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$. □

Let $\mathcal{G} = (G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ be a Menger $(2,n)$-semigroup. A binary relation $\rho \subset G \times G$ is called

- *v-regular*, if

$$(x,y), (x_1,y_1), \ldots, (x_n,y_n) \in \rho \longrightarrow \begin{cases} (g[x_1^n], g[y_1^n]) \in \rho, \\ (g \underset{i}{\oplus} x, g \underset{i}{\oplus} y) \in \rho \end{cases}$$

for all $g, x, y, x_i, y_i \in G$, $i = 1, \ldots, n$,

- *l-regular*, if

$$(x,y) \in \rho \longrightarrow \begin{cases} (x[z_1^n], y[z_1^n]) \in \rho, \\ (x \underset{i}{\oplus} z, y \underset{i}{\oplus} z) \in \rho \end{cases}$$

for all $x, y, z, z_i \in G$, $i = 1, \ldots, n$,

- *stable*, if

$$(x,y), (x_1,y_1), \ldots, (x_n,y_n) \in \rho \longrightarrow \begin{cases} (x[x_1^n], y[y_1^n]) \in \rho, \\ (x_1 \underset{i}{\oplus} x_2, y_1 \underset{i}{\oplus} y_2) \in \rho \end{cases}$$

for all $x, y, x_i, y_i \in G$, $i = 1, \ldots, n$.

Note that a reflexive and transitive relation ρ is stable if and only if it is v-regular and l-regular.

A nonempty subset W of G is called an *l-ideal* of a Menger $(2,n)$-semigroup $\mathcal{G}$ if

$$g[\bar{w}|_i x] \in W \ \wedge \ g \underset{i}{\oplus} x \in W$$

for all $g \in G$, $\bar{w} \in G^n$, $i = 1,\dots,n$ and $x \in W$.

Definition 5.2.8. By *determining pair* of a Menger $(2,n)$-semigroup $\mathcal{G}$ we mean an ordered pair $(\mathcal{E}, W)$, where $\mathcal{E}$ is a symmetric and transitive binary relation defined on a unitary extension $\mathcal{G}^*$ of $\mathcal{G}$ and W is a subset of G^* such that:

(i) $G \cup \{e_1,\dots,e_n\} \subset \operatorname{pr}_1 \mathcal{E}$,

(ii) $\{e_1,\dots,e_n\} \cap W = \varnothing$,

(iii) $g[\mathcal{E}\langle e_1\rangle,\dots,\mathcal{E}\langle e_n\rangle] \subset \mathcal{E}\langle g\rangle$ for all $g \in G$, where $\mathcal{E}\langle g\rangle$ is the $\mathcal{E}$-class containing g,

(iv) $g[\,\mathcal{E}\langle \mu_1^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)\rangle \cdots \mathcal{E}\langle \mu_n^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)\rangle] \subset \mathcal{E}\langle g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s\rangle$ for all $y_1,\dots,y_s \in G$, $i_1,\dots,i_s \in \{1,\dots,n\}$,

(v) $g[\mathcal{E}\langle g_1\rangle \cdots \mathcal{E}\langle g_n\rangle] \subset \mathcal{E}\langle g[g_1 \cdots g_n]\rangle$ for all $g, g_1,\dots,g_n \in G$,[4]

(vi) if $W \neq \varnothing$, then W is an $\mathcal{E}$-class and $W \cap G$ is an l-ideal of $\mathcal{G}$.

Let $(H_a)_{a\in A}$ be a collection of $\mathcal{E}$-classes (uniquely indexed by elements of A) such that $H_a \neq W$ and $H_a \cap (G \cup \{e_1,\dots,e_n\}) \neq \varnothing$ for all $a \in A$. Using this collection we define the set $\mathfrak{A} \subset A^n$ in the following way:

(a) if $g_i \in G$ and $H_{a_i} = \mathcal{E}\langle g_i\rangle$ for $i = 1,\dots,n$, then $(a_1,\dots,a_n) \in \mathfrak{A}$,

(b) if $H_{a_i} = \mathcal{E}\langle e_i\rangle$ for $i = 1,\dots,n$, then $(a_1,\dots,a_n) \in \mathfrak{A}$,

(c) if $H_{a_i} = \mathcal{E}\langle \mu^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)\rangle$ for $i = 1,\dots,n$ and some $y_1,\dots,y_s \in G$, then $(a_1,\dots,n) \in \mathfrak{A}$,

(d) $(a_1,\dots,n) \in \mathfrak{A}$ if and only if $(a_1,\dots,a_n)$ is determined by (a), (b), or (c).

Next, for any $g \in G$ we define on A an $(n+1)$-ary relation $P_{(\mathcal{E},W)}(g)$ letting:

$$(a_1^n, b) \in P_{(\mathcal{E},W)}(g) \longleftrightarrow (a_1,\dots,a_n) \in \mathfrak{A} \wedge g[H_{a_1} \cdots H_{a_n}] \subset H_b. \qquad (5.2.10)$$

Proposition 5.2.9. *If* $(a_1,\dots,a_n) \in \mathfrak{A}$ *and* $g[H_{a_1} \cdots H_{a_n}] \subset H_b$, *then* $H_b \cap G \neq \varnothing$.

Proof. Let $g, g_1,\dots,g_n \in G$. If $H_{a_i} = \mathcal{E}\langle g_i\rangle$ for $i = 1,\dots,n$, then $g[g_1^n] \in H_b \cap G$, whence $H_b \cap G \neq \varnothing$. If $H_{a_i} = \mathcal{E}\langle e_i\rangle$ for $i = 1,\dots,n$, then $g[e_1^n] \in H_b$

[4] Note that from the conditions (iv) and (v) follows that the relation $\mathcal{E} \cap G \times G$ is v-regular in $\mathcal{G}$.

and $g[e_1^n] = g \in G$. Thus $H_b \cap G \neq \varnothing$. Now if $H_{a_i} = \mathcal{E}\langle \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)\rangle$ for $i = 1,\ldots,n$ and some $y_1,\ldots,y_s \in G$, then $g[\mu_1^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \cdots \mu_n^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)] \in H_b$. But $g[\mu_1^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \cdots \mu_n^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)] = g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \in G$, so, $H_b \cap G \neq \varnothing$, which completes the proof. □

According to the definition of $P_{(\mathcal{E},W)}(g)$,

$$(a_1^n, b), (a_1^n, c) \in P_{(\mathcal{E},W)}(g) \longrightarrow b = c.$$

So, $P_{(\mathcal{E},W)}(g)$ is an n-place function such that

$$(a_1,\ldots,a_n) \in \mathrm{pr}_1\, P_{(\mathcal{E},W)}(g) \longleftrightarrow \begin{cases} (a_1,\ldots,a_n) \in \mathfrak{A}, \\ g[H_{a_1} \cdots H_{a_n}] \cap W = \varnothing. \end{cases} \tag{5.2.11}$$

Proposition 5.2.10. *If $(\mathcal{E}, W)$ is a determining pair of a Menger $(2,n)$-semigroup $\mathcal{G}$ satisfying the assumptions of Theorem 5.2.3, then*

$$P_{(\mathcal{E},W)}(g[g_1^n]) = P_{(\mathcal{E},W)}(g)\left[P_{(\mathcal{E},W)}(g_1) \cdots P_{(\mathcal{E},W)}(g_n)\right], \tag{5.2.12}$$

$$P_{(\mathcal{E},W)}(g_1 \underset{i}{\oplus} g_2) = P_{(\mathcal{E},W)}(g_1) \underset{i}{\oplus} P_{(\mathcal{E},W)}(g_2) \tag{5.2.13}$$

for all $g, g_1,\ldots,g_n \in G$ and $i = 1,\ldots,n$.

Proof. We prove only (5.2.12). The proof of (5.2.13) is similar.

Let $g, g_1,\ldots,g_n \in G$. If $(a_1^n, c) \in P_{(\mathcal{E},W)}(g)[P_{(\mathcal{E},W)}(g_1) \cdots P_{(\mathcal{E},W)}(g_n)]$, then there are $b_1,\ldots,b_n \in A$ such that

$$(a_1^n, b_i) \in P_{(\mathcal{E},W)}(g_i) \quad \text{and} \quad (b_1^n, c) \in P_{(\mathcal{E},W)}(g)$$

for $i = 1,\ldots,n$. This, by (5.2.10), is equivalent to

$$(a_1,\ldots,a_n) \in \mathfrak{A} \ \wedge\ g_i[H_{a_1} \cdots H_{a_n}] \subset H_{b_i},\ \ i = 1,\ldots,n,$$

$$(b_1,\ldots,b_n) \in \mathfrak{A} \ \wedge\ g[H_{b_1} \cdots H_{b_n}] \subset H_c\,.$$

Let $h_i \in H_{b_i}$, $p_i \in H_{a_i}$, where $i = 1,\ldots,n$. The following three cases are possible: 1) all p_i are in G; 2) $p_i = e_i$ for all $i = 1,\ldots,n$; 3) $p_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$ for all $i = 1,\ldots,n$.

(1) If all p_i are in G, then $g[h_1^n] \in H_c$ and $g_i[p_1^n] \in H_{b_i} \cap G$, $i = 1,\ldots,n$. Since $\big(h_i, g_1[p_1^n]\big) \in \mathcal{E}$ for all $i = 1,\ldots,n$, the v-regularity of $\mathcal{E}$ implies $\big(g[h_1^n], g[g_1[p_1^n] \cdots g_n[p_1^n]]\big) \in \mathcal{E}$, which gives us $\big(g[h_1^n], g[g_1^n][p_1^n]\big) \in \mathcal{E}$. Hence $g[g_1^n][p_1^n] \in H_c$. Therefore $g[g_1^n][H_{a_1} \cdots H_{a_n}] \subset H_c$.

(2) If $p_1 = e_1, \ldots, p_n = e_n$, then $g[h_1^n] \in H_c$, $g_i \in H_{b_i}$ and $\big(h_i, g_i\big) \in \mathcal{E}$ for $i = 1, \ldots, n$. Thus $\big(g[h_1^n], g[g_1^n]\big) \in \mathcal{E}$, which implies $g[g_1^n] \in H_c$. Hence $g[g_1^n][H_{a_1} \cdots H_{a_n}] \subset H_c$.

(3) If $p_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$ for all $i = 1, \ldots, n$ and some $y_1, \ldots, y_s \in G$, then $g[h_1^n] \in H_c$ and $g_i[p_1^n] = g_i \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \in H_{b_i}$ for $i = 1, \ldots, n$. Thus $\big(g[h_1^n], g[g_1 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \cdots g_n \overset{i_s}{\underset{i_1}{\oplus}} y_1^s]\big) \in \mathcal{E}$, which, after application of (5.2.6), proves $\big(g[h_1^n], g[g_1^n] \overset{i_s}{\underset{i_1}{\oplus}} y_1^s\big) \in \mathcal{E}$, i.e., $\big(g[h_1^n], g[g_1^n][p_1^n]\big) \in \mathcal{E}$. Therefore, as in the previous case, $g[g_1^n][H_{a_1} \cdots H_{a_n}] \subset H_c$.

From the above considerations, we obtain

$$(a_1, \ldots, a_n) \in \mathfrak{A} \ \wedge \ g[g_1^n][H_{a_1} \cdots H_{a_n}] \subset H_c \,.$$

Hence $(a_1^n, c) \in P_{(\mathcal{E},W)}([g, g_1^n])$. This completes the proof of the inclusion

$$P_{(\mathcal{E},W)}(g)[P_{(\mathcal{E},W)}(g_1) \cdots P_{(\mathcal{E},W)}(g_n)] \subset P_{(\mathcal{E},W)}([g, g_1^n]) \,.$$

To prove the converse inclusion, consider an arbitrary element (a_1^n, c) from $P_{(\mathcal{E},W)}([g, g_1^n])$. Then $a_1, \ldots, a_n \in G$ and $g[g_1^n][H_{a_1} \cdots H_{a_n}] \subset H_c$.

(1) If $H_{a_i} = \mathcal{E}\langle h_i \rangle$ and $h_i \in G$ for all $i = 1, \ldots, n$, then $g[g_1^n][h_1^n] \in H_c$, which implies $g[g_1[h_1^n] \cdots g_n[h_1^n]] \in H_c$. Obviously $g_i[h_1^n] \in G$ for all i. Moreover $H_{b_i} = \mathcal{E}\langle g_i[h_1^n] \rangle$ for $i = 1, \ldots, n$ gives $(b_1, \ldots, b_n) \in \mathfrak{A}$ and $g[H_{b_1} \cdots H_{b_n}] \subset H_c$. Clearly $(a_1, \ldots, a_n) \in \mathfrak{A}$ and $g_i[H_{a_1} \cdots H_{a_n}] \subset H_{b_i}$ for every $i = 1, \ldots, n$. Thus $(a_1^n, b_i) \in P_{(\mathcal{E},W)}(g_i)$, $i = 1, \ldots n$ and $(b_1^n, c) \in P_{(\mathcal{E},W)}(g)$. Hence $(a_1^n, c) \in P_{(\mathcal{E},W)}(g)[P_{(\mathcal{E},W)}(g_1) \cdots P_{(\mathcal{E},W)}(g_n)]$.

(2) If $H_{a_i} = \mathcal{E}\langle e_i \rangle$ for all $i = 1, \ldots, n$, then $g[g_1^n] \in H_c$. For $g_i \in H_{b_i}$, $i = 1, \ldots, n$ we have $g_i[e_1^n] \in H_{b_i}$ and $g_i[H_{a_1} \cdots H_{a_n}] \subset H_{b_i}$. But $(b_1, \ldots, b_n) \in \mathfrak{A}$ and $g[H_{b_1} \cdots H_{b_n}] \subset H_c$. Thus $(a_1^n, b_i) \in P_{(\mathcal{E},W)}(g_i)$ and $(b_1^n, c) \in P_{(\mathcal{E},W)}(g)$. Hence $(a_1^n, c) \in P_{(\mathcal{E},W)}(g)[P_{(\mathcal{E},W)}(g_1), \ldots, P_{(\mathcal{E},W)}(g_n)]$.

(3) If $H_{a_i} = \mathcal{E}\langle \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \rangle$ for all $i = 1, \ldots, n$ and some $y_1, \ldots, y_s \in G$, then $g[g_1^n][\mu_1^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), \ldots, \mu_n^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)] \in H_c$, which, by (5.2.8), is equivalent to $g[g_1^n] \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \in H_c$. This, by (5.2.6), implies $g[g_1 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \cdots g_n \overset{i_s}{\underset{i_1}{\oplus}} y_1^s] \in H_c$, where $g_i \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \in G$. But for $H_{b_i} = \mathcal{E}\langle g_i \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \rangle$, $i = 1, \ldots, n$, we have $(b_1, \ldots, b_n) \in \mathfrak{A}$ and $g[H_{b_1} \cdots H_{b_n}] \subset H_c$. Since, according to (5.2.8), we have $g_i \overset{i_s}{\underset{i_1}{\oplus}} y_1^s = g_i[\mu_1^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \cdots \mu_n^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)]$, we have $g_i[H_{a_1} \cdots H_{a_n}] \subset H_{b_i}$ for $i = 1, \ldots, n$. Hence $(a_1^n, b_i) \in$

$P_{(\mathcal{E},W)}(g_i)$, $i = 1,\ldots,n$. Therefore $(b_1^n, c) \in P_{(\mathcal{E},W)}(g)$. So, $(a_1^n, c) \in P_{(\mathcal{E},W)}(g)[P_{(\mathcal{E},W)}(g_1)\cdots P_{(\mathcal{E},W)}(g_n)]$.

This completes the proof of the inclusion

$$P_{(\mathcal{E},W)}(g[g_1^n]) \subset P_{(\mathcal{E},W)}(g)[P_{(\mathcal{E},W)}(g_1)\cdots P_{(\mathcal{E},W)}(g_n)]$$

and the proof of (5.2.12). □

The mapping $P_{(\mathcal{E},W)} : g \mapsto P_{(\mathcal{E},W)}(g)$, where $g \in G$, is by Proposition 5.2.10 a representation of a Menger $(2,n)$-semigroup $\mathcal{G}$ by n-place functions. This representation will be called *simplest*.

Let $(P_i)_{i\in I}$ be a family of representations of a Menger $(2,n)$-semigroup $\mathcal{G}$ by n-place functions defined on sets $(A_i)_{i\in I}$, respectively. We call *union* of this family the mapping $P : g \mapsto P(g)$, where $g \in G$, and $P(g)$ is an n-place function on $A = \bigcup\limits_{i\in I} A_i$ defined by

$$P(g) = \bigcup_{i\in I} P_i(g)\,.$$

If $A_i \cap A_j = \varnothing$ for all $i, j \in I$, $i \neq j$, then P is called the *sum* of $(P_i)_{i\in I}$ and is denoted by $P = \sum_{i\in I} P_i$. It is not difficult to see that the sum of representations is a representation, but the union of representations may not be a representation.

Theorem 5.2.11. *Any representation of a Menger $(2,n)$-semigroup $\mathcal{G}$ by n-place functions is a union of some family of simplest representations.*

Proof. Let P be a representation of a Menger $(2,n)$-semigroup $\mathcal{G}$ by n-place functions defined on A, and let $\alpha \notin A$ be some fixed element. For every $g \in G$ we define on $A^* = A \cup \{\alpha\}$ an n-place function $P^*(g)$ putting

$$P^*(g)(a_1^n) = \begin{cases} P(g)(a_1^n) & \text{if } (a_1,\ldots,a_n) \in \operatorname{pr}_1 P(g), \\ \alpha & \text{if } (a_1,\ldots,a_n) \notin \operatorname{pr}_1 P(g). \end{cases}$$

It is not difficult to see that P^* is a representation of $\mathcal{G}$ by n-place functions defined on A^*, and $P(g) \mapsto P^*(g)$, where $g \in G$, is an isomorphism of $(P(G), O, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ onto $(P^*(G), O, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$. Since $G \cup \{e_1,\ldots,e_n\}$ is the generating set of a unitary extension $\mathcal{G}^*$ with selectors $e_1,\ldots,e_n$, we obtain by putting $P^*(e_i) = I_i^n$, $i = 1,\ldots,n$, where I_i^n is the ith n-place projector of A^*, a unique extension of P^* from $\mathcal{G}$ to $\mathcal{G}^*$.

For any $a_1,\ldots,a_n \in A$ we define on G^* a relation $\Theta_{a_1^n}$ such that

$$\big(x, y\big) \in \Theta_{a_1^n} \longleftrightarrow P^*(x)(a_1^n) = P^*(y)(a_1^n).$$

It is easily to verify that it is a v-regular equivalence relation such that its abstract classes have the form $H_b^{a_1^n} = \{x \in G^* \mid (a_1^n, b) \in P^*(x)\}$. The pair $(\mathcal{E}_{a_1^n}, W_{a_1^n})$, where

$$\mathcal{E}_{a_1^n} = \Theta_{a_1^n} \cap \left(\Theta_{a_1^n}(G \cup \{e_1, \ldots, e_n\}) \times \Theta_{a_1^n}(G \cup \{e_1, \ldots, e_n\})\right),$$

$$W_{a_1^n} = \{x \in G^* \mid P^*(x)(a_1^n) = \alpha\},$$

is the determining pair of $\mathcal{G}$.

We prove that

$$P(g) = \bigcup_{a_1^n \in A^n} P_{a_1^n}(g)$$

for every $g \in G$, where $(P_{a_1^n})_{a_1^n \in A^n}$ is the family of the simplest representations $P_{a_1^n}$ induced by the determining pair $(\mathcal{E}_{a_1^n}, W_{a_1^n})$.

Let $(b_1^n, c) \in P(g)$. Then $g \in H_c^{b_1^n}$, $g[e_1^n] = g$ and $e_i \in H_{b_i}^{b_1^n}$ for all $i = 1, \ldots, n$. The v-regularity of $\Theta_{a_1^n}$ implies that $(b_1, \ldots, b_n) \in \mathfrak{A}$ and $g[H_{b_1}^{b_1^n}, \ldots, H_{b_n}^{b_1^n}] \subset H_c^{b_1^n}$. Hence

$$(b_1^n, c) \in P_{b_1^n}(g) \subset \bigcup_{a_1^n \in A^n} P_{a_1^n}(g).$$

So,

$$P(g) \subset \bigcup_{a_1^n \in A^n} P_{a_1^n}(g).$$

To prove the converse inclusion let $(b_1^n, c) \in P_{a_1^n}(g)$ for some $a_1^n \in A^n$. Then

$$(b_1, \ldots, b_n) \in \mathfrak{A} \wedge g[H_{b_1}^{a_1^n} \cdots H_{b_n}^{a_1^n}] \subset H_c^{a_1^n}.$$

(1) If $h_i \in H_{b_i}^{a_1^n} \cap G$, then $g[h_1^n] \in H_c^{a_1^n}$, and, consequently,

$$(a_1^n, c) \in P(g[h_1^n]) = P(g)[P(h_1) \cdots P(h_n)].$$

Thus, for some $d_1, \ldots, d_n \in A$ we have $(a_1^n, d_i) \in P(h_i)$, $i = 1, \ldots, n$ and $(d_1^n, c) \in P(g)$. But $(a_1^n, b_i) \in P(h_i)$ implies $b_i = d_i$. Hence $(b_1^n, c) \in P(g)$.

(2) If $e_i \in H_{b_i}^{a_1^n}$, then $g = g[e_1^n] \in H_c^{a_1^n}$, which gives $(a_1^n, c) \in P(g)$. But for $e_i \in H_{b_i}^{a_1^n}$, we have $(a_1^n, b_i) \in P(e_i) = P^*(e_i) = I_i^n$. Thus $b_i = I_i^n(a_1^n) = a_i$ for $i = 1, \ldots, n$. Therefore $(b_1^n, c) \in P(g)$.

(3) If $\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \in H_{b_i}^{a_1^n}$ for all $i = 1, \ldots, n$ and some $y_1, \ldots, y_s \in G$, then $g[\mu_1^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \cdots \mu_n^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)] \in H_c^{a_1^n}$, which proves $g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \in H_c^{a_1^n}$. Thus

$$(a_1^n, c) \in P(g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = P(g)[P(\mu_1^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)) \cdots P(\mu_n^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s))].$$

But there are $d_1,\dots,d_n \in A$ such that $(a_1^n, d_i) \in P(\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s))$, $i = 1,\dots,n$, $(d_1^n, c) \in P(g)$. This, together with $(a_1^n, b_i) \in P(\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s))$, $i = 1,\dots,n$, implies $d_i = b_i$ for all $i = 1,\dots,n$. Therefore $(b_1^n, c) \in P(g)$.

Summarizing, we see that in any case $P_{a_1^n}(g) \subset P(g)$. This proves inclusion $\bigcup\limits_{a_1^n\in A^n} P_{a_1^n}(g) \subset P(g)$ and completes the proof. □

Let $\mathcal{G}$ be a Menger $(2, n)$-semigroup, x – an individual variable. We denote by $T(G)$ the set of polynomials over $\mathcal{G}$ such that:

(a) $x \in T(G)$,

(b) if $t(x) \in T(G)$, $g, g_1,\dots,g_n \in G$, then $g[g_1^{i-1}\, t(x)\, g_{i+1}^n] \in T(G)$ for all $i = 1,\dots,n$,

(c) if $t(x) \in T(G)$, $g \in G$, then $g \underset{i}{\oplus} t(x) \in T(G)$ for all $i = 1,\dots,n$,

(d) $T(G)$ contains only elements determined in (a), (b) and (c).

For a nonempty subset H of a Menger $(2, n)$-semigroup $\mathcal{G}$ we define the relation $\mathcal{E}_H$ and the set W_H letting

$$(x, y) \in \mathcal{E}_H \longleftrightarrow \Big(\forall\, t \in T(G)\Big)\Big(t(x) \in H \longleftrightarrow t(y) \in H\Big), \tag{5.2.14}$$

$$x \in W_H \longleftrightarrow \Big(\forall\, t \in T(G)\Big)\Big(t(x) \notin H\Big). \tag{5.2.15}$$

Proposition 5.2.12. *$\mathcal{E}_H$ is a v-regular equivalence relation on $\mathcal{G}$ and W_H is an $\mathcal{E}_H$-class which is an l-ideal or empty set.*

Proof. The fact that $\mathcal{E}_H$ is an equivalence relation is obvious. We prove that $\mathcal{E}_H$ is v-regular.

Let $(x_i, y_i) \in \mathcal{E}_H$ for all $i = 1,\dots,n$ and some $x_i, y_i \in G$. Then, by (5.2.14), we have

$$\Big(\forall\, t \in T(G)\Big)\Big(t(x_i) \in H \longleftrightarrow t(y_i) \in H\Big), \tag{5.2.16}$$

for $i = 1,\dots,n$. For all $u \in G$, $\bar{w} \in G^n$ and polynomials of the form $t_1\,(u[\bar{w}|_i x])$, where $t_1(x) \in T(G)$, this gives us

$$\Big(t_1 \in T(G)\Big)\Big(t_1\big(u[\bar{w}|_i x]\big) \in H \longleftrightarrow t_1\big(u[\bar{w}|_i y]\big) \in H\Big).$$

Hence

$$\big(u[\bar{w}|_i x], u[\bar{w}|_i y]\big) \in \mathcal{E}_H \tag{5.2.17}$$

for all $i = 1,\dots,n$. In particular, for $(x_1, y_1) \in \mathcal{E}_H$ we get

$$\big(u[x_1\, x_2^n], u[y_1\, x_2^n]\big) \in \mathcal{E}_H\,. \tag{5.2.18}$$

Similarly, for $(x_2, y_2) \in \mathcal{E}_H$ we have

$$\big(u[y_1 x_2\, x_3^n], u[y_1^2\, x_3^n]\big) \in \mathcal{E}_H\,. \tag{5.2.19}$$

Since the relation $\mathcal{E}_H$ is transitive, from (5.2.18) and (5.2.19) we conclude $\big(u[x_1^2, x_3^n], u[y_1^2, x_3^n]\big) \in \mathcal{E}_H$. By repeating this procedure we obtain

$$\big(u[x_1^n], u[y_1^n]\big) \in \mathcal{E}_H.$$

Now let $(x, y) \in \mathcal{E}_H$, i.e., $\Big(\forall\, t \in T(G)\Big)\Big(t(x) \in H \longleftrightarrow t(y) \in H\Big)$. This, for $t(x) = t_1(u \underset{i}{\oplus} x) \in T(G)$, gives

$$\Big(\forall\ t_1 \in T(G)\Big)\Big(t_1(u \underset{i}{\oplus} x) \in H \longleftrightarrow t(u \underset{i}{\oplus} y) \in H\Big).$$

Hence $\big(u \underset{i}{\oplus} x, u \underset{i}{\oplus} y\big) \in \mathcal{E}_H$, which completes the proof of the v-regularity of $\mathcal{E}_H$.

Assume that $W_H \neq \varnothing$. For $x, y \in W_H$, we get $\Big(\forall\, t \in T(G)\Big)\Big(t(x) \notin H\Big)$ and $\Big(\forall\, t \in T(G)\Big)\Big(t(y) \notin H\Big)$. Thus

$$\Big(\forall\, t \in T(G)\Big)\Big(t(x) \notin H \wedge t(y) \notin H\Big),$$

and consequently,

$$\Big(\forall\, t \in T(G)\Big)\Big(t(x) \notin H \longleftrightarrow t(y) \notin H\Big).$$

Therefore

$$\Big(\forall\, t \in T(G)\Big)\Big(t(x) \in H \longleftrightarrow t(y) \in H\Big),$$

i.e., $(x, y) \in \mathcal{E}_H$. If $x \in W_H$ and $(x, y) \in \mathcal{E}_H$, then, as it is easy to see, $y \in W_H$. This proves that W_H is an $\mathcal{E}_H$-class.

Now, according to the definition of W_H, for every $x \in W_H$ we have

$$\Big(\forall\, t \in T(G)\Big)\Big(t(x) \notin H\Big).$$

Hence, $t_1\big(u[\bar{w}|_i x]\big) \notin H$, where $t(x) = t_1(u[\bar{w}|_i x])$, $u \in G$, $\bar{w} \in G^n$, $i = 1, \ldots, n$. So, $u[\bar{w}|_i x] \in W_H$.

Analogously one can prove that $u \underset{i}{\oplus} x \in W_H$ for all $i = 1, \ldots, n$. Thus, W_H is an l-ideal of $\mathcal{G}$. □

Let P be a fixed representation of a Menger $(2,n)$-semigroup $\mathcal{G}$ by n-place functions. Consider the relation ζ_P defined on G by

$$(g_1, g_2) \in \zeta_P \longleftrightarrow P(g_1) \subset P(g_2)\,. \tag{5.2.20}$$

It is clear that ζ_P is a quasi-order. If P is a faithful representation, then ζ_P is an order, i.e., an anti-symmetric quasi-order. If P is a sum of the family $(P_i)_{i\in I}$ of representations of $\mathcal{G}$, then

$$\zeta_P = \bigcap_{i\in I} \zeta_{P_i}. \tag{5.2.21}$$

In the case when $P = P_{(\mathcal{E},W)}$ is the simplest representation induced by a determining pair $(\mathcal{E},W)$ (see (5.2.10) and Proposition 5.2.10) the quasi-order defined by (5.2.20) will be denoted by $\zeta_{(\mathcal{E},W)}$.

Proposition 5.2.13. *Let $(\mathcal{E},W)$ be a determining pair of a Menger $(2,n)$-semigroup $\mathcal{G}$ satisfying all assumptions of Theorem* 5.2.3. *Then $(g_1,g_2)\in\zeta_{(\mathcal{E},W)}$ if and only if*

$$\Big(\forall\,(x_1,\ldots,x_n)\in B\Big)\Big(g_1[x_1^n]\notin W\longrightarrow \big(g_1[x_1^n],g_2[x_1^n]\big)\in\mathcal{E}\Big), \tag{5.2.22}$$

$$\Big(\forall\, y_1,\ldots,y_s\in G\Big)\Big(g_1\overset{i_s}{\underset{i_1}{\oplus}} y_1^s\notin W\longrightarrow \big(g_1\overset{i_s}{\underset{i_1}{\oplus}} y_1^s, g_2\overset{i_s}{\underset{i_1}{\oplus}} y_1^s\big)\in\mathcal{E}\Big) \tag{5.2.23}$$

for $i_1,\ldots,i_s\in\{1,\ldots,n\}$, $s\in\mathbb{N}$, $B=G^n\cup\{(e_1,\ldots,e_n)\}$.

Proof. If $(g_1,g_2)\in\zeta_{(\mathcal{E},W)}$, then $P_{(\mathcal{E},W)}(g_1)\subset P_{(\mathcal{E},W)}(g_2)$. Thus for all $(a_1,\ldots,a_n)\in\mathfrak{A}$ we have

$$(a_1,\ldots,a_n)\in \mathrm{pr}_1\, P_{(\mathcal{E},W)}(g_1)\longrightarrow P_{(\mathcal{E},W)}(g_1)(a_1^n)=P_{(\mathcal{E},W)}(g_2)(a_1^n),$$

which, by (5.2.10) and (5.2.11), is equivalent to

$$g_1[H_{a_1}\cdots H_{a_n}]\cap W=\varnothing\longrightarrow g_1[H_{a_1}\cdots H_{a_n}]\cup g_2[H_{a_1}\cdots H_{a_n}]\subset H_c \tag{5.2.24}$$

for some $c\in A$. The last implication applied to $(x_1,\ldots,x_n)\in B$ gives (5.2.22).

For $(x_1,\ldots,x_n)=(\mu_1^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s),\ldots,\mu_n^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s))$, where $y_1,\ldots,y_s\in G$ and $i_1,\ldots,i_s\in\{1,\ldots,n\}$, by applying (5.2.8) and (5.2.24) we obtain

$$g_1\overset{i_s}{\underset{i_1}{\oplus}} y_1^s\notin W\longrightarrow g_1\overset{i_s}{\underset{i_1}{\oplus}} y_1^s\in H_c\ \wedge\ g_2\overset{i_s}{\underset{i_1}{\oplus}} y_1^s\in H_c\,,$$

and consequently (5.2.23).

In a similar way one can prove the converse statement. □

We say that a Menger $(2,n)$-semigroup $(\Phi,O,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ of n-place functions is *fundamentally ordered* if there is a relation ζ_Φ on Φ such that

$$(f,g)\in\zeta_\Phi\longleftrightarrow f\subset g\,.$$

Theorem 5.2.14. *An algebraic system* $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \zeta)$ *with one* $(n+1)$*-ary operation* o*,* n *binary operations* $\underset{1}{\oplus}, \ldots, \underset{n}{\oplus}$ *and one relation* $\zeta \subset G \times G$*, is isomorphic to a fundamentally ordered Menger* $(2,n)$*-semigroup of* n*-place functions if and only if* $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ *is isomorphic to some Menger* $(2,n)$*-semigroup of* n*-place functions and* ζ *is a stable order such that*

$$\big(g_1, g_2\big),\ \big(g, t_1(g_1)\big),\ \big(g, t_2(g_2)\big) \in \zeta \longrightarrow \big(g, t_2(g_1)\big) \in \zeta \tag{5.2.25}$$

for all $g, g_1, g_2 \in G$ *and* $t_1, t_2 \in T(G)$*.*

Proof. Necessity. Let $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \zeta)$ be isomorphic to a fundamentally ordered Menger $(2,n)$-semigroup $(\mathbf{\Phi}, O, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \zeta_{\mathbf{\Phi}})$ of n-place functions. We need only to verify (5.2.25). The rest is a consequence of Theorem 5.2.3.

Assume that the elements $g, g_1, g_2 \in G$, $t_1(x), t_2(x) \in T(G)$ correspond to the functions $f, f_1, f_2 \in \mathbf{\Phi}$, $t_1(x), t_2(x) \in T(\mathbf{\Phi})$. If $\big(g_1, g_2\big), \big(g, t_1(g_1)\big), \big(g, t_2(g_2)\big) \in \zeta$, then $\big(f_1, f_2\big)$, $\big(f, t_1(f_1)\big)$, $\big(f, t_2(f_2)\big)$ are in $\zeta_{\mathbf{\Phi}}$. Hence $f_1 = f_2 \circ \triangle_{\mathrm{pr}_1 f_1}$, where $\triangle_H = \{(a,a) \mid a \in H\}$, because $f_1 \subset f_2$. By analogy, we obtain $f = t_1(f_1) \circ \triangle_{\mathrm{pr}_1 f}$ and $f = t_2(f_2) \circ \triangle_{\mathrm{pr}_1 f}$. Obviously $f \subset t_1(f_1)$ implies $\triangle_{\mathrm{pr}_1 f} \subset \triangle_{\mathrm{pr}_1 f_1}$. Thus

$$\begin{aligned} f &= f \circ \triangle_{\mathrm{pr}_1 f_1} = t_2(f_2) \circ \triangle_{\mathrm{pr}_1 f} \circ \triangle_{\mathrm{pr}_1 f_1} = t_2(f_2) \circ \triangle_{\mathrm{pr}_1 f_1} \circ \triangle_{\mathrm{pr}_1 f} \\ &= t_2(f_2 \circ \triangle_{\mathrm{pr}_1 f_1}) \circ \triangle_{\mathrm{pr}_1 f} = t_2(f_1) \circ \triangle_{\mathrm{pr}_1 f}. \end{aligned}$$

Hence $f \subset t_2(f_1)$, i.e., $\big(f, t_2(f_1)\big) \in \zeta_{\mathbf{\Phi}}$. This proves (5.2.25).

Sufficiency. Consider the family $(P_g)_{g \in G}$ of representations of a Menger $(2,n)$-semigroup $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$, where P_g is the simplest representation induced by the determining pair $(\mathcal{E}^*_{\zeta\langle g\rangle}, W_{\zeta\langle g\rangle})$, where $\mathcal{E}_{\zeta\langle g\rangle}$, $W_{\zeta\langle g\rangle}$ are defined by (5.2.14), (5.2.15), $\zeta\langle g\rangle = \{x \in G \mid \big(g, x\big) \in \zeta\}$, $e_1, \ldots, e_n$ — selectors of $(G^*, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ and $\mathcal{E}^*_{\zeta\langle g\rangle} = \mathcal{E}_{\zeta\langle g\rangle} \cup \{(e_1, e_1), \ldots, (e_n, e_n)\}$.

It is clear that

$$P = \sum_{g \in G} P_g \tag{5.2.26}$$

is a representation of $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ by n-place functions.

We must prove that $\zeta = \zeta_P$. Let $(g_1, g_2) \in \zeta_P$, where $g_1, g_2 \in G$. Then $(g_1, g_2) \in \bigcap\limits_{g \in G} \zeta_{P_g}$, by (5.2.26) and (5.2.21). This together with Proposition 5.2.13 gives us

$$\Big(\forall\ (x_1, \ldots, x_n) \in B\Big)\Big(g_1[x_1^n] \notin W_{\zeta\langle g\rangle} \longrightarrow \big(g_1[x_1^n], g_2[x_1^n]\big) \in \mathcal{E}_{\zeta\langle g\rangle}\Big) \tag{5.2.27}$$

for every $g \in G$. Replacing all x_i in (5.2.27) with e_i, $i = 1, \dots, n$, we obtain the implication

$$g_1 \notin W_{\zeta\langle g\rangle} \longrightarrow \big(g_1, g_2\big) \in \mathcal{E}_{\zeta\langle g\rangle}\,,$$

which for $g = g_1$ gives $\big(g_1, g_2\big) \in \mathcal{E}_{\zeta\langle g_1\rangle}$, because $g_1 \notin W_{\zeta\langle g_1\rangle}$. Therefore

$$\big(g_1, t(g_1)\big) \in \zeta \longleftrightarrow \big(g_1, t(g_2)\big) \in \zeta$$

for all $t \in T(G)$. In particular, for $t(x) = x$, we get

$$\big(g_1, g_1\big) \in \zeta \longleftrightarrow \big(g_1, g_2\big) \in \zeta.$$

Applying the reflexivity of ζ we obtain $\big(g_1, g_2\big) \in \zeta$. Hence $\zeta_P \subset \zeta$.

Conversely, let $(g_1, g_2) \in \zeta$ for some $g_1, g_2 \in G$. First, we prove (5.2.22). Consider $g_1[x_1^n] \notin W_{\zeta\langle g\rangle}$, where $g \in G$ and $(x_1, \dots, x_n) \in B$. Then there exists $t_1 \in T(G)$ such that $\big(g, t_1(g_1[x_1^n])\big) \in \zeta$. But $\big(g_1, g_2\big) \in \zeta$ and the stability of ζ imply $\big(t(g_1[x_1^n]), t(g_2[x_1^n])\big) \in \zeta$ for all $t \in T(G)$ and $x_1^n \in B$. Thus $\big(g, t(g_1[x_1^n])\big) \in \zeta$ implies $\big(g, t(g_2[x_1^n])\big) \in \zeta$. This, together with $\big(g_1, g_2\big) \in \zeta$, $\big(g, t_1(g_1[x_1^n])\big) \in \zeta$ and (5.2.25), gives $\big(g, t(g_1[x_1^n])\big) \in \zeta$. Therefore $\big(g_1[x_1^n], g_2[x_1^n]\big) \in \mathcal{E}_{\zeta\langle g\rangle}$, which proves (5.2.22).

In a similar way we can prove (5.2.23). This condition, by Proposition 5.2.13, shows that $(g_1, g_2) \in \zeta_{P_g}$ for every $g \in G$, i.e., $(g_1, g_2) \in \zeta_P$. Hence $\zeta \subset \zeta_P$, and consequently $\zeta = \zeta_P$.

Now if $P(g_1) = P(g_2)$ for some $g_1, g_2 \in G$, then $P(g_1) \subset P(g_2)$ and $P(g_2) \subset P(g_1)$. Thus $\big(g_1, g_2\big)$, $\big(g_2, g_1\big) \in \zeta$, which by the anti-symmetry of ζ, gives $g_1 = g_2$. Hence P is a faithful representation and $\mathcal{G}$ is isomorphic to a fundamentally ordered Menger $(2, n)$-semigroup $(\mathbf{\Phi}, O, \underset{1}{\oplus}, \dots, \underset{n}{\oplus}, \zeta_{\mathbf{\Phi}})$ of n-place functions. □

5.3 Projection relations on $(2, n)$-semigroups

Let $\mathbf{\Phi}$ be some set of n-place functions, i.e., $\mathbf{\Phi} \subset \mathcal{F}(A^n, A)$. Consider the following three binary relations on $\mathbf{\Phi}$:

$$\chi_{\mathbf{\Phi}} = \{(f, g) \in \mathbf{\Phi} \times \mathbf{\Phi} \,|\, \mathrm{pr}_1 f \subset \mathrm{pr}_1 g\},$$

$$\gamma_{\mathbf{\Phi}} = \{(f, g) \in \mathbf{\Phi} \times \mathbf{\Phi} \,|\, \mathrm{pr}_1 f \cap \mathrm{pr}_1 g \neq \varnothing\},$$

$$\pi_{\mathbf{\Phi}} = \{(f, g) \in \mathbf{\Phi} \times \mathbf{\Phi} \,|\, \mathrm{pr}_1 f = \mathrm{pr}_1 g\},$$

called respectively: *inclusion of domains*, *co-definability* and *equality of domains*. A Menger $(2, n)$-semigroup $(\mathbf{\Phi}, O, \underset{1}{\oplus}, \dots, \underset{n}{\oplus})$ with a relation $\chi_{\mathbf{\Phi}}$ is called a *projection quasi-ordered Menger $(2, n)$-semigroup* and is denoted by $(\mathbf{\Phi}, O, \underset{1}{\oplus}, \dots, \underset{n}{\oplus}, \chi_{\mathbf{\Phi}})$.

Consider also a representable (Menger) $(2, n)$-semigroup $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ (respectively, $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$) and its representation P by n-place functions. On the set G we define the following four binary relations:

$$\chi_P = \{(g_1, g_2) \mid \mathrm{pr}_1 P(g_1) \subset \mathrm{pr}_1 P(g_2)\},$$
$$\gamma_P = \{(g_1, g_2) \mid \mathrm{pr}_1 P(g_1) \cap \mathrm{pr}_1 P(g_2) \neq \varnothing\},$$
$$\pi_P = \{(g_1, g_2) \mid \mathrm{pr}_1 P(g_1) = \mathrm{pr}_1 P(g_2)\},$$
$$\varepsilon_P = \{(g_1, g_2) \mid P(g_1) = P(g_2)\}.$$

It is not difficult to see that χ_P is a quasi-order and π_P is an equivalence relation such that $\pi_P = \chi_P \cap \chi_P^{-1}$, where $\chi_P^{-1} = \{(b, a) \mid (a, b) \in \chi_P\}$. If P is a faithful representation, then ε_P coincides with $\triangle_G$. Of course, in the case when P is a representation by n-ary operations we have $\chi_P = G \times G$.

If P is the sum of the family $(P_i)_{i \in I}$ of representations by n-place functions, then, obviously, P is also a representation by n-place functions. Moreover, in this case

$$\chi_P = \bigcap_{i \in I} \chi_{P_i}, \quad \gamma_P = \bigcup_{i \in I} \gamma_{P_i}, \quad \pi_P = \bigcap_{i \in I} \pi_{P_i}, \quad \varepsilon_P = \bigcap_{i \in I} \varepsilon_{P_i}. \tag{5.3.1}$$

Let 0 be a zero of a $(2, n)$-semigroup $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ (respectively, Menger $(2, n)$-semigroup $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$), i.e., $0 \underset{i}{\oplus} g = g \underset{i}{\oplus} 0 = 0$ (respectively, $0 \underset{i}{\oplus} g = g \underset{i}{\oplus} 0 = 0$ and $0[g_1^n] = g[g_1^{i-1}\, 0\, g_{i+1}^n] = 0$) for all $i = 1, \ldots, n$ and $g, g_1, \ldots, g_n \in G$. We say that a binary relation $\rho \subset G \times G$ is 0-*reflexive*, if $(g, g) \in \rho$ for all $g \in G \setminus \{0\}$. A symmetric relation ρ which is reflexive if $0 \in \mathrm{pr}_1 \rho$, and 0-reflexive if $0 \notin \mathrm{pr}_1 \rho$, is called a 0-*quasi-equivalence*.

A binary relation ρ on a Menger $(2, n)$-semigroup $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ is called

- l-*regular*, if

$$(x, y) \in \rho \longrightarrow (x[z_1^n], y[z_1^n]) \in \rho \;\wedge\; (x \underset{i}{\oplus} z, y \underset{i}{\oplus} z) \in \rho$$

  for all $x, y, z, z_1, \ldots, z_n \in G$ and $i = 1 \ldots, n$,

- l-*cancellative*, if

$$(x[z_1^n], y[z_1^n]) \in \rho \longrightarrow (x, y) \in \rho,$$
$$(x \underset{i}{\oplus} z, y \underset{i}{\oplus} z) \in \rho \longrightarrow (x, y) \in \rho$$

  for all $x, y, z, z_1, \ldots, z_n \in G$ and $i = 1, \ldots, n$,

- *v-negative*, if

$$(x[y_1^n], y_i) \in \rho \;\wedge\; \left(x \overset{i_s}{\underset{i_1}{\oplus}} y_1^n,\, \mu_j(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)\right) \in \rho$$

for all $x, y_1, \ldots, y_k \in G$, where $k = \max\{n, s\}$, $i = 1, \ldots, n$ and $j \in \{i_1, \ldots, i_s\}$.

Now we prove the first result of this section.

Theorem 5.3.1. *Let χ be a binary relation on a Menger $(2,n)$-semigroup $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$. Then an algebraic system $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \chi)$ is isomorphic to some projection quasi-ordered Menger $(2,n)$-semigroup of n-place functions if and only if $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ is a representable Menger $(2,n)$-semigroup, and χ is an l-regular v-negative quasi-order.*

Proof. We only prove the sufficiency of the above conditions because the proof of their necessity is trivial. For this assume that the algebraic system $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \chi)$ satisfies all the conditions of the theorem and consider pairwise distinct elements $e_1, \ldots, e_n \notin G$.

Let $G^* = G \cup \{e_1, \ldots, e_n\}$. Denote by $\mathbf{A_0}$ the set of all n-tuples $(x_1, \ldots, x_n) \in (G^*)^n$ for which there exist $y_1, \ldots, y_s \in G$ and $i_1, \ldots, i_s \in \{1, \ldots, n\}$ such that $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$, $i = 1, \ldots, n$. Next we fix some element $a \in G$ and for every $g \in G$ define on $\mathbf{A}^* = \mathbf{A_0} \cup \{(e_1, \ldots, e_n)\}$ an n-place function $P_a(g)$ in the following way:

$$P_a(g)(x_1^n) = y \longleftrightarrow \begin{cases} a \sqsubset y = g[x_1^n] & \text{if} \quad (x_1, \ldots, x_n) \in G^n, \\ a \sqsubset y = g & \text{if} \quad (x_1, \ldots, x_n) = (e_1, \ldots, e_n), \\ a \sqsubset y = g \overset{i_s}{\underset{i_1}{\oplus}} z_1^s & \text{if} \quad x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} z_1^s),\ i = 1, \ldots, n, \\ & \quad\;\; \text{for some} \;\; z_1, \ldots, z_s \in G, \end{cases}$$

where $a \sqsubset y$ means $(a, y) \in \chi$.

Let us prove that $P_a : g \mapsto P_a(g)$ is a homomorphism. To prove this fact, assume that $g, g_1, \ldots, g_n \in G$ and $P_a(g[g_1^n])(x_1^n) = y$.

(1) If $(x_1, \ldots, x_n) \in G^n$, then

$$a \sqsubset y = g[g_1^n][x_1^n] = g[g_1[x_1^n] \cdots g_n[x_1^n]] \sqsubset g_i[x_1^n]$$

for every $i = 1, \ldots, n$. Therefore $P_a(g_i)(x_1^n) = g_i[x_1^n]$, $i = 1, \ldots, n$, and $P_a(g)(g_1[x_1^n], \ldots, g_n[x_1^n]) = y$, whence

$$P_a(g)[P_a(g_1) \cdots P_a(g_n)](x_1^n) = y.$$

(2) If $(x_1,\ldots,x_n)=(e_1,\ldots,e_n)$, then $a\sqsubset y=g[g_1^n]\sqsubset g_i$, $i=1,\ldots,n$. Thus $P_a(g_i)(e_1^n)=g_i$, $i=1,\ldots,n$, $P_a(g)(g_1^n)=y$, whence

$$P_a(g)[P_a(g_1)\cdots P_a(g_n)](e_1^n)=y.$$

(3) Let $x_i=\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} z_1^s)$, $i=1,\ldots,n$, where $z_1,\ldots,z_s\in G$. Then

$$a\sqsubset g=g[g_1^n]\overset{i_s}{\underset{i_1}{\oplus}} z_1^s=g[\,(g_1\overset{i_s}{\underset{i_1}{\oplus}} z_1^s)\cdots(g_n\overset{i_s}{\underset{i_1}{\oplus}} z_1^s)\,]\sqsubset g_i\overset{i_s}{\underset{i_1}{\oplus}} z_1^s.$$

From this, we conclude that $P_a(g_i)(x_1^n)=g_i\overset{i_s}{\underset{i_1}{\oplus}} z_1^s$ for every $i=1,\ldots,n$, and $P_a(g)\Big((g_1\overset{i_s}{\underset{i_1}{\oplus}} z_1^s),\ldots,(g_n\overset{i_s}{\underset{i_1}{\oplus}} z_1^s)\Big)=y$. Thus

$$P_a(g)[P_a(g_1)\cdots P_a(g_n)](x_1^n)=y.$$

So, in any case we have

$$P_a(g[g_1^n])\subset P_a(g)[P_a(g_1)\cdots P_a(g_n)].$$

To prove the converse inclusion, let $P_a(g)[P_a(g_1)\cdots P_a(g_n)](x_1^n)=y$. Then there are $z_1,\ldots,z_n\in G$ for which $P_a(g_i)(x_1^n)=z_i$, $i=1,\ldots,n$, and $P_a(g)(z_1^n)=y$. From this, according to the definition of $P_a(g)$, we obtain

$$a\sqsubset y=g[z_1^n]. \tag{5.3.2}$$

(1) If $(x_1,\ldots,x_n)\in G^n$, then $a\sqsubset z_i=g_1[x_1^n]$, $i=1,\ldots,n$. This, together with (5.3.2), gives us

$$a\sqsubset y=g[g_1[x_1^n]\cdots g_n[x_1^n]]=g[g_1^n][x_1^n].$$

Thus $P_a(g[g_1^n])(x_1^n)=y$.

(2) If $(x_1,\ldots,x_n)=(e_1,\ldots,e_n)$, then $a\sqsubset z_i=g_i$, $i=1,\ldots,n$, whence, according to (5.3.2), we have $a\sqsubset y=g[g_1^n]$, i.e., $P_a(g[g_1^n])(e_1^n)=y$.

(3) Now let $x_i=\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} u_1^s)$, $i=1,\ldots,n$, where $u_1,\ldots,u_s\in G$. From $P_a(g_i)(x_1^n)=z_i$, $i=1,\ldots,n$, it follows that $a\sqsubset z_i=g_i\overset{i_s}{\underset{i_1}{\oplus}} u_1^s$, $i=1,\ldots,n$, whence, applying (5.3.2), we obtain

$$a\sqsubset y=g[(g_1\overset{i_s}{\underset{i_1}{\oplus}} u_1^s)\cdots(g_n\overset{i_s}{\underset{i_1}{\oplus}} u_1^s)]=g[g_1^n]\overset{i_s}{\underset{i_1}{\oplus}} u_1^s.$$

Thus $P_a(g[g_1^n])(x_1^n)=y$.

So, in all cases, we have $P_a(g[g_1^n])(x_1^n) = y$. This completes the proof of the inclusion

$$P_a(g)[P_a(g_1) \cdots P_a(g_n)] \subset P_a(g[g_1^n]),$$

and consequently, the proof of the identity

$$P_a(g[g_1^n]) = P_a(g)[P_a(g_1) \cdots P_a(g_n)]. \tag{5.3.3}$$

Now let $g_1, g_2 \in G$ and $P_a(g_1 \underset{i}{\oplus} g_2)(x_1^n) = y$.

(1) If $(x_1, \dots, x_n) \in G^n$, then, in view of the v-negativity of the relation χ, we have

$$a \sqsubset y = (g_1 \underset{i}{\oplus} g_2)[x_1^n] = g_1[x_1^{i-1}\, g_2[x_1^n]\, x_{i+1}^n] \sqsubset g_2[x_1^n] \in G.$$

So, $P_a(g_2)(x_1^n) = g_2[x_1^n]$ and $P_a(g_1)(x_1^{i-1} g_2[x_1^n]\, x_{i+1}^n) = y$. This implies $P_a(g_1) \underset{i}{\oplus} P_a(g_2)(x_1^n) = y$.

(2) If $(x_1, \dots, x_n) = (e_1, \dots, e_n)$, then $a \sqsubset y = g_1 \underset{i}{\oplus} g_2 \sqsubset \mu_i(\underset{i}{\oplus} g_2) = g_2$. Consequently $P_a(g_1)(e_1^n) = g_2$. But $\mu_j(\underset{i}{\oplus} g_2) = e_j$ for $j \neq i,\ j = 1, \dots, n$, thus $P_a(g_1)(e_1^{i-1} g_2 e_{i+1}^n) = y$, whence $P_a(g_1) \underset{i}{\oplus} P_a(g_2)(e_1^n) = y$.

(3) If $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} z_1^s)$ for every $i = 1, \dots, n$ and some $z_1, \dots, z_s \in G$, then

$$a \sqsubset y = (g_1 \underset{i}{\oplus} g_2) \overset{i_s}{\underset{i_1}{\oplus}} z_1^s \sqsubset \mu_i(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} z_1^s) = g_2 \overset{i_s}{\underset{i_1}{\oplus}} z_1^s.$$

This implies $P_a(g_2)(x_1^n) = g_2 \overset{i_s}{\underset{i_1}{\oplus}} z_1^s$. Since $\mu_j^*(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} z_1^s) = \mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}} z_1^s) = x_j$ for $j \neq i = 1, \dots, n$, we have also $P_a(g_1)(x_1^{i-1}\, g_2 \overset{i_s}{\underset{i_1}{\oplus}} z_1^s x_{i+1}^n) = y$. Therefore $P_a(g_1) \underset{i}{\oplus} P_a(g_2)(x_1^n) = y$.

So, we have proved the inclusion

$$P_a(g_1 \underset{i}{\oplus} g_2) \subset P_a(g_1) \underset{i}{\oplus} P_a(g_2).$$

To prove the converse inclusion, let $P_a(g_1) \underset{i}{\oplus} P_a(g_2)(x_1^n) = y$. In this case, $P_a(g_2)(x_1^n) = z$ and $P_a(g_1)(x_1^{i-1} z x_{i+1}^n) = y$ for some $z \in G$.

(1) If $(x_1, \dots, x_n) \in G^n$, then $(x_1^{i-1}\, z\, x_{i+1}^n) \in G^n$, i.e.,

$$a \sqsubset z = g_2[x_1^n] \quad \text{and} \quad a \sqsubset y = g_1[x_1^{i-1}\, z\, x_{i+1}^n].$$

Hence $a \sqsubset y = g_1[x_1^{i-1}\, g_2[x_1^n]\, x_{i+1}^n] = (g_1 \underset{i}{\oplus} g_2)[x_1^n]$. Thus, we have the equality $P_a(g_1 \underset{i}{\oplus} g_2)(x_1^n) = y$.

(2) If $(x_1,\ldots,x_n)=(e_1,\ldots,e_n)$, then $a\sqsubset z=g_2$. Since $\mu_i^*(\underset{i}{\oplus}g_2)=g_2$, $\mu_j^*(\underset{i}{\oplus}g_2)=e_j$ for $j\neq i=1,\ldots,n$, and $P_a(g_1)(e_1^{i-1}\,g_2\,e_{i+1}^n)=y$, from the above we get $a\sqsubset y=g_1\underset{i}{\oplus}g_2$, which shows us that $P_a(g_1\underset{i}{\oplus}g_2)(e_1^n)=y$.

(3) Now let $x_i=\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}}u_1^s)$ for all $i=1,\ldots,n$ and some $u_1,\ldots,u_s\in G$. Then $a\sqsubset z=g_2\overset{i_s}{\underset{i_1}{\oplus}}u_1^s$. Since $\mu_i^*(\underset{i}{\oplus}g_2\overset{i_s}{\underset{i_1}{\oplus}}u_1^s)=g_2\overset{i_s}{\underset{i_1}{\oplus}}u_1^s$ and $\mu_j^*(\underset{i}{\oplus}g_2\overset{i_s}{\underset{i_1}{\oplus}}u_1^s)$ $=\mu_j^*(\overset{i_s}{\underset{i_1}{\oplus}}u_1^s)=x_j$ for $j\neq i$, and $P_a(g_1)(x_1^{i-1}\,g_2\overset{i_s}{\underset{i_1}{\oplus}}u_1^s\,x_{i+1}^n)=y$, from the above we have $a\sqsubset y=(g_1\underset{i}{\oplus}g_2)\overset{i_s}{\underset{i_1}{\oplus}}u_1^s$. This proves $P_a(g_1\underset{i}{\oplus}g_2)(x_1^n)=y$.

So, in any case we have $P_a(g_1\underset{i}{\oplus}g_2)(x_1^n)=y$, i.e.,

$$P_a(g_1)\underset{i}{\oplus}P_a(g_2)\subset P_a(g_1\underset{i}{\oplus}g_2),$$

which completes the proof of the identity

$$P_a(g_1\underset{i}{\oplus}g_2)=P_a(g_1)\underset{i}{\oplus}P_a(g_2). \tag{5.3.4}$$

From (5.3.3) and (5.3.4), it follows that P_a is a representation of a Menger $(2,n)$-semigroup $(G,o,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ by partial n-place functions. According to Theorem 5.2.5, $(G,o,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ has a representation by n-ary operations too.

Let P_0 be the sum of the family of representations P_a, $a\in G$. It is clear that P_0 is a representation of this Menger $(2,n)$-semigroup by partial n-place functions. Let us prove that $\chi=\chi_{P_0}$.

Indeed, if $(g_1,g_2)\in\chi_{P_0}$, then $(g_1,g_2)\in\bigcap\limits_{a\in G}\chi_{P_a}$, according to (5.3.1), i.e.,

$$(\forall a\in G)\Big(\mathrm{pr}_1\,P_a(g_1)\subset\mathrm{pr}_1\,P_a(g_2)\Big),$$

which means that for every $(x_1,\ldots,x_n)\in\mathbf{A}^*$

$$(\forall a\in G)\Big((x_1,\ldots,x_n)\in\mathrm{pr}_1\,P_a(g_1)\longrightarrow(x_1,\ldots,x_n)\in\mathrm{pr}_1\,P_a(g_2)\Big).$$

This, in particular, for $(x_1,\ldots,x_n)=(e_1,\ldots,e_n)$ gives

$$(\forall a\in G)\Big((e_1,\ldots,e_n)\in\mathrm{pr}_1\,P_a(g_1)\longrightarrow(e_1,\ldots,e_n)\in\mathrm{pr}_1\,P_a(g_2)\Big),$$

i.e.,

$$(\forall a\in G)\Big((\exists y\in G)\;P_a(g_1)(e_1^n)=y\longrightarrow(\exists z\in G)\;P_a(g_2)(e_1^n)=z\Big),$$

which is equivalent to

$$(\forall a \in G)(\forall y \in G)\Big(a \sqsubset y = g_1 \longrightarrow (\exists z \in G)\, a \sqsubset z = g_2\Big).$$

Putting in this condition $a = y = g_1$ and using the reflexivity of the relation χ we see that there exists $z \in G$ such that $g_1 \sqsubset z = g_2$, which proves $(g_1, g_2) \in \chi$. So, $\chi_{P_0} \subset \chi$.

To prove the converse inclusion let $a \in G$, $(x_1, \ldots, x_n) \in \mathrm{pr}_1\, P_a(g_1)$ and $(g_1, g_2) \in \chi$.

(1) If $(x_1, \ldots, x_n) \in G^n$, then for some $y \in G$ we have $P_a(g_1)(x_1^n) = y$, i.e., $a \sqsubset y = g_1[x_1^n]$. Since from $g_1 \sqsubset g_2$, by the l-regularity of χ, we get $g_1[x_1^n] \sqsubset g_2[x_1^n]$, the above condition implies $a \sqsubset g_2[x_1^n]$. So, there exists an element $z = g_2[x_1^n]$ such that $a \sqsubset z = g_2[x_1^n]$, i.e., $P_a(g_2)(x_1^n) = z$, whence $(x_1, \ldots, x_n) \in \mathrm{pr}_1\, P_a(g_2)$.

(2) If $(x_1, \ldots, x_n) = (e_1, \ldots, e_n)$, then there exists $y \in G$ such that $a \sqsubset y = g_1$, which, in view of $g_1 \sqsubset g_2$, implies $a \sqsubset g_2$. Thus, in the consequence, we obtain $(e_1, \ldots, e_n) \in \mathrm{pr}_1\, P_a(g_2)$.

(3) If $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} u_1^s)$, $i = 1, \ldots, n$, where $u_1, \ldots, u_s \in G$, then for some $y \in G$ we have $a \sqsubset y = g_1 \overset{i_s}{\underset{i_1}{\oplus}} u_1^s$. Applying the l-regularity of χ to $g_1 \sqsubset g_2$ we get $g_1 \overset{i_s}{\underset{i_1}{\oplus}} u_1^s \sqsubset g_2 \overset{i_s}{\underset{i_1}{\oplus}} u_1^s$. So, $a \sqsubset g_2 \overset{i_s}{\underset{i_1}{\oplus}} u_1^s$, i.e., there exists $z = g_2 \overset{i_s}{\underset{i_1}{\oplus}} u_1^s$ such that $a \sqsubset z$. Therefore $P_a(g_2)(x_1^n) = z$, whence $(x_1, \ldots, x_n) \in \mathrm{pr}_1\, P_a(g_2)$.

Summarizing we see that in any case $(x_1, \ldots, x_n) \in \mathrm{pr}_1\, P_a(g_1)$ and $(g_1, g_2) \in \chi$ imply $(x_1, \ldots, x_n) \in \mathrm{pr}_1\, P_a(g_2)$. Thus $\mathrm{pr}_1\, P_a(g_1) \subset \mathrm{pr}_1\, P_a(g_2)$ for every $a \in G$, whence $(g_1, g_2) \in \chi_{P_0}$. This completes the proof of the inclusion $\chi \subset \chi_{P_0}$, and consequently, the proof of the equality $\chi = \chi_{P_0}$.

Since a Menger $(2, n)$-semigroup $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ satisfies the condition (5.1.19), it follows from Theorems 5.2.3 and 5.2.5, that there exists a faithful representation Λ of $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ by n-ary operations. Obviously $\chi_\Lambda = G \times G$ and $\varepsilon_\Lambda = \triangle_G$.

Let P be a representation of a given Menger $(2, n)$-semigroup such that $P = \Lambda + P_0$. Then

$$\chi_P = \chi_\Lambda \cap \chi_{P_0} = G \times G \cap \chi = \chi,$$

$$\varepsilon_P = \varepsilon_\Lambda \cap \varepsilon_{P_0} = \triangle_G \cap \varepsilon_{P_0} = \triangle_G.$$

So, P is an isomorphism of the system $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \chi)$ onto some projection quasi-ordered $(2, n)$-semigroup of n-place functions. The theorem is proved. □

Let $\mathcal{G} = (G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ be a representable Menger $(2,n)$-semigroup, χ, γ, π – binary relations on G. We say that the triplet (χ, γ, π) is (*faithful*) *projection representable for* $\mathcal{G}$, if there exists a (faithful) representation P of $\mathcal{G}$ by n-place functions such that $\chi = \chi_P$, $\gamma = \gamma_P$ and $\pi = \pi_P$. Similarly we define projection representable pairs and separate relations.

From this point on, instead of $(g_1, g_2) \in \chi$, $(g_1, g_2) \in \gamma$ and $(g_1, g_2) \in \pi$ we will write $g_1 \sqsubset g_2$, $g_1 \top g_2$ and $g_1 \equiv g_2$, respectively.

Theorem 5.3.2. *A triplet (χ, γ, π) of binary relations on G is projection representable for a representable Menger $(2,n)$-semigroup $\mathcal{G}$ if and only if the following conditions are satisfied:*
(a) *χ is an l-regular and v-negative quasi-order,*
(b) *γ is an l-cancellative 0-quasi-equivalence,*
(c) *$\pi = \chi \cap \chi^{-1}$ and*

$$h_1 \top h_2 \,\wedge\, h_1 \sqsubset g_1 \,\wedge\, h_2 \sqsubset g_2 \longrightarrow g_1 \top g_2 \tag{5.3.5}$$

for all $h_1, h_2, g_1, g_2 \in G$.

Proof. Necessity. Let $(\Phi, O, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ be a Menger $(2,n)$-semigroup of n-place functions on the set A. Let us show that the triplet $(\chi_\Phi, \gamma_\Phi, \pi_\Phi)$ satisfies all the conditions of the theorem.

At first, we prove condition (a). The relation χ_Φ is obviously a quasi-order. Let $f, g, h_1, \ldots, h_n \in \Phi$ and $(f, g) \in \chi_\Phi$, i.e., $\mathrm{pr}_1 f \subset \mathrm{pr}_1 g$. Suppose that $\bar{a} \in \mathrm{pr}_1 f[h_1^n]$ for some $\bar{a} \in A^n$. Then $\{f[h_1^n](\bar{a})\} \neq \varnothing$, i.e., $\{f(h_1(\bar{a}), \ldots, h_n(\bar{a}))\} \neq \varnothing$. Thus $(h_1(\bar{a}), \ldots, h_n(\bar{a})) \in \mathrm{pr}_1 f$ and, consequently, $(h_1(\bar{a}), \ldots, h_n(\bar{a})) \in \mathrm{pr}_1 g$. Therefore $\{g(h_1(\bar{a}), \ldots, h_n(\bar{a}))\} \neq \varnothing$, whence $\{g[h_1^n](\bar{a})\} \neq \varnothing$, i.e., $\bar{a} \in \mathrm{pr}_1 g[h_1^n]$. So, $\mathrm{pr}_1 f[h_1^n] \subset \mathrm{pr}_1 g[h_1^n]$, which implies $(f[h_1^n], g[h_1^n]) \in \chi_\Phi$. Similarly we can prove that for all $f, g, h \in \Phi$ and $i = 1, \ldots, n$, from $(f, g) \in \chi_\Phi$, it follows that $(f \underset{i}{\oplus} h\,, g \underset{i}{\oplus} h) \in \chi_\Phi$. This means that the relation χ_Φ is l-regular. The proof of the v-negativity is similar.

To prove (b), let Θ be a zero of a Menger $(2,n)$-semigroup $(\Phi, O, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$. If $\Theta \neq \varnothing$, then $\mathrm{pr}_1 \Theta \neq \varnothing$, whence $(\Theta, \Theta) \in \gamma_\Phi$. Thus $\Theta \in \mathrm{pr}_1 \gamma_\Phi$. So, in this case γ_Φ is reflexive. For $\Theta = \varnothing$ we have $\mathrm{pr}_1 \Theta = \varnothing$. Therefore $\Theta \notin \mathrm{pr}_1 \gamma_\Phi$, i.e., $(f, f) \in \gamma_\Phi$ for every $f \neq \Theta$. Hence γ_Φ is Θ-reflexive. Since γ_Φ is symmetric, the above means that γ_Φ is a Θ-quasi-equivalence.

Suppose now that $(f[h_1^n], g[h_1^n]) \in \gamma_\Phi$ for some $f, g \in \Phi$, $h_1, \ldots, h_n \in \Phi$. Then $\mathrm{pr}_1 f[h_1^n] \cap \mathrm{pr}_1 g[h_1^n] \neq \varnothing$, i.e., there exists $\bar{a} \in A^n$ such that $\bar{a} \in \mathrm{pr}_1 f[h_1^n]$ and $\bar{a} \in \mathrm{pr}_1 g[h_1^n]$. Therefore $\{f[h_1^n](\bar{a})\} \neq \varnothing$ and $\{g[h_1^n](\bar{a})\} \neq \varnothing$. Thus $\{f(h_1(\bar{a}), \ldots, h_n(\bar{a}))\} \neq \varnothing$ and $\{g(h_1(\bar{a}), \ldots, h_n(\bar{a}))\} \neq \varnothing$, which shows that

$(h_1(\bar{a}),\ldots,h_n(\bar{a}))\in \mathrm{pr}_1\, f\cap \mathrm{pr}_1\, g$. So, $(f,g)\in\gamma_\Phi$. Similarly, for $f,g,h\in\Phi$, $i=1,\ldots,n$, from $(f\underset{i}{\oplus} h, g\underset{i}{\oplus} h)\in\gamma_\Phi$, it follows that $(f,g)\in\gamma_\Phi$. So, γ_Φ is l-cancellative.

Since in (c) the first condition is obvious, we prove (5.3.5) only. In order to do this let $(h_1,h_2)\in\gamma_\Phi$, $(h_1,g_1)\in\chi_\Phi$ and $(h_2,g_2)\in\chi_\Phi$ for some $h_1,h_2,g_1,g_2\in\Phi$. Then $\mathrm{pr}_1\, h_1\cap \mathrm{pr}_1\, h_2\neq\varnothing$, $\mathrm{pr}_1\, h_1\subset \mathrm{pr}_1\, g_1$ and $\mathrm{pr}_1\, h_2\subset \mathrm{pr}_1\, g_2$, whence $\varnothing\neq \mathrm{pr}_1\, h_1\cap \mathrm{pr}_1\, h_2\subset \mathrm{pr}_1\, g_1\cap \mathrm{pr}_1\, g_2$. Thus $\mathrm{pr}_1\, g_1\cap\mathrm{pr}_1\, g_2\neq\varnothing$, i.e., $(g_1,g_2)\in\gamma_\Phi$, which proves (5.3.5) and completes the proof of the necessity of the conditions formulated in the theorem. □

To prove the sufficiency of these conditions we must introduce some additional constructions. Consider the triplet (χ,γ,π) of binary relations on a representable Menger $(2,n)$-semigroup $\mathcal{G}$ satisfying all the conditions of the theorem. Let $e_1,\ldots,e_n$ be pairwise distinct elements not belonging to G. For all $x_1,\ldots,x_s\in G$, $i=1,\ldots,n$, and operations $\underset{i_1}{\oplus},\ldots,\underset{i_s}{\oplus}$ defined on G, we denote by $\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} x_1^s)$ an element of $G^*=G\cup\{e_1,\ldots,e_n\}$ such that

$$\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} x_1^s)=\begin{cases}\mu_i(\overset{i_s}{\underset{i_1}{\oplus}} x_1^s) & \text{if } \ i\in\{i_1,\ldots,i_s\},\\ e_i & \text{if } \ i\notin\{i_1,\ldots,i_s\}.\end{cases}$$

Consider the set $\mathfrak{A}^*=G^n\cup\mathfrak{A}_0\cup\{(e_1,\ldots,e_n)\}$, where $\mathfrak{A}_0$ is the collection of all n-tuples $(x_1,\ldots,x_n)\in(G^*)^n$ for which there exists $y_1,\ldots,y_s\in G$ and $i_1,\ldots,i_n\in\{1,\ldots,n\}$ such that $x_i=\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$. Let $(h_1,h_2)\in G^2$ be fixed. For each $g\in G$ we define a partial n-place function $P_{(h_1,h_2)}(g):\mathfrak{A}^*\to G$ such that $(x_1,\ldots,x_n)\in \mathrm{pr}_1\, P_{(h_1,h_2)}(g)$ only in the case when

$$\begin{cases} h_1\sqsubset g[x_1^n]\,\vee\, h_2\sqsubset g[x_1^n] & \text{if } \ (x_1,\ldots,x_n)\in G^n,\\ h_1\sqsubset g\,\vee\, h_2\sqsubset g & \text{if } \ (x_1,\ldots,x_n)=(e_1,\ldots,e_n),\\ h_1\sqsubset g\overset{i_s}{\underset{i_1}{\oplus}} y_1^s\,\vee\, h_2\sqsubset g\overset{i_s}{\underset{i_1}{\oplus}} y_1^s & \text{if } \ x_i=\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s),\\ & i=1,\ldots,n, \ \text{for some}\\ & y_1,\ldots,y_s\in G \ \text{and}\\ & i_1,\ldots,i_s\in\{1,\ldots,n\}.\end{cases}$$

For $(x_1,\ldots,x_n)\in \mathrm{pr}_1\, P_{(h_1,h_2)}(g)$ we put

$$P_{(h_1,h_2)}(g)(x_1^n)=\begin{cases} g[x_1^n] & \text{if } (x_1,\ldots,x_n)\in G^n,\\ g & \text{if } (x_1,\ldots,x_n)=(e_1,\ldots,e_n),\\ g\overset{i_s}{\underset{i_1}{\oplus}} y_1^s & \text{if } x_i=\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s),\\ & i=1,\ldots,n, \text{ for some } y_1,\ldots,y_s\in G \text{ and}\\ & i_1,\ldots,i_s\in\{1,\ldots,n\}.\end{cases} \tag{5.3.6}$$

Let us show that $P_{(h_1,h_2)}$ is a representation of $\mathcal{G}$ by n-place functions.

Proposition 5.3.3. *The function $P_{(h_1,h_2)}(g)$ is single-valued.*

Proof. Let $(x_1,\ldots,x_n)\in \mathrm{pr}_1\, P_{(h_1,h_2)}(g)$, where $g,h_1,h_2\in G$ are fixed. Since for $(x_1,\ldots,x_n)\in G^n$ and $(x_1,\ldots,x_n)=(e_1,\ldots,e_n)$ the value of $P_{(h_1,h_2)}(g)(x_1^n)$ is uniquely determined, we verify only the case when $x_i=\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$, $i=1,\ldots,n$, for some $y_1,\ldots,y_s\in G$. If for some $z_1,\ldots,z_k\in G$ and $j_1,\ldots,j_k\in\{1,\ldots,n\}$ we also have $x_i=\mu_i^*(\overset{j_k}{\underset{j_1}{\oplus}} z_1^k)$, $i=1,\ldots,n$, then $\mu_i(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)=\mu_i(\overset{j_k}{\underset{j_1}{\oplus}} z_1^k)$ for every $i=1,\ldots,n$, which, according to (5.1.19), implies $g\overset{i_s}{\underset{i_1}{\oplus}} y_1^s=g\overset{j_k}{\underset{j_1}{\oplus}} z_1^k$. This means that also in this case $P_{(h_1,h_2)}(g)(x_1^n)$ is uniquely determined. Thus, the function $P_{(h_1,h_2)}(g)$ is single-valued. □

Proposition 5.3.4. *For all $g,g_1,\ldots,g_n,h_1,h_2\in G$ we have*

$$P_{(h_1,h_2)}(g[g_1^n])=P_{(h_1,h_2)}(g)[P_{(h_1,h_2)}(g_1)\cdots P_{(h_1,h_2)}(g_n)].$$

Proof. Let $g,g_1,\ldots,g_n\in G$ and $(x_1,\ldots,x_n)\in \mathrm{pr}_1\, P_{(h_1,h_2)}(g[g_1^n])$. If all $x_1,\ldots,x_n$ are in G, then

$$h_1\sqsubset g[g_1^n][x_1^n]\ \vee\ h_2\sqsubset g[g_1^n][x_1^n],$$

whence we obtain

$$h_1\sqsubset g[g_1[x_1^n]\cdots g_n[x_1^n]]\ \vee\ h_2\sqsubset g[g_1[x_1^n]\cdots g_n[x_1^n]]. \tag{5.3.7}$$

This together with the v-negativity of χ implies

$$h_1\sqsubset g_i[x_1^n]\ \vee\ h_2\sqsubset g_i[x_1^n],\ i=1,\ldots,n. \tag{5.3.8}$$

From (5.3.7), it follows that $(g_1[x_1^n],\ldots,g_n[x_1^n])\in\mathrm{pr}_1 P_{(h_1,h_2)}(g)$ and from (5.3.8) that $(x_1,\ldots,x_n)\in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_i)$, $i=1,\ldots,n$. So, if $(x_1,\ldots,x_n)\in G^n$, then

$$(x_1,\ldots,x_n)\in \mathrm{pr}_1\, P_{(h_1,h_2)}(g[g_1^n])\longleftrightarrow\begin{cases}(g_1[x_1^n],\ldots,g_n[x_1^n])\in \mathrm{pr}_1 P_{(h_1,h_2)}(g),\\ \bigwedge\limits_{i=1}^{n}(x_1,\ldots,x_n)\in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_i).\end{cases} \tag{5.3.9}$$

Similarly we can verify that

$$(e_1,\ldots,e_n)\in \mathrm{pr}_1\,P_{(h_1,h_2)}(g[g_1^n]) \longleftrightarrow \begin{cases}(g_1,\ldots,g_n)\in \mathrm{pr}_1\,P_{(h_1,h_2)}(g),\\ \bigwedge\limits_{i=1}^{n}(e_1,\ldots,e_n)\in \mathrm{pr}_1\,P_{(h_1,h_2)}(g_i).\end{cases} \tag{5.3.10}$$

Now let $x_i=\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)$, $i=1,\ldots,n$, for some $i_1,\ldots,i_s\in\{1,\ldots,n\}$ and $y_1,\ldots,y_s\in G$. Then $(x_1,\ldots,x_n)\in \mathrm{pr}_1\,P_{(h_1,h_2)}(g[g_1^n])$ implies

$$h_1\sqsubset g[g_1^n]\overset{i_s}{\underset{i_1}{\oplus}}y_1^s \;\vee\; h_2\sqsubset g[g_1^n]\overset{i_s}{\underset{i_1}{\oplus}}y_1^s,$$

which, by (5.2.6), is equivalent to

$$h_1\sqsubset g[(g_1\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)\cdots(g_n\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)] \;\vee\; h_2\sqsubset g[(g_1\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)\cdots(g_n\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)]. \tag{5.3.11}$$

From this, applying the v-negativity of χ, we obtain

$$h_1\sqsubset g_i\overset{i_s}{\underset{i_1}{\oplus}}y_1^s \;\vee\; h_2\sqsubset g_i\overset{i_s}{\underset{i_1}{\oplus}}y_1^s \tag{5.3.12}$$

for every $i=1,\ldots,n$.

The condition (5.3.11) means that $(g_1\overset{i_s}{\underset{i_1}{\oplus}}y_1^s,\ldots,g_n\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)$ belongs to $\mathrm{pr}_1P_{(h_1,h_2)}(g)$. The condition (5.3.12) — that $(x_1,\ldots,x_n)\in \mathrm{pr}_1\,P_{(h_1,h_2)}(g_i)$ for every $i=1,\ldots,n$, where $x_i=\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)$, $i=1,\ldots,n$. So,

$$(x_1,\ldots,x_n)\in \mathrm{pr}_1P_{(h_1,h_2)}(g[g_1^n])\longleftrightarrow \begin{cases}(g_1\overset{i_s}{\underset{i_1}{\oplus}}y_1^s,\ldots,g_n\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)\in \mathrm{pr}_1P_{(h_1,h_2)}(g),\\ \bigwedge\limits_{i=1}^{n}(x_1,\ldots,x_n)\in \mathrm{pr}_1\,P_{(h_1,h_2)}(g_i),\end{cases} \tag{5.3.13}$$

where $x_i=\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)$, $i=1,\ldots,n$.

Let $(x_1,\ldots,x_n)\in \mathrm{pr}_1\,P_{(h_1,h_2)}(g[g_1^n])$. If $(x_1,\ldots,x_n)\in G^n$, then, according to (5.3.6) and (5.3.9), we have

$$\begin{aligned}P_{(h_1,h_2)}(g[g_1^n])(x_1^n)&=g[g_1^n][x_1^n]=g[g_1[x_1^n]\cdots g_n[x_1^n]]\\&=P_{(h_1,h_2)}(g)(g_1[x_1^n],\ldots,g_n[x_1^n])\\&=P_{(h_1,h_2)}(g)\left(P_{(h_1,h_2)}(g_1)(x_1^n),\ldots,P_{(h_1,h_2)}(g_n)(x_1^n)\right)\\&=P_{(h_1,h_2)}(g)\left[P_{(h_1,h_2)}(g_1)\cdots P_{(h_1,h_2)}(g_n)\right](x_1^n).\end{aligned}$$

Similarly, we can prove that

$$P_{(h_1,h_2)}(g[g_1^n])(e_1^n) = P_{(h_1,h_2)}(g)\left[P_{(h_1,h_2)}(g_1)\,\cdots\,P_{(h_1,h_2)}(g_n)\right](e_1^n)$$

for $(e_1,\dots,e_n)\in \mathrm{pr}_1\,P_{(h_1,h_2)}(g[g_1^n])$.

If $(x_1,\dots,x_n)\in \mathrm{pr}_1\,P_{(h_1,h_2)}(g[g_1^n])$, where $x_i=\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)$, $i=1,\dots,n$, for some $y_1,\dots,y_s\in G$, $i_1,\dots,i_s\in\{1,\dots,n\}$, then, according to (5.3.6) and (5.3.13), we obtain

$$\begin{aligned}
P_{(h_1,h_2)}(g[g_1^n])(x_1^n) &= g[g_1^n]\overset{i_s}{\underset{i_1}{\oplus}}y_1^s = g[(g_1\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)\cdots(g_n\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)]\\
&= P_{(h_1,h_2)}(g)(g_1\overset{i_s}{\underset{i_1}{\oplus}}y_1^s,\dots,g_n\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)\\
&= P_{(h_1,h_2)}(g)\left(P_{(h_1,h_2)}(g_1)(x_1^n),\dots,P_{(h_1,h_2)}(g_n)(x_1^n)\right)\\
&= P_{(h_1,h_2)}(g)\left[P_{(h_1,h_2)}(g_1)\,\cdots\,P_{(h_1,h_2)}(g_n)\right](x_1^n).
\end{aligned}$$

The proof is complete. □

Proposition 5.3.5. *For all $g_1,g_2,h_1,h_2\in G$ and $i=1,\dots,n$ we have*

$$P_{(h_1,h_2)}(g_1\underset{i}{\oplus}g_2)=P_{(h_1,h_2)}(g_1)\underset{i}{\oplus}P_{(h_1,h_2)}(g_2).$$

Proof. Let $(x_1,\dots,x_n)\in \mathrm{pr}_1\,P_{(h_1,h_2)}(g_1\underset{i}{\oplus}g_2)$. If $(x_1,\dots,x_n)\in G^n$, then

$$h_1\sqsubset(g_1\underset{i}{\oplus}g_2)[x_1^n]\,\vee\,h_2\sqsubset(g_1\underset{i}{\oplus}g_2)[x_1^n],$$

which, by (5.3.5), is equivalent to

$$h_1\sqsubset g_1[x_1^{i-1}g_2[x_1^n]\,x_{i+1}^n]\,\vee\,h_2\sqsubset g_1[x_1^{i-1}g_2[x_1^n]\,x_{i+1}^n]. \tag{5.3.14}$$

This, according to the v-negativity of χ, implies

$$h_1\sqsubset g_2[x_1^n]\,\vee\,h_2\sqsubset g_2[x_1^n]. \tag{5.3.15}$$

The condition (5.3.14) means that $(x_1^{i-1},g_2[x_1^n],x_{i+1}^n)\in \mathrm{pr}_1\,P_{(h_1,h_2)}(g_1)$. From (5.3.15) we obtain $(x_1,\dots,x_n)\in \mathrm{pr}_1\,P_{(h_1,h_2)}(g_2)$. So, for every $(x_1,\dots,x_n)\in G^n$ we have

$$(x_1,\dots,x_n)\in \mathrm{pr}_1 P_{(h_1,h_2)}(g_1\underset{i}{\oplus}g_2)\longleftrightarrow\begin{cases}(x_1^{i-1},g_2[x_1^n],x_{i+1}^n)\in \mathrm{pr}_1 P_{(h_1,h_2)}(g_1)\\ (x_1,\dots,x_n)\in \mathrm{pr}_1\,P_{(h_1,h_2)}(g_2).\end{cases} \tag{5.3.16}$$

Consider now the case when $(x_1,\dots,x_n)=(e_1,\dots,e_n)$. In this case $(e_1,\dots,e_n)\in \mathrm{pr}_1\,P_{(h_1,h_2)}(g_1\underset{i}{\oplus}g_2)$ means, by (5.3.7), that

$$h_1\sqsubset g_1\underset{i}{\oplus}g_2\,\vee\,h_2\sqsubset g_1\underset{i}{\oplus}g_2. \tag{5.3.17}$$

Since $g_1 \underset{i}{\oplus} g_2 \sqsubset \mu_i(\underset{i}{\oplus} g_2) = g_2$, by the v-negativity of χ, the above condition gives us

$$h_1 \sqsubset g_2 \vee h_2 \sqsubset g_2. \tag{5.3.18}$$

But $\mu_i^*(\underset{i}{\oplus} g_2) = \mu_i(\underset{i}{\oplus} g_2) = g_2$ and $\mu_k^*(\underset{i}{\oplus} g_2) = e_k$ for $k \in \{1,\ldots,n\} \setminus \{i\}$, so, (5.3.17) implies $(e_1^{i-1}, g_2, e_{i+1}^n) \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_1)$. On the other hand, from (5.3.18), it follows that $(e_1,\ldots,e_n) \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_2)$. Therefore

$$(e_1,\ldots,e_n) \in \mathrm{pr}_1 P_{(h_1,h_2)}(g_1 \underset{i}{\oplus} g_2) \longleftrightarrow \begin{cases} (e_1^{i-1}, g_2, e_{i+1}^n) \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_1) \\ (e_1,\ldots,e_n) \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_2). \end{cases} \tag{5.3.19}$$

In the third case when $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$, $i = 1,\ldots,n$, for some $y_1,\ldots,y_s$ in G and $i_1,\ldots,i_s \in \{1,\ldots,n\}$, from $(x_1,\ldots,x_n) \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_1 \underset{i}{\oplus} g_2)$ we conclude

$$h_1 \sqsubset (g_1 \underset{i}{\oplus} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \vee h_2 \sqsubset (g_1 \underset{i}{\oplus} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s. \tag{5.3.20}$$

Since χ is v-negative, we have $(g_1 \underset{i}{\oplus} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \sqsubset \mu_i(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s$, which means that (5.3.20) can be written in the form

$$h_1 \sqsubset g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \vee h_2 \sqsubset g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s. \tag{5.3.21}$$

But $\mu_i^*(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = \mu_i(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s$ and $\mu_k^*(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = \mu_k^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$ for all $k \in \{1,\ldots,n\} \setminus \{i\}$. This, together with condition (5.3.20), proves $(x_1^{i-1},\, g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, x_{i+1}^n) \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_1)$. Similarly, from (5.3.21), we can deduce $(x_1,\ldots,x_n) \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_2)$. Therefore

$$(x_1,\ldots,x_n) \in \mathrm{pr}_1 P_{(h_1,h_2)}(g_1 \underset{i}{\oplus} g_2) \longleftrightarrow \begin{cases} (x_1^{i-1}, g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, x_{i+1}^n) \in \mathrm{pr}_1 P_{(h_1,h_2)}(g_1) \\ (x_1,\ldots,x_n) \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_2), \end{cases}$$

where $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$, $i = 1,\ldots,n$.

Let $(x_1,\ldots,x_n) \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_1 \underset{i}{\oplus} g_2)$. If $(x_1,\ldots,x_n) \in G^n$, then, according to (5.3.6) and (5.3.16), we have

$$\begin{aligned} P_{(h_1,h_2)}(g_1 \underset{i}{\oplus} g_2)(x_1^n) &= (g_1 \underset{i}{\oplus} g_2)[x_1^n] = g_1[x_1^{i-1} g_2[x_1^n] x_{i+1}^n] \\ &= P_{(h_1,h_2)}(g_1)\left(x_1^{i-1}, g_2[x_1^n], x_{i+1}^n\right) \\ &= P_{(h_1,h_2)}(g_1)\left(x_1^{i-1}, P_{(h_1,h_2)}(g_2)(x_1^n), x_{i+1}^n\right) \\ &= P_{(h_1,h_2)}(g_1) \underset{i}{\oplus} P_{(h_1,h_2)}(g_2)(x_1^n). \end{aligned}$$

If $(x_1, \ldots, x_n) = (e_1, \ldots, e_n)$, then, analogously as in the previous case, using (5.3.6) and (5.3.19) we obtain

$$P_{(h_1, h_2)}(g_1 \underset{i}{\oplus} g_2)(e_1^n) = P_{(h_1, h_2)}(g_1) \underset{i}{\oplus} P_{(h_1, h_2)}(g_2)(e_1^n).$$

Similarly, in the case when $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$, $i = 1, \ldots, n$, for some $y_1, \ldots, y_s \in G$, $i_1, \ldots, i_s \in \{1, \ldots, n\}$, we have

$$\begin{aligned} P_{(h_1, h_2)}(g_1 \underset{i}{\oplus} g_2)(x_1^n) &= (g_1 \underset{i}{\oplus} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \\ &= P_{(h_1, h_2)}(g_1)(x_1^{i-1}, g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, x_{i+1}^n) \\ &= P_{(h_1, h_2)}(g_1)\left(x_1^{i-1}, P_{(h_1, h_2)}(g_2)(x_1^n), x_{i+1}^n\right) \\ &= P_{(h_1, h_2)}(g_1) \underset{i}{\oplus} P_{(h_1, h_2)}(g_2)(x_1^n). \end{aligned}$$

This completes our proof. □

Based on these propositions, we are able to prove the sufficiency of the conditions of Theorem 5.3.2.

Sufficiency. Let the triplet (χ, γ, π) of binary relations on a representable Menger $(2, n)$-semigroup $\mathcal{G}$ satisfy all the conditions of the theorem. Then, as it follows from Propositions 5.3.3–5.3.5, for all $h_1, h_2 \in G$, the mapping $P_{(h_1, h_2)}$ is a representation of $\mathcal{G}$ by n-place functions. Consider the family of representations $P_{(h_1, h_2)}$ such that $(h_1, h_2) \in \gamma$. Let P be the sum of this family, i.e., $P = \sum\limits_{(h_1, h_2) \in \gamma} P_{(h_1, h_2)}$. Of course, P is a representation of $\mathcal{G}$ by n-place functions. Let us show that $\chi = \chi_P$, $\gamma = \gamma_P$ and $\pi = \pi_P$.

Let $(g_1, g_2) \in \chi_P$. Then, according to (5.3.1), $(g_1, g_2) \in \chi_{(h_1, h_2)}$[5] are defined in a similar way for all $(h_1, h_2) \in \gamma$, i.e.,

$$(\forall (h_1, h_2) \in \gamma)\ \left(\mathrm{pr}_1\, P_{(h_1, h_2)}(g_1) \subset \mathrm{pr}_1\, P_{(h_1, h_2)}(g_2)\right),$$

which means that for all $(h_1, h_2) \in \gamma$ the following implication

$$(x_1, \ldots, x_n) \in \mathrm{pr}_1\, P_{(h_1, h_2)}(g_1) \longrightarrow (x_1, \ldots, x_n) \in \mathrm{pr}_1\, P_{(h_1, h_2)}(g_2)$$

is true. This, in particular, applied to $(x_1, \ldots, x_n) = (e_1, \ldots, e_n)$, gives the implication

$$(e_1, \ldots, e_n) \in \mathrm{pr}_1\, P_{(h_1, h_2)}(g_1) \longrightarrow (e_1, \ldots, e_n) \in \mathrm{pr}_1\, P_{(h_1, h_2)}(g_2)$$

[5] $\chi_{(h_1, h_2)}$ denotes this quasi-order which corresponds to the representation $P_{(h_1, h_2)}$. $\gamma_{(h_1,h_2)}$ and $\pi_{(h_1, h_2)}$ have the similar sense.

for all $(h_1, h_2) \in \gamma$. Hence

$$(\forall (h_1, h_2) \in \gamma)\ \ (h_1 \sqsubset g_1 \ \vee\ h_2 \sqsubset g_1 \longrightarrow h_1 \sqsubset g_2 \ \vee\ h_2 \sqsubset g_2)\,.$$

If $g_1 \neq 0$, then $(g_1, g_1) \in \gamma$, and, according to the above implication, $g_1 \sqsubset g_1 \longrightarrow g_1 \sqsubset g_2$. This proves $(g_1, g_2) \in \chi$ because χ is reflexive. For $g_1 = 0$, by the v-negativity of χ, we have $0 = 0[g_2 \cdots g_2] \sqsubset g_2$. Hence $(0, g_2) \in \chi$. So, $(g_1, g_2) \in \chi$, i.e., $\chi_P \subset \chi$.

Conversely, let $(g_1, g_2) \in \chi$, $(h_1, h_2) \in \gamma$, $(x_1, \ldots, x_n) \in \mathrm{pr}_1\, P_{(h_1,\,h_2)}(g_1)$. If $(x_1, \ldots, x_n) \in G^n$, then $h_1 \sqsubset g_1[x_1^n] \ \vee\ h_2 \sqsubset g_1[x_1^n]$. Since the l-regularity of χ together with $g_1 \sqsubset g_2$ imply $g_1[x_1^n] \sqsubset g_2[x_1^n]$, from the above we conclude $h_1 \sqsubset g_2[x_1^n] \vee h_2 \sqsubset g_2[x_1^n]$, i.e., $(x_1, \ldots, x_n) \in \mathrm{pr}_1\, P_{(h_1,\,h_2)}(g_2)$. In the case when $(x_1, \ldots, x_n) = (e_1, \ldots, e_n)$, from $(e_1, \ldots, e_n) \in \mathrm{pr}_1\, P_{(h_1,\,h_2)}(g_1)$ it follows that $(e_1, \ldots, e_n) \in \mathrm{pr}_1\, P_{(h_1,\,h_2)}(g_2)$.

In the case when $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$, $i = 1, \ldots, n$, for some $y_1, \ldots, y_s \in G$, $i_1, \ldots, i_s \in \{1, \ldots, n\}$, applying the l-regularity of χ to $g_1 \sqsubset g_2$, we have $g_1 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \sqsubset g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s$, whence, in view of $h_1 \sqsubset g_1 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \ \vee\ h_2 \sqsubset g_1 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s$, we obtain $h_1 \sqsubset g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \vee h_2 \sqsubset g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s$. Therefore $x_1^n \in \mathrm{pr}_1\, P_{(h_1,\,h_2)}(g_2)$, which proves $\mathrm{pr}_1\, P_{(h_1,\,h_2)}(g_1) \subset \mathrm{pr}_1\, P_{(h_1,\,h_2)}(g_2)$ for all $(h_1,\, h_2) \in \gamma$. Thus $(g_1, g_2) \in \chi_P$, i.e., $\chi \subset \chi_P$. Consequently, $\chi = \chi_P$, which, together with condition (c) formulated in the theorem, gives $\pi = \chi \cap \chi^{-1} = \chi_P \cap \chi_P^{-1} = \pi_P$. So, $\pi = \pi_P$.

Now let $(g_1, g_2) \in \gamma_P$. Then, according to (5.3.1), $(g_1, g_2) \in \gamma_{(h_1,\,h_2)}$ for some $(h_1, h_2) \in \gamma$, i.e.,

$$(\exists (h_1, h_2) \in \gamma)\ \left(\mathrm{pr}_1\, P_{(h_1,\,h_2)}(g_1) \cap \mathrm{pr}_1\, P_{(h_1,\,h_2)}(g_2) \neq \varnothing\right),$$

which means that there exists $(h_1, h_2) \in \gamma$ and $(x_1, \ldots, x_n)$ such that

$$(x_1, \ldots, x_n) \in \mathrm{pr}_1\, P_{(h_1,\,h_2)}(g_1) \ \wedge\ (x_1, \ldots, x_n) \in \mathrm{pr}_1\, P_{(h_1,\,h_2)}(g_2).$$

This, for $(x_1, \ldots, x_n) \in G^n$ implies $h_1 \sqsubset g_1[x_1^n] \ \vee\ h_2 \sqsubset g_1[x_1^n]$ and $h_1 \sqsubset g_2[x_1^n] \ \vee\ h_2 \sqsubset g_2[x_1^n]$. From the above, in view of $h_1 \top h_2$ and (5.3.5), we obtain $g_1[x_1^n] \top g_2[x_1^n]$, whence, applying the l-cancellativity of γ, we get $g_1 \top g_2$, i.e., $(g_1, g_2) \in \gamma$.

In a similar way, we can see that in the case $(x_1, \ldots, x_n) = (e_1, \ldots, e_n)$ the condition $(g_1, g_2) \in \gamma$ holds, too.

If $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$, $i = 1, \ldots, n$, for some $i_1, \ldots, i_s \in \{1, \ldots, n\}$ and $y_1, \ldots,$ $y_s \in G$, then

$$h_1 \sqsubset g_1 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \ \vee\ h_2 \sqsubset g_1 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \ \ \text{and}\ \ h_1 \sqsubset g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \ \vee\ h_2 \sqsubset g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s,$$

whence, using $h_1 \top h_2$ and (5.3.5), we deduce $g_1 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \top g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s$. Thus $g_1 \top g_2$, because γ is l-cancellative. In this way we have proved that $\gamma_P \subset \gamma$ in any case.

Conversely, let $(g_1, g_2) \in \gamma$. Then $g_1 \sqsubset g_1$ and $g_2 \sqsubset g_2$, because χ is reflexive. Hence $g_1 \sqsubset g_1 \vee g_2 \sqsubset g_1$ and $g_1 \sqsubset g_2 \vee g_2 \sqsubset g_2$. Consequently, $(e_1, \ldots, e_n) \in \mathrm{pr}_1 P_{(g_1, g_2)}(g_1)$ and $(e_1, \ldots, e_n) \in \mathrm{pr}_1 P_{(g_1, g_2)}(g_2)$. Thus $(g_1, g_2) \in \gamma_{(g_1, g_2)} \subset \gamma_P$, i.e., $\gamma \subset \gamma_P$. So, $\gamma = \gamma_P$.

This completes the proof of the theorem. □

If we delete from Theorem 5.3.2 the equality $\pi = \chi \cap \chi^{-1}$ we obtain the necessary and sufficient conditions under which the pair (χ, γ) of binary relations is projection representable for a representable Menger $(2, n)$-semigroup. Furthermore, all parts of the proof of this theorem connected with these two relations are true. So, we have the following.

Theorem 5.3.6. *A pair (χ, γ) of binary relations on G is projection representable for a representable Menger $(2, n)$-semigroup $\mathcal{G}$ if and only if χ is an l-regular and v-negative quasi-order, γ is an l-cancellative 0-quasi-equivalence and the implication* (5.3.5) *is satisfied.*

Let $\mathcal{G}$ be a representable Menger $(2, n)$-semigroup. Let us consider the set $T_n(G)$ of mappings $t : x \mapsto t(x)$ defined as follows:

(a) $x \in T_n(G)$, i.e., $T_n(G)$ contains the identity transformation of G,

(b) if $i \in \{1, \ldots, n\}$, $a, b_1, \ldots, b_n \in G$, then $a[b_1^{i-1} t(x)\, b_{i+1}^n] \in T_n(G)$ for all $t(x) \in T_n(G)$,

(c) $T_n(G)$ contains those and only those mappings which are defined by (a) and (b).

Let us consider on G two binary relations δ_1 and δ_2 defined in the following way:

(1) $(g_1, g_2) \in \delta_1 \longleftrightarrow g_1 = t(g_2)$ for some $t \in T_n(G)$,

(2) $(g_1, g_2) \in \delta_2 \longleftrightarrow \begin{cases} g_1 = (x \overset{i_s}{\underset{i_1}{\oplus}} y_1^s)[\bar{z}] \ \text{ and } \ g_2 = \mu_i(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)[\bar{z}] \ \text{ for some} \\ x, y_1, \ldots, y_s \in G, \bar{z} \in G^n, \ i, i_1, \ldots, i_s \in \{1, \ldots, n\}, \\ \text{where the symbol } [\bar{z}] \text{ can be empty.} \end{cases}$

It is not difficult to see that δ_1 and δ_2 are l-regular relations, in addition δ_1 is a quasi-order. Moreover, *a binary relation $\rho \subset G \times G$ is v-negative if and only if it contains δ_1 and δ_2.*

Let π be an l-regular equivalence on a representable Menger $(2, n)$-semigroup $\mathcal{G}$. Denote by $\chi(\pi)$ the binary relation $f_t(f_R(\delta_2) \circ \delta_1 \circ \pi)$, where f_R and

f_t are respectively reflexive and transitive closure operations, i.e.,

$$\chi(\pi) = f_t(f_R(\delta_2) \circ \delta_1 \circ \pi) = \bigcup_{n=1}^{\infty} ((\delta_2 \cup \triangle_G) \circ \delta_1 \circ \pi)^n . \tag{5.3.22}$$

Since π, δ_1 and $f_R(\delta_2)$ are reflexive l-regular relations, $\chi(\pi)$ is an l-regular quasi-order containing π, δ_1 and δ_2. So, $\chi(\pi)$ is a v-negative quasi-order.

Proposition 5.3.7. *$\chi(\pi)$ is the least l-regular and v-negative quasi-order containing π.*

Proof. Let χ be an arbitrary l-regular and v-negative quasi-order containing π. Then $\delta_1 \subset \chi$ and $\delta_2 \subset \chi$, because χ is v-negative. Thus, $\pi \subset \chi$, $\delta_1 \subset \chi$ and $f_R(\delta_2) \subset \chi$, whence $f_R(\delta_2) \circ \delta_1 \circ \pi \subset \chi \circ \chi \circ \chi \subset \chi$. From this, applying the transitivity of χ, we obtain $(f_R(\delta_2) \circ \delta_1 \circ \pi)^n \subset \chi^n \subset \chi$ for every natural n. Therefore $\bigcup\limits_{n=1}^{\infty} ((\delta_2 \cup \triangle_G) \circ \delta_1 \circ \pi)^n \subset \chi$, i.e., $\chi(\pi) \subset \chi$. □

Theorem 5.3.8. *A pair (γ, π) of binary relations on a representable Menger $(2, n)$-semigroup $\mathcal{G}$ is projection representable if and only if*
(a) *γ is an l-cancellative 0-quasi-equivalence,*
(b) *π is an l-regular equivalence such that $\chi(\pi) \cap (\chi(\pi))^{-1} \subset \pi$,*
(c) *the following condition*

$$h_1 \top h_2 \ \wedge \ h_1 \sqsubset_\pi g_1 \ \wedge \ h_2 \sqsubset_\pi g_2 \longrightarrow g_1 \top g_2, \tag{5.3.23}$$

where $h \sqsubset_\pi g$ means $(h, g) \in \chi(\pi)$, is satisfied for all $g_1, g_2, h_1, h_2 \in G$.

Proof. Let P be a representation on a representable Menger $(2, n)$-semigroup $\mathcal{G}$ for which $\gamma = \gamma_P$ and $\pi = \pi_P$. Then, by Proposition 5.3.5, we have $\chi(\pi) \subset \chi_P$, whence $\chi(\pi) \cap (\chi(\pi))^{-1} \subset \chi_P \cap \chi_P^{-1} = \pi_P = \pi$.

Assume now that the premise of (5.3.23) is satisfied. Then $(h_1, h_2) \in \gamma$, $(h_1, g_1) \in \chi(\pi)$ and $(h_2, g_2) \in \chi(\pi)$. Hence, $(h_1, h_2) \in \gamma_P$, $(h_1, g_1) \in \chi_P$ and $(h_2, g_2) \in \chi_P$, i.e., $\mathrm{pr}_1 P(h_1) \cap \mathrm{pr}_1 P(h_2) \neq \varnothing$, $\mathrm{pr}_1 P(h_1) \subset \mathrm{pr}_1 P(g_1)$ and $\mathrm{pr}_1 P(h_2) \subset \mathrm{pr}_1 P(g_2)$, whence $\mathrm{pr}_1 P(g_1) \cap \mathrm{pr}_1 P(g_2) \neq \varnothing$. Therefore, $(g_1, g_2) \in \gamma_P = \gamma$, which means that condition (5.3.23) is valid. The necessity is proved.

To prove the sufficiency, assume that the pair (γ, π) of binary relations satisfies all the conditions of the theorem and consider the triplet $(\chi(\pi), \gamma, \pi)$. Then $\pi = \pi^{-1} \subset (\chi(\pi))^{-1}$, because $\pi \subset \chi(\pi)$. Consequently $\pi \subset \chi(\pi) \cap (\chi(\pi))^{-1}$, which, together with condition (b), gives $\pi = \chi(\pi) \cap (\chi(\pi))^{-1}$. This means that the triplet $(\chi(\pi), \gamma, \pi)$ satisfies all the conditions of Theorem 5.3.2. So, $(\chi(\pi), \gamma, \pi)$ is projection representable, therefore (γ, π) is also projection representable. The theorem is proved. □

Applying the method of mathematical induction to (5.3.22), we can prove the following proposition.

Proposition 5.3.9. *The condition* $(g_1, g_2) \in \chi(\pi)$, *where* $g_1, g_2 \in G$, *means that the system of conditions*

$$\left.\begin{array}{c} g_1 = x_0 \wedge g_2 = x_n, \\ \bigwedge\limits_{i=0}^{n-1} \left(\begin{pmatrix} x_i \equiv t_i((y_i \overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i})[\bar{w}_i]), \\ x_{i+1} = \mu_{k_i}(\overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i})[\bar{w}_i] \end{pmatrix} \vee x_i \equiv t_i(x_{i+1}) \right) \end{array}\right\} \tag{5.3.24}$$

is valid for some $n \in \mathbb{N}$, $x_i, y_i, z_{k_i} \in G$, $\bar{w}_i \in G^n$ *and* $t_i \in T_n(G)$, *where* $i \in \{1, \ldots, n\}$.

The formula

$$\bigwedge\limits_{i=m}^{n} \left(\begin{pmatrix} x_i \equiv t_i((y_i \overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i})[\bar{w}_i]), \\ x_{i+1} = \mu_{k_i}(\overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i})[\bar{w}_i] \end{pmatrix} \vee x_i \equiv t_i(x_{i+1}) \right)$$

will be denoted by $\mathfrak{M}(m, n)$.

The inclusion $\chi(\pi) \cap (\chi(\pi))^{-1} \subset \pi$ means that for all $g_1, g_2 \in G$ we have

$$(g_1, g_2) \in \chi(\pi) \wedge (g_2, g_1) \in \chi(\pi) \longrightarrow g_1 \equiv g_2,$$

which, according to Proposition 5.3.9, can be written as the system of conditions $(A_{n,m})_{n,m \in \mathbb{N}}$, where

$$A_{n,m} : \mathfrak{M}(0, n-1) \wedge \mathfrak{M}(n+1, n+m) \wedge x_0 = x_{n+m} \longrightarrow x_0 \equiv x_n.$$

The system $(A_{n,m})_{n,m \in \mathbb{N}}$ is equivalent to the system $(A_n)_{n \in \mathbb{N}}$, where

$$A_n : \mathfrak{M}(0, n-1) \wedge x_0 = x_n \longrightarrow x_0 \equiv x_1.$$

Consider now the implication (5.3.23). According to (5.3.24) condition $(h_1, g_1) \in \chi(\pi)$ means that

$$h_1 = x_0 \wedge \mathfrak{M}(0, n-1) \wedge x_n = g_1 \tag{5.3.25}$$

for some $x_i, y_i, z_{k_i}, t_i, k_i, \bar{w}_i$. Similarly, condition $(h_2, g_2) \in \chi(\pi)$ means that

$$h_2 = x_{n+1} \wedge \mathfrak{M}(n+1, n+m) \wedge x_{n+m+1} = g_2 \tag{5.3.26}$$

for some $x_i, y_i, z_{k_i}, t_i, k_i, \bar{w}_i$. So, (5.3.23) can be written as the system $(B_{n,m})_{n,m\in\mathbb{N}}$ of conditions

$$B_{n,m}: \ x_0 \top x_{n+1} \wedge \mathfrak{M}(0, n-1) \wedge \mathfrak{M}(n+1, n+m) \longrightarrow x_n \top x_{n+m+1}.$$

In this way we have proved the following.

Theorem 5.3.10. *A pair (γ, π) of binary relations on a representable Menger $(2, n)$-semigroup $\mathcal{G}$ is projection representable if and only if*
(a) *γ is an l-cancellative 0-quasi-equivalence,*
(b) *π is an l-regular equivalence,*
(c) *the systems of conditions $(A_n)_{n\in\mathbb{N}}$ and $(B_{n,m})_{n,m\in\mathbb{N}}$ are satisfied.*

Theorem 5.3.11. *A pair (χ, π) of binary relations is (faithful) projection representable for a representable Menger $(2, n)$-semigroup $\mathcal{G}$ if and only if χ is an l-regular and v-negative quasi-order such that $\pi = \chi \cap \chi^{-1}$.*

Proof. The necessity of these conditions follows from the proof of Theorem 5.3.2. To prove their sufficiency, for every element $g \in G$ we define an n-place function $P_a(g) : \mathfrak{A}^* \to G$, where $a \in G$, putting

$$P_a(g)(x_1^n) = \begin{cases} g[x_1^n] & \text{if } \ a \sqsubset g[x_1^n] \ \text{ and } \ (x_1, \ldots, x_n) \in G^n, \\ g & \text{if } \ a \sqsubset g \ \text{ and } \ (x_1, \ldots, x_n) = (e_1, \ldots, e_n), \\ g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s & \text{if } \ a \sqsubset g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \ \text{ and } \ x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), \\ & i = 1, \ldots, n, \ \text{for some } y_1, \ldots, y_s \in G, \\ & i_1, \ldots, i_s \in \{1, \ldots, n\}. \end{cases} \tag{5.3.27}$$

Since, for $h_1 = h_2 = a \in G$, the function $P_{(h_1,h_2)}(g)$ defined by (5.3.6) coincides with the function $P_a(g)$, from Propositions 5.3.3–5.3.5 it follows that the mapping $P_a : g \mapsto P_a(g)$ is a representation of $\mathcal{G}$ by n-place functions. Further, similarly as in the proof of Theorem 5.3.2, we can prove that $P_0 = \sum_{a\in G} P_a$ is a representation of $\mathcal{G}$ for which $\chi = \chi_{P_0}$ and $\pi = \pi_{P_0}$. So, the pair (χ, π) is projection representable for $\mathcal{G}$.

Let us show that (χ, π) is faithful projection representable. From Theorem 5.3.1, it follows that $(G, o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}, \chi)$ has a faithful representation by n-place functions. Let Λ be such a representation. Then obviously $\chi_\Lambda = G \times G$ and $\pi_\Lambda = G \times G$. Consider the representation $P = \Lambda + P_0$. Since Λ is a faithful representation, P is also faithful. Moreover $\chi_P = \chi_\Lambda \cap \chi_{P_0} = G \times G \cap \chi = \chi$ and $\pi_P = \pi_\Lambda \cap \pi_{P_0} = G \times G \cap \pi = \pi$. So, (χ, π) is faithful projection representable for $\mathcal{G}$. □

From the Theorem 5.3.1 we obtain the next theorem.

Theorem 5.3.12. *A binary relation χ is (faithful) projection representable for a representable Menger $(2, n)$-semigroup if and only if it is an l-regular, v-negative quasi-order.*

Theorem 5.3.13. *A binary relation π is (faithful) projection representable for a representable Menger $(2, n)$-semigroup if and only if it is an l-regular equivalence such that $\chi(\pi) \cap (\chi(\pi))^{-1} \subset \pi$.*

Proof. Consider the pair $(\chi(\pi), \pi)$ of binary relations, where $\chi(\pi)$ is defined by (5.3.22). In a way similar as in the proof of Theorem 5.3.8, we can prove that this pair satisfies all the conditions of Theorem 5.3.11, whence we deduce the validity of Theorem 5.3.13. □

Since, as it was shown above, the inclusion $\chi(\pi) \cap (\chi(\pi))^{-1} \subset \pi$ is equivalent to the system of conditions $(A_n)_{n\in\mathbb{N}}$, the last theorem can be rewritten in the following form.

Theorem 5.3.14. *A binary relation π is (faithful) projection representable for a representable Menger $(2, n)$-semigroup if and only if it is an l-regular equivalence and the system of conditions $(A_n)_{n\in\mathbb{N}}$ is satisfied.*

Consider the binary relation χ_0 on a Menger $(2, n)$-semigroup $\mathcal{G}$ defined in the following way:

$$\chi_0 = f_t(f_R(\delta_2) \circ \delta_1) = \bigcup_{n=1}^{\infty} ((\delta_2 \cup \triangle_G) \circ \delta_1)^n , \tag{5.3.28}$$

where f_R and f_t are reflexive and transitive closure operations.

Proposition 5.3.15. *χ_0 is the least l-regular and v-negative quasi-order on a Menger $(2, n)$-semigroup $\mathcal{G}$.*

The proof of this proposition is analogous to the proof of Proposition 5.3.7.

Theorem 5.3.16. *A binary relation γ is projection representable for a representable Menger $(2, n)$-semigroup if and only if it is an l-cancellative 0-quasi-equivalence and the following implication*

$$h_1 \top h_2 \wedge h_1 \sqsubset_0 g_1 \wedge h_2 \sqsubset_0 g_2 \longrightarrow g_1 \top g_2 \tag{5.3.29}$$

is satisfied for all $h_1, h_2, g_1, g_2 \in G$, where $h \sqsubset_0 g$ means $(h, g) \in \chi_0$.

Proof. The necessity of (5.3.29) can be proved analogously as the necessity of (5.3.23) in the proof of Theorem 5.3.8. To prove the sufficiency we consider the pair (χ_0, γ). By Proposition 5.3.15, this pair satisfies all demands of Theorem 5.3.6, whence we conclude that Theorem 5.3.16 is true. □

Based on the formula (5.3.28) we can prove the following proposition.

Proposition 5.3.17. *From* $(g_1, g_2) \in \chi_0$, *where* $g_1, g_2 \in G$, *it follows that the system of conditions*

$$g_1 = x_0 \wedge g_2 = x_n \wedge \bigwedge_{i=0}^{n-1} \left(\begin{pmatrix} x_i = t_i((y_i \overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i})[\bar{w}_i]), \\ x_{i+1} = \mu_{k_i}(\overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i})[\bar{w}_i] \end{pmatrix} \vee x_i = t_i(x_{i+1}) \right)$$

is satisfied for $n \in \mathbb{N},\ x_i, y_i, z_i \in G,\ \bar{w}_i \in G^n,\ t_i \in T_n,\ k_i \in \{1, \ldots, n\}$.

Denoting by $\mathfrak{N}(m, n)$ the formula

$$\bigwedge_{i=m}^{n} \left(\begin{pmatrix} x_i = t_i((y_i \overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i})[\bar{w}_i]), \\ x_{i+1} = \mu_{k_i}(\overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i})[\bar{w}_i] \end{pmatrix} \vee x_i = t_i(x_{i+1}) \right),$$

and using the same argumentation as in the proof of Theorem 5.3.8, we can prove that the implication (5.3.29) is equivalent to the system of conditions $(C_{n,m})_{n,m\in\mathbb{N}}$, where

$$C_{n,m}:\ x_0 \top x_{n+1} \wedge \mathfrak{N}(0, n-1) \wedge \mathfrak{N}(n+1, n+m) \longrightarrow x_0 \top x_{n+m+1}.$$

So, the following theorem is true.

Theorem 5.3.18. *A binary relation* γ *is projection representable for a representable Menger* $(2, n)$*-semigroup if and only if it is an* l*-cancellative* 0*-quasi-equivalence and the system of conditions* $(C_{n,m})_{n,m\in\mathbb{N}}$ *is valid.*

A similar result can be proved for $(2, n)$-semigroups. Indeed, let χ, γ and π be binary relations on a $(2, n)$-semigroup $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$. Similarly as in the case of Menger $(2, n)$-semigroups we will say that the triplet (χ, γ, π) is *(faithful) projection representable for a* $(2, n)$*-semigroup* $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$, if there exists such (faithful) representation P of $(G, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ by n-place functions for which $\chi = \chi_P$, $\gamma = \gamma_P$ and $\pi = \pi_P$. The projection representable pairs and separate relations can be defined analogously.

It is not difficult to verify that Theorem 5.3.2 formulated for representable Menger $(2, n)$-semigroup is also valid for representable $(2, n)$-semigroups. The proof of this version of Theorem 5.3.2 is similar to the proof of the previous version, but in the proof of the sufficiency instead of the representation P, we must consider the representation $P^{\bullet}$, which is the sum of the family of representations $(P^{\bullet}_{(h_1, h_2)})_{(h_1, h_2)\in\gamma}$, where for every $g \in G$ $P^{\bullet}_{(h_1, h_2)}(g) :\ \mathfrak{A}_0^* \to G$

($\mathfrak{A}_0^* = \mathfrak{A}_0 \cup \{(e_1,\ldots,e_n)\}$ (see p. 272) is a partial n-place function such that $(x_1,\ldots,x_n) \in \mathrm{pr}_1\, P^{\bullet}_{(h_1,h_2)}(g)$ only in the case when

$$\begin{cases} h_1 \sqsubset g \,\vee\, h_2 \sqsubset g & \text{if } (x_1,\ldots,x_n) = (e_1,\ldots,x_n), \\ h_1 \sqsubset g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \,\vee\, h_2 \sqsubset g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s & \text{if } x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s),\ i = 1,\ldots,n, \\ & \text{for some } y_1,\ldots,y_s \in G \\ & \text{and } i_1 \cdots, i_s \in \{1,\ldots,n\}. \end{cases}$$

Then

$$P^{\bullet}_{(h_1,h_2)}(g)(x_1^n) = \begin{cases} g & \text{if } (x_1,\ldots,x_n) = (e_1,\ldots,e_n), \\ g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s & \text{if } x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s),\ i = 1,\ldots,n, \\ & \text{for some } y_1,\ldots,y_s \in G \\ & \text{and } i_1,\ldots,i_s \in \{1,\ldots,n\}. \end{cases}$$

Theorem 5.3.6 is also valid for $(2,n)$-semigroups. Theorem 5.3.8 will be valid for $(2,n)$-semigroups if we replace the relation $\chi(\pi)$ by the relation

$$\chi^{\bullet}(\pi) = f_t(f_R(\delta_2) \circ \pi) = \bigcup_{n=1}^{\infty} ((\delta_2 \cup \triangle_G) \circ \pi)^n\,, \tag{5.3.30}$$

i.e., if we delete δ_1 from the formula (5.3.22).

Proposition 5.3.9 for $(2,n)$-semigroups has the following form.

Proposition 5.3.19. *The condition $(g_1,g_2) \in \chi^{\bullet}(\pi)$, where $g_1,g_2 \in G$, means that the system of conditions*

$$g_1 = x_0 \,\wedge\, g_2 = x_n \,\wedge\, \bigwedge_{i=0}^{n-1} \left(\begin{pmatrix} x_i \equiv y_i \overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i}, \\ x_{i+1} = \mu_{k_i}(\overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i}) \end{pmatrix} \vee\, x_i \equiv x_{i+1} \right)$$

is satisfied for some $n \in \mathbb{N}$, $x_i, y_i, z_i \in G$, $k_i \in \{1,\ldots,n\}$.

Denoting by $\mathfrak{X}(m,n)$ the formula

$$\bigwedge_{i=m}^{n} \left(\begin{pmatrix} x_i \equiv y_i \overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i}, \\ x_{i+1} = \mu_{k_i}(\overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i}) \end{pmatrix} \vee\, x_i \equiv x_{i+1} \right)$$

and using the same argumentation as in the proof of Theorem 5.3.10, we can prove the following.

Theorem 5.3.20. *A pair (γ, π) of binary relations on a representable $(2, n)$-semigroup is projection representable if and only if γ is an l-cancellative 0-quasi-equivalence, π is an l-regular equivalence, and the systems of conditions $A_n^\bullet$ and $B_{n,m}^\bullet$, where*

$$A_n^\bullet: \ \mathfrak{X}(0, n-1) \wedge x_0 = x_n \longrightarrow x_0 \equiv x_1,$$

$$B_{n,m}^\bullet: \ x_0 \top x_{n+1} \wedge \mathfrak{X}(0, n-1) \wedge \mathfrak{X}(n+1, n+m) \longrightarrow x_n \top x_{n+m+1}$$

are satisfied.

Theorem 5.3.11 is true for $(2, n)$-semigroups too, but in the proof, the representation P_a defined by (5.3.27), must be replaced by the representation $P_a^\bullet$, where

$$P_a^\bullet(g)(x_1^n) = \begin{cases} g & \text{if } \ a \sqsubset g \ \text{ and } \ (x_1, \ldots, x_n) = (e_1, \ldots, e_n), \\ g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s & \text{if } \ a \sqsubset g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \ \text{ and } \ x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), \\ & i = 1, \ldots, n, \ \text{for some } \ y_1, \ldots, y_s \in G, \\ & \text{and } \ i_1, \ldots, i_s \in \{1, \ldots, n\}. \end{cases}$$

For $(2, n)$-semigroups Theorem 5.3.12 has the same form as for Menger $(2, n)$-semigroup, in Theorem 5.3.13 the relation $\chi(\pi)$ must be replaced by $\chi^\bullet(\pi)$, and in Theorem 5.3.14 instead of A_n we must use $A_n^\bullet$.

Furthermore, using the same argumentation as in the proof of Proposition 5.3.7 we can prove that the relation

$$\chi_0^\bullet = f_t(f_R(\delta_2)) = \bigcup_{n=1}^{\infty} (\delta_2 \cup \triangle_G)^n,$$

where f_R and f_t are reflexive and transitive closure operations, is the least l-regular and v-negative quasi-order on a given $(2, n)$-semigroup. Using this relation, we can prove the analog of Theorem 5.3.18 for $(2, n)$-semigroups.

Proposition 5.3.17 for $(2, n)$-semigroups has the following form.

Proposition 5.3.21. *The condition $(g_1, g_2) \in \chi_0^\bullet$, where $g_1, g_2 \in G$, means that the system of conditions*

$$g_1 = x_0 \wedge g_2 = x_n \wedge \bigwedge_{i=0}^{n-1} \left(\begin{pmatrix} x_i = y_i \overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i}, \\ x_{i+1} = \mu_{k_i}(\overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i}) \end{pmatrix} \vee x_i = x_{i+1} \right)$$

is valid for $n \in \mathbb{N}$, $x_i, y_i, z_i \in G$.

Furthermore, denoting by $\mathfrak{B}(m,n)$ the formula

$$\bigwedge_{i=m}^{n}\left(\begin{pmatrix} x_i = y_i \overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i}, \\ \\ x_{i+1} = \mu_{k_i}(\overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i}) \end{pmatrix} \vee x_i = x_{i+1}\right)$$

and using the same argumentation as in the proof of Theorem 5.3.16, we can prove the following.

Theorem 5.3.22. *A binary relation γ is projection representable for a representable $(2,n)$-semigroup if and only if it is an l-cancellative 0-quasi-equivalence and the system of conditions $(C^{\bullet}_{n,m})_{n,m\in\mathbb{N}}$, where*

$$C^{\bullet}_{n,m}:\ x_0 \top x_{n+1} \wedge \mathfrak{B}(0,n-1) \wedge \mathfrak{B}(n+1,n+m) \longrightarrow x_0 \top x_{n+m+1}$$

is satisfied.

5.4 Notes on Chapter 5

Mann's compositions of multiplace functions were introduced in [109]. Afterwards these compositions were used by V. D. Belousov for the investigation of systems orthogonal operations [5] and the study of decompositions of quasigroup operations [6]. The systematic investigation of $(2,n)$-semigroups of n-place operations was undertaken by T. O. Yakubov [260] who was one of the students of V. D. Belousov. The first abstract characterization of $(2,n)$-semigroups of n-ary operations was obtained by F. N. Sokhatskii [199]. In this paper, the first abstract characterization of $(2,n)$-semigroups of operations closed under Menger composition is also presented. Further study was conducted in [205]. Unfortunately, part of the results presented in this paper is incorrect. For example, in Theorem 10, part of the axioms (used later in the proof) is omitted.

Representations of abstract $(2,n)$-semigroups by n-place functions are described in [33, 35, 36].

Bisemigroups of binary operations studied by K. A. Zareckii [264] and other authors are in fact $(2,2)$-semigroups in the above sense. $(2,2)$-Semigroups of binary relations are characterized in [217]. Bisemigroups of binary operations with some additional properties are studied by Yu. M. Movsisyan [124] and J. Pashazadeh [137]. In the study of $(2,2)$ and $(2,3)$-semigroups, an important role play left null elements defined as elements θ such that $\theta \underset{i}{\oplus} f = \theta$ for each i and $f \in G$. A $(2,2)$-semigroup G containing left null elements is isomorphic to a $(2,2)$-semigroup of binary functions defined on a set A if and only the set A and the set of all null elements of G have the same cardinality and some identities with θ are satisfied. A similar result is valid for $(2,3)$ and $(2,4)$-semigroups (see [104] and [139]).

Chapter 6

Systems of multiplace functions

Unlike the previous chapters, in which we studied the sets of functions with a fixed number of variables, in this chapter we will be investigate sets of functions with a variable number of variables. Sets of such functions have application in multivalued logics, in automata theory and programming (see, for example, [50, 106, 108, 145]). Below we present different abstract characterizations of such sets of functions closed with respect to basic compositions of functions. It is clear that the results presented in this chapter are essential generalizations of the results of the previous chapters.

6.1 Menger systems

Let A be some nonempty set, I – a nonempty set of positive integers, $\mathcal{T}_n(A) = \mathcal{T}(A^n, A)$ and $\mathcal{F}_n(A) = \mathcal{F}(A^n, A)$. On the family of sets $(\mathcal{F}_m(A))_{m\in I}$ we define $(n+1)$-ary operations (compositions of functions) O_n, where $n \in I$, in the following way:

Let $f \in \mathcal{F}_n(A)$, $g_1, \ldots, g_n \in \mathcal{F}_m(A)$, $(a_1, \ldots, a_m) \in A^m$, then by $O_n(f, g_1, \ldots, g_n)$ (or shortly by $f[g_1^n]$) we shall denote the function from $\mathcal{F}_m(A)$ such that

$$f[g_1^n](a_1^m) = f(g_1(a_1^m), \ldots, g_n(a_1^m)). \tag{6.1.1}$$

It is easy to check that for any $n, m, l \in I$, $f \in \mathcal{F}_n(A)$, $g_1, \ldots, g_n \in \mathcal{F}_m(A)$ and $h_1, \ldots, h_m \in \mathcal{F}_l(A)$ the following identity (called the *superassociativity*) is satisfied:

$$f[g_1^n][h_1^m] = f[g_1[h_1^m] \ldots g_n[h_1^m]]. \tag{6.1.2}$$

Algebraic systems $((\mathcal{T}_n(A))_{n\in I}, (O_n)_{n\in I})$ and $((\mathcal{F}_n(A))_{n\in I}, (O_n)_{n\in I})$ are called *symmetric Menger systems of rank I of full* (*respectively, partial*) *multiplace functions on the set A.* Their subsystems are called *Menger systems of rank I of full* (*respectively partial*) *multiplace functions.* It is not difficult to see that in the case $I = \{1\}$ we obtain a semigroup of transformations of A, in the case $I = \{n\}$ – a Menger algebra of n-place functions.

Consider the family $(G_n)_{n\in I}$ of nonempty pairwise disjoint sets and the family $(o_n)_{n\in I}$ of partial $(n+1)$-ary operations on $G = \bigcup\limits_{n\in I} G_n$ satisfying the following conditions:

- if $n, m \in I$, $x \in G_n$, $y_1, \ldots, y_n \in G_m$, then $o_n(x, y_1^n)$ is defined and $o_n(x, y_1^n) \in G_m$,

- for all $n, m, l \in I$, $x \in G_n$, $y_1, \ldots, y_n \in G_m$, $z_1, \ldots, z_m \in G_l$ the following *superassociativity property* holds:

$$o_m(o_n(x, y_1^n), z_1^m) = o_n(x, o_m(y_1, z_1^m), \ldots, o_m(y_n, z_1^m)).$$

The system $\mathfrak{G} = ((G_n)_{n\in I}, (o_n)_{n\in I})$ is called a *Menger system of rank* I. For $I = \{n\}$ the system $\mathfrak{G}$ is a Menger algebra of rank n; for $I = \{1\}$ it is a semigroup. So, further, by analogy with the previous chapters, instead of $o_n(x, y_1^n)$ we will write $x[y_1^n]$. In this convention, the above superassociativity can be written in the form

$$x[y_1^n][z_1^m] = x[y_1[z_1^m] \ldots y_n[z_1^m]].$$

A Menger system $\mathfrak{G}$ of rank I is called *unitary* if it contains a *complete collection of selectors* of all arities, i.e., if for every $n \in I$ there are elements $e_1^{(n)}, \ldots, e_n^{(n)} \in G_n$, called *selectors*, such that

$$x[e_1^{(n)} \ldots e_n^{(n)}] = x,$$
$$e_i^{(n)}[y_1 \ldots y_n] = y_i$$

for all $i = 1, \ldots, n$ and $x \in G_n$, $y_1 \ldots, y_n \in G_m$. In a symmetrical Menger system of full multiplace functions the role of selectors is played by *projectors*, i.e., n-place functions $I_i^n \in \mathcal{T}_n(A)$ such that $I_i^n(a_1^n) = a_i$.

By a *homomorphism of a Menger system* $((G_n)_{n\in I}, (o_n)_{n\in I})$ into a Menger system $((G'_n)_{n\in I}, (o'_n)_{n\in I})$ we mean a family $(P_n)_{n\in I}$ of mappings P_n from G_n into G'_n such that

$$P_m(o_n(x, y_1^n)) = o'_n(P_n(x), P_m(y_1), \ldots, P_m(y_n))$$

holds for all $x \in G_n$, $y_1, \ldots, y_n \in G_m$. A homomorphism $(P_n)_{n\in I}$ in which all P_n are one-to-one is called an *isomorphism.*

Theorem 6.1.1. *Every unitary Menger system of rank I is isomorphic to some Menger system (of the same rank) of full multiplace functions on some set in such a way that its selectors correspond to the projectors of this set.*

Proof. Let $\mathfrak{G} = ((G_n)_{n\in I}, (o_n)_{n\in I})$ be a fixed unitary Menger system of rank I, $e_1^{(n)}, \ldots, e_n^{(n)}$ – selectors on G_n.

Denote by $\overline{G} = \bigtimes_{n\in I} G_n$ the Cartesian product of the family $(G_n)_{n\in I}$ and by $\bar{x}(n)$ the component of $\bar{x} \in \overline{G}$ which belongs to G_n.

For every $n \in I$ and every $g \in G_n$ we define on $\overline{G}$ an n-place function λ_g putting

$$\lambda_g(\bar{x}_1, \ldots, \bar{x}_n) = \bar{y} \longleftrightarrow (\forall i \in I)\Big(g[\bar{x}_1(i) \ldots \bar{x}_n(i)] = \bar{y}(i)\Big), \tag{6.1.3}$$

where $\bar{x}_1, \ldots, \bar{x}_n, \bar{y} \in \overline{G}$.

Let us show that the family $(P_n)_{n \in I}$, where $P_n : g \mapsto \lambda_g$, is an isomorphism of the system $\mathfrak{G}$ onto a Menger system $((\mathcal{T}_n(\overline{G}))_{n \in I}, (O_n)_{n \in I})$. Indeed, for $g \in G_n$, $g_1, \ldots, g_n \in G_m$, $\bar{x}_1, \ldots, \bar{x}_m, \bar{y} \in \overline{G}$, the condition $\lambda_{g[g_1^n]}(\bar{x}_1, \ldots, \bar{x}_m) = \bar{y}$ is equivalent to the condition

$$(\forall i \in I)\Big(g[g_1^n][\bar{x}_1(i) \ldots \bar{x}_m(i)] = \bar{y}(i)\Big), \tag{6.1.4}$$

which, by superassociativity, can be written in the form

$$(\forall i \in I)\Big(g[g_1[\bar{x}_1(i) \ldots \bar{x}_m(i)] \ldots g_n[\bar{x}_1(i) \ldots \bar{x}_m(i)]\,] = \bar{y}(i)\Big). \tag{6.1.5}$$

Denoting by $\bar{z}_1, \ldots, \bar{z}_n$ the elements of $\overline{G}$ such that

$$(\forall i \in I)\Big(\bar{z}_k(i) = g_k[\bar{x}_1(i) \ldots \bar{x}_m(i)]\Big) \tag{6.1.6}$$

for $k = 1, \ldots, n$, and using (6.1.5) we see that

$$(\forall i \in I)\Big(g[\bar{z}_1(i) \ldots \bar{z}_n(i)] = \bar{y}(i)\Big). \tag{6.1.7}$$

But, according to (6.1.3), the condition (6.1.6) gives $\lambda_{g_k}(\bar{x}_1, \ldots, \bar{x}_m) = \bar{z}_k$ for every $k = 1, \ldots, n$. Similarly from (6.1.7) we get $\lambda_g(\bar{z}_1, \ldots, \bar{z}_n) = \bar{y}$. Thus

$$\lambda_g(\lambda_{g_1}(\bar{x}_1, \ldots, \bar{x}_m), \ldots, \lambda_{g_n}(\bar{x}_1, \ldots, \bar{x}_m)) = \bar{y},$$

i.e.,

$$\lambda_g[\lambda_{g_1} \ldots \lambda_{g_n}](\bar{x}_1, \ldots, \bar{x}_m) = \bar{y}.$$

Therefore

$$\lambda_{g[g_1^n]}(\bar{x}_1, \ldots, \bar{x}_m) = \lambda_g[\lambda_{g_1} \ldots \lambda_{g_n}](\bar{x}_1, \ldots, \bar{x}_m)$$

for all $\bar{x}_1, \ldots, \bar{x}_m \in \overline{G}$, whence $\lambda_{g[g_1^n]} = \lambda_g[\lambda_{g_1} \ldots \lambda_{g_n}]$, i.e.,

$$P_m(g[g_1^n]) = P_n(g)[P_m(g_1) \ldots P_m(g_n)].$$

So, $(P_n)_{n \in I}$ is a homomorphism.

Now let $P_n(g_1) = P_n(g_2)$ for some $g_1, g_2 \in G_n$, $n \in I$. Then

$$P_n(g_1)(\bar{x}_1, \ldots, \bar{x}_n) = P_n(g_2)(\bar{x}_1, \ldots, \bar{x}_n)$$

for all $\bar{x}_1, \ldots, \bar{x}_n \in \overline{G}$. Thus, in particular,

$$g_1[\bar{x}_1(n) \ldots \bar{x}_n(n)] = g_2[\bar{x}_1(n) \ldots \bar{x}_n(n)].$$

Replacing each element $x_k(n)$ in this equality by the corresponding selector $e_k^{(n)}$ of G_n, we obtain $g_1[e_1^{(n)}, \dots, e_n^{(n)}] = g_2[e_1^{(n)}, \dots, e_n^{(n)}]$, which implies $g_1 = g_2$. This means that $(P_n)_{n\in I}$ is an isomorphism.

Moreover, for all $\bar{x}_1, \dots, \bar{x}_n \in \overline{G}$ and $i \in I$ we have

$$P(e_k^{(n)})(\bar{x}_1, \dots, \bar{x}_n)(i) = e_k^{(n)}[\bar{x}_1(i) \dots \bar{x}_n(i)] = \bar{x}_k(i).$$

Thus $P(e_k^{(n)})$ is the n-place projector of $\overline{G}$. This completes the proof. □

Theorem 6.1.2. *Any Menger system of rank I is isomorphic to some Menger system of the same rank of partial multiplace functions.*

Proof. Let $\mathfrak{G} = ((G_n)_{n\in I}, (o_n)_{n\in I})$ be a Menger system of rank I and let $G = \bigcup_{n\in I} G_n$, $\overline{G} = \bigtimes_{n\in I} G_n$, e – some fixed element not belonging to G. Consider the set $\mathfrak{A}_n = \overline{G}^{\,n} \cup \{(\underbrace{(e)_{n\in I}, \dots, (e)_{n\in I}}_{n})\}$, where $(e)_{n\in I} = \bigtimes_{n\in I} \{e\}$, and define for each $n \in I$ and each element $g \in G_n$ the partial n-place function $\lambda'_g : \mathfrak{A}_n \to \overline{G}$ putting

(a) $\lambda'_g((e)_{n\in I}, \dots, (e)_{n\in I}) = (g)_{n\in I} = \bigtimes_{n\in I} \{g\}$ and

(b) $\lambda'_g(\bar{x}_1, \dots, \bar{x}_n) = \bar{y} \in \overline{G}$ for $\bar{x}_1, \dots, \bar{x}_n \in \overline{G}$ such that

$$(\forall i \in I)\, (\bar{y}(i) = g[\bar{x}_1(i) \dots \bar{x}_n(i)])\,.$$

In the other cases, the function λ'_g is not defined.

Let us show that the family $(P_n)_{n\in I}$, where $P_n : g \mapsto \lambda'_g$, is an isomorphism of the system $\mathfrak{G}$ onto a Menger system $((\mathcal{F}_n(\overline{G_0}))_{n\in I}, (O_n)_{n\in I})$, where $G_0 = G \cup \{e\}$ and $\overline{G_0} = \bigtimes_{n\in I} G_0$, i.e., that for any $g \in G_n$, $g_1, \dots, g_n \in G_m$, where $n, m \in I$, the following equality

$$\lambda'_{g[g_1 \dots g_n]} = \lambda'_g[\lambda'_{g_1} \dots \lambda'_{g_n}] \tag{6.1.8}$$

is satisfied.

Using the same argumentation as in the proof of the previous theorem we can prove that for $\bar{x}_1, \dots, \bar{x}_m \in \overline{G}$ we have

$$\lambda'_{g[g_1 \dots g_n]}(\bar{x}_1, \dots, \bar{x}_m) = \lambda'_g[\lambda'_{g_1} \dots \lambda'_{g_n}](\bar{x}_1, \dots, \bar{x}_m).$$

Moreover, according to the definition of λ'_g we have also

$$\lambda'_{g[g_1 \dots g_n]}(\underbrace{(e)_{n\in I}, \dots, (e)_{n\in I}}_{m}) = (g[g_1 \dots g_n])_{n\in I}.$$

On the other hand

$$\lambda'_g[\lambda'_{g_1}\dots\lambda'_{g_n}](\underbrace{(e)_{n\in I},\dots,(e)_{n\in I}}_{m}) =$$
$$\lambda'_g\Big(\lambda_{g_1}(\underbrace{(e)_{n\in I},\dots,(e)_{n\in I}}_{m}),\dots,\lambda'_{g_n}(\underbrace{(e)_{n\in I},\dots,(e)_{n\in I}}_{m})\Big) =$$
$$\lambda'_g(\underbrace{(g_1)_{n\in I},\dots,(g_n)_{n\in I}}_{m}) = (g[g_1\dots g_n])_{n\in I}.$$

So,

$$\lambda'_{g[g_1\dots g_n]}(\underbrace{(e)_{n\in I},\dots,(e)_{n\in I}}_{m}) = \lambda'_g[\lambda'_{g_1}\dots\lambda'_{g_n}](\underbrace{(e)_{n\in I},\dots,(e)_{n\in I}}_{m}),$$

which proves that (6.1.8) is satisfied in any case. Thus $(P_n)_{n\in I}$ is a homomorphism. In fact, it is an isomorphism because $\lambda'_{g_1} = \lambda'_{g_2}$ for some $g_1, g_2 \in G_n$, $n \in I$, implies

$$\lambda'_{g_1}((e)_{n\in I},\dots,(e)_{n\in I}) = \lambda'_{g_2}((e)_{n\in I},\dots,(e)_{n\in I}),$$

whence $g_1 = g_2$. □

Theorem 6.1.3. *Any Menger system of rank I of partial multiplace functions is isomorphic to some Menger system of the same rank of full multiplace functions.*

Proof. Let $((\Phi_n)_{n\in I}, (O_n)_{n\in I})$, where $\Phi_n \subset \mathcal{F}_n(A)$, be a Menger system of partial multiplace functions of rank I and let $c \notin A$ be some fixed element. We extend every partial function $f \in \Phi_n$ to a full function f^c of the same arity defined on the set $A^c = A \cup \{c\}$ putting

$$f^c(x_1,\dots,x_n) = \begin{cases} f(x_1,\dots,x_n) & \text{if } (x_1,\dots,x_n) \in \mathrm{pr}_1 f, \\ c, & \text{if } (x_1,\dots,x_n) \notin \mathrm{pr}_1 f. \end{cases}$$

An extension $f \mapsto f^c$ defined in this way is an isomorphism. To prove this fact assume that $f \in \Phi_n$ and $g_1,\dots,g_n \in \Phi_m$ are arbitrary and consider two cases:

(a) $\bar{x} = (x_1,\dots,x_m) \in \mathrm{pr}_1 f[g_1\dots g_n]$,

(b) $\bar{x} = (x_1,\dots,x_m) \notin \mathrm{pr}_1 f[g_1\dots g_n]$.

In the first case we have

$$(f[g_1\dots g_n])^c(\bar{x}) = f[g_1\dots g_n](\bar{x}). \tag{6.1.9}$$

But from $\bar{x} \in \mathrm{pr}_1 f[f, g_1 \dots g_n]$ it follows $\bar{x} \in \mathrm{pr}_1 g_k$ for all $k = 1, \dots, n$, and $(g_1(\bar{x}), \dots, g_n(\bar{x})) \in \mathrm{pr}_1 f$. Consequently, $g_k^c(\bar{x}) = g_k(\bar{x})$, for every $k = 1, \dots, n$, and $f^c(g_1(\bar{x}), \dots, g_n(\bar{x})) = f(g_1(\bar{x}), \dots, g_n(\bar{x}))$. Hence

$$\begin{aligned} f^c[g_1^c \dots g_n^c](\bar{x}) &= f^c(g_1^c(\bar{x}), \dots, g_n^c(\bar{x})) = f^c(g_1(\bar{x}), \dots, g_n(\bar{x})) \\ &= f(g_1(\bar{x}), \dots, g_n(\bar{x})) = f[g_1 \dots g_n](\bar{x}). \end{aligned}$$

So, according to (6.1.9), we obtain

$$(f[g_1 \dots g_n])^c(\bar{x}) = f^c[g_1^c \dots g_n^c](\bar{x}) \tag{6.1.10}$$

for all $\bar{x} \in \mathrm{pr}_1 f[g_1 \dots g_n]$.

For $\bar{x} \notin \mathrm{pr}_1 f[g_1 \dots g_n]$ we have $(f[g_1 \dots g_n])^c(\bar{x}) = c$. But from $\bar{x} \notin \mathrm{pr}_1 f[g_1 \dots g_n]$ it follows that either (i) $\bar{x} \notin \mathrm{pr}_1 g_k$ for some $k \in \{1, \dots, n\}$ or (ii) $\bar{x} \in \mathrm{pr}_1 g_k$ for every $k = 1, \dots, n$ and $(g_1(\bar{x}), \dots, g_n(\bar{x})) \notin \mathrm{pr}_1 f$. If $\bar{x} \notin \mathrm{pr}_1 g_k$, then $g_k^c(\bar{x}) = c$, whence $f^c[g_1^c \dots g_n^c](g_1^c(\bar{x}) \dots c \dots g_n^c(\bar{x})) = c$. In the second case $(g_1(\bar{x}), \dots, g_n(\bar{x})) \notin \mathrm{pr}_1 f$ implies $f^c(g_1(\bar{x}), \dots, g_n(\bar{x})) = c$. So, in view of $g_k^c(\bar{x}) = g_k(\bar{x})$ for all $k = 1, \dots, n$, we get

$$f^c[g_1^c \dots g_n^c](\bar{x}) = f^c(g_1^c(\bar{x}), \dots, g_n^c(\bar{x})) = f^c(g_1(\bar{x}), \dots, g_n(\bar{x})) = c,$$

which means that (6.1.10) is satisfied in this case too. Thus, we have proved

$$(f[g_1 \dots g_n])^c = f^c[g_1^c \dots g_n^c]$$

for all $f \in \mathbf{\Phi}_n$, $g_1, \dots, g_n \in \mathbf{\Phi}_m$, $n, m \in I$. Hence $f \mapsto f^c$ is a homomorphism. It is clear that it is one-to-one. □

Corollary 6.1.4. *Any Menger system is isomorphic to some Menger system of full multiplace functions.*

Theorem 6.1.5. *Any Menger system of full multiplace functions can be isomorphically embedded into a Menger system* (*of the same rank*) *of full multiplace functions containing a complete collection of projectors.*

Proof. Let $((\mathbf{\Phi}_n)_{n\in I}, (O_n)_{n\in I})$ be a Menger system of rank I of full multiplace functions on the set A, i.e., $\mathbf{\Phi}_n \subset \mathcal{T}_n(A)$ for every $n \in I$. Denote by $(\mathbf{\Phi}_n^0)_{n\in I}$ the family of subsets of $\mathcal{T}_n(A)$ such that

$$\mathbf{\Phi}_n^0 = \mathbf{\Phi}_n \cup \{I_1^n, \dots, I_n^n\},$$

where $I_1^n, \dots, I_n^n$ are n-ary projectors of A. Next, for every $k \in \mathbb{N}$, we define the family $(\mathbf{\Phi}_n^{k+1})_{n\in I}$ of subsets of $\mathcal{T}_n(A)$ satisfying the condition

$$f \in \mathbf{\Phi}_n^{k+1} \longleftrightarrow (\exists m \in I)(\exists g \in \mathbf{\Phi}_m^k)(\exists h_1, \dots, h_m \in \mathbf{\Phi}_n^k) f = g[h_1 \dots h_m].$$

It is clear that $I_1^n, \dots, I_n^n \in \Phi_n^k$ and $\Phi_n^k \subset \Phi_n^{k+1}$ for every $k \in \mathbb{N}$, $n \in I$. Thus, we have the sequence $(\Phi_n^0)_{n\in I}, (\Phi_n^1)_{n\in I}, \dots, (\Phi_n^k)_{n\in I}, \dots$ Using this sequence we define the family of subsets $(\Phi_n^*)_{n\in I}$ putting

$$\Phi_n^* = \bigcup_{k=0}^{\infty} \Phi_n^k.$$

This family is stable with respect to the superpositions $(O_n)_{n\in I}$. Indeed, if $f \in \Phi_n^*$, $g_1, \dots, g_n \in \Phi_m^*$ for some $n, m \in I$, then, according to the definition of Φ_n^*, there are $k_0, k_1, \dots, k_n \in \mathbb{N}$ such that

$$f \in \Phi_n^{k_0},\ g_1 \in \Phi_m^{k_1},\ \dots, g_n \in \Phi_m^{k_n}.$$

For $k = \max\{k_0, k_1, \dots, k_n\}$ we have $f \in \Phi_n^k$, $g_1, \dots, g_n \in \Phi_m^k$, whence $f[g_1 \dots g_n] \in \Phi_m^{k+1} \subset \Phi_m^*$, consequently $f[g_1 \dots g_n] \in \Phi_m^*$.

So, $((\Phi_n^*)_{n\in I}, (O_n)_{n\in I})$ is a unitary Menger system of rank I of full multiplace functions in which the role of selectors play projectors of A. Obviously this system contains $((\Phi_n)_{n\in I}, (O_n)_{n\in I})$ as a subsystem. □

Corollary 6.1.6. *Any Menger system $\mathcal{G}$ can be isomorphically embedded into a unitary Menger system $\mathcal{G}^*$ of the same rank, whose selectors do not belong to $\mathcal{G}$.*

Further the system, $\mathcal{G}^*$ will be called a *Menger system obtained from $\mathcal{G}$ by addition of a complete collection of selectors.*

Let $\mathcal{G} = ((G_n)_{n\in I}, (o_n)_{n\in I})$ be a Menger system of rank I. On the set $G = \bigcup_{n\in I} G_n$ we define a binary operation $\cdot$ putting $x \cdot y = x[y \dots y]$ for all $x, y \in G$. Then $(G, \cdot)$ is a semigroup which is called the *diagonal semigroup* of a Menger system $\mathcal{G}$.

Consider a homomorphism P of an arbitrary semigroup $(G, \cdot)$ onto a right zero semigroup I, where $I \subset \mathbb{N}$.[1] Let $G_n = \{g \in G \,|\, P(g) = n\}$. It is clear that $G = \bigcup_{n\in I} G_n$ and $G_n \cap G_m = \varnothing$ for $n \neq m$.

Theorem 6.1.7. *If there exists a homomorphism of a semigroup $(G, \cdot)$ onto a left zero semigroup I, where $I \subset \mathbb{N}$, and for each $n \in I$ there exist an n-place function f_n defined on G such that*
(a) $(g_1, \dots, g_n) \in pr_1 f_n \longleftrightarrow (\exists m \in I)\,(g_1, \dots, g_n \in G_m)$,
(b) $f_n(g_1, \dots, g_n) \in G_m$, *if* $g_1, \dots g_n \in G_m$,
(c) $f_n(g, \dots, g) = g$,
(d) $f_n(g_1, \dots, g_n) \cdot g = f_n(g_1 g, \dots, g_n g)$ *for all* $(g_1, \dots, g_n) \in pr_1 f_n$,
then $(G, \cdot)$ is the diagonal semigroup of some Menger system of rank I.

[1] Remind that by a *right zero semigroup* is mean a semigroup S with the multiplication defined by $ab = b$ for all $a, b \in S$.

Proof. Let all the conditions of the theorem be satisfied and let P be a homomorphism of a semigroup $(G, \cdot)$ onto a right zero semigroup I. Then for any $x \in G_n$, $y \in G_m$ we have $P(x \cdot y) = P(x)P(y) = P(y)$, which means that $x \cdot y$ and y are in the same G_m. Next, for every $n \in I$ we define an $(n+1)$-ary operator o_n putting

$$o_n(x, y_1, \ldots, y_n) = x \cdot f_n(y_1, \ldots, y_n)$$

for all $x \in G_n$ and $y_1, \ldots, y_n \in G_m$. Obviously $o_n(x, y_1, \ldots, y_n) \in G_m$.

Denoting $o_n(x, y_1, \ldots, y_n)$ by $x[y_1^n]$ and using conditions (a), (b), and (d), we can see that for all $x \in G_n$, $y_1, \ldots, y_n \in G_m$, $z_1, \ldots, z_m \in G_l$ we have

$$\begin{aligned} x[y_1^n][z_1^m] &= (x \cdot f_n(y_1, \ldots, y_n)) \cdot f_m(z_1, \ldots, z_m) \\ &= x \cdot (f_n(y_1, \ldots, y_n) \cdot f_m(z_1, \ldots, z_m)) \\ &= x \cdot f_n(y_1 \cdot f_m(z_1, \ldots, z_m), \ldots, y_n \cdot f_m(z_1, \ldots, z_m)) \\ &= x \cdot f_n(y_1[z_1^m] \ldots y_n[z_1^m]) = x[y_1[z_1^m] \ldots y_n[z_1^m]], \end{aligned}$$

i.e.,

$$x[y_1^n][z_1^m] = x[y_1[z_1^m] \ldots y_n[z_1^m]],$$

which means that $\mathcal{G} = ((G_n)_{n\in I}, (o_n)_{n\in I})$ is a Menger system of rank I.

Let $(G, *)$ be the diagonal semigroup of this Menger system. Then $G = \bigcup_{n\in I} G_n$, $x * y = x[y \ldots y]$ and

$$x * y = o_n(x, y, \ldots, y) = x \cdot f_n(y, \ldots, y) = x \cdot y,$$

according to the definition of o_n and condition (c) of the theorem. So, $(G, *)$ coincides with $(G, \cdot)$. □

Definition 6.1.8. A Menger system $((G_n)_{n\in I}, (o_n)_{n\in I})$ of rank I is called *group-like*, if for any $n, m \in I$, $a_0 \in G_n$, $a_1, \ldots, a_n, b \in G_m$, $i = 1, \ldots, n$ the following two equations

$$x_0[a_1] \ldots a_n] = b,$$
$$a_0[a_1 \ldots a_{i-1} x_i a_{i+1} \ldots a_n] = b$$

have unique solutions $x_0 \in G_n$, $x_i \in G_m$.

Using similar methods as those in section 2.5 we can prove that *a Menger system $\mathcal{G} = ((G_n)_{n\in I}, (o_n)_{n\in I})$ of rank I is group-like if and only if there exists a family of elements $(e_n)_{n\in I}$, $e_n \in G_n$, such that $e_m[b \ldots b] = b$, $a[e_n \ldots e_n] = a$ for all $n, m \in I$, $a \in G_n$, $b \in G_m$, and the equations*

$$x_0[a_1 \ldots a_n] = e_m,$$
$$a_0[a_1 \ldots a_{i-1} x_i a_{i+1} \ldots a_n] = e_m$$

have unique solutions $x_0 \in G_n$, $x_i \in G_m$ *for all* $n, m \in I$, $a_0 \in G_n$, $a_1, \ldots, a_n \in G_m$, $i = 1, \ldots, n$.

It is clear that the diagonal semigroup $(G, \cdot)$ of a group-like Menger system $\mathcal{G}$ is a union of the diagonal groups $(G_n, \cdot)$. Moreover, any two groups $(G_n, \cdot)$, $(G_m, \cdot)$, $n, m \in I$, are isomorphic. This isomorphism has the form $x \mapsto x \cdot e_m$, where $x \in G_n$ and e_m is the unit of $(G_m, \cdot)$. The group $(G_{n_0}, \cdot)$, where $n_0 = \min I$, is called the *main diagonal group* of $\mathcal{G}$ and is denoted by $(G_0, \cdot)$. The isomorphism of $(G_n, \cdot)$ onto $(G_0, \cdot)$ is denoted by g_n. It is not difficult to see that $g_m(xy) = g_n(x)g_m(y)$ for any $n, m \in I$, $x \in G_n$, $y \in G_m$.

Proposition 6.1.9. *The diagonal semigroup* $(G, \cdot)$ *of a group-like Menger system* $\mathcal{G}$ *is isomorphic to a direct product of its main diagonal group and* I *considered as the right zero semigroup.*

Proof. Indeed, if $x \in G$, then $x \in G_n$ for some $n \in I$ and, as it is not difficult to see, the mapping $P : x \mapsto (g_n(x), n)$ is an isomorphism of the diagonal semigroup $(G, \cdot)$ onto a direct product $G_0 \times I$. □

Let $\mathcal{G} = ((G_n)_{n \in I}, (o_n)_{n \in I})$ be a group-like Menger system. Consider on $\mathcal{G}$ a family $(f_n^m)_{n \in I}$ of $(n-1)$-ary operations defined in the following way:

$$f_n^m(x_1, \ldots, x_{n-1}) = e_n[x_1 \ldots x_{n-1} e_m], \tag{6.1.11}$$

where $x_1, \ldots, x_{n-1} \in G_m$ and e_n, e_m are identities of the corresponding groups $(G_n, \cdot)$ and $(G_m, \cdot)$. Since $\mathcal{G}$ is group-like, each groupoid (G_m, f_n^m) is an $(n-1)$-ary quasigroup. Moreover, for all $x \in G_n$, $y_1, \ldots, y_n \in G_m$, $n, m \in I$, we have

$$x[y_1 \ldots y_n] = x \cdot f_n^m(y_1 \cdot y_n^{-1}, \ldots, y_{n-1} \cdot y_n^{-1}) \cdot y_n,$$

where y_n^{-1} is inverse element in $(G_m, \cdot)$. In the case $m = \min I$ the operation f_n^m will be denoted by f_n.

Directly from the definition of f_n^m, it follows that any two $(n-1)$-ary quasigroups (G_m, f_n^m) and (G_k, f_n^k) are isomorphic.

The system $(G_0, \cdot, (f_n)_{n \in I})$, where $(G_0, \cdot)$ is the main diagonal group of $\mathcal{G}$ is called the *main rigged group* of $\mathcal{G}$.

From the above discussion we conclude the following proposition.

Proposition 6.1.10. *Two group-like Menger systems are isomorphic if and only if their main rigged groups are isomorphic.*

The next theorem gives a characterization of these semigroup which are diagonal group of group-like Menger systems.

Theorem 6.1.11. *A semigroup* $(S, \cdot)$ *is isomorphic to the diagonal semigroup of a group-like Menger system* $\mathcal{G}$ *of rank* I *if and only if*

(a) *it is isomorphic to the direct product of some group* $(G,\cdot)$ *and the right zero semigroup* I,

(b) *for any* $n \in I$, $n > 1$, *on* $(G,\cdot)$ *can be defined an* $(n-1)$*-ary quasigroup operation* f_n *with the property* $f_n(e,\dots,e) = e$, *where* e *is the identity of* $(G,\cdot)$,

(c) *the equation*

$$f_n(a_1 \cdot x, \dots, a_{n-1} \cdot x) = b \cdot x$$

has a unique solution $x \in G$ *for any* $a_1,\dots,a_{n-1}, b \in G$.

Proof. Necessity. Assume that a semigroup $(S,\cdot)$ is isomorphic to the diagonal semigroup of a group-like Menger system $\mathcal{G} = ((G_n)_{n\in I}, (o_n)_{n\in I})$. Then, according to Proposition 6.1.9, it is isomorphic to the direct product of the main diagonal group $(G_0,\cdot)$ of $\mathcal{G}$ and the right zero semigroup I. So, the condition (a) is satisfied. Next, using the identity e_n of a group $(G_n,\cdot)$ and (6.1.11), for any $n > 1$, $n \in I$, we define an $(n-1)$-ary quasigroup operation f_n such that $f_n(e,\dots,e) = e_n[e\dots ee] = e$ for the identity e of $(G_0,\cdot)$. Since the equation $a_0[a_1\dots a_{n-1}x] = b$ has a unique solution x, $a_0 \cdot f_n(a_1\cdot x^{-1},\dots,a_{n-1}\cdot x^{-1})\cdot x = b$ has also a unique solution. Hence, there is only one x satisfying the equation

$$f_n(a_1 \cdot x^{-1}, \dots, a_{n-1} \cdot x^{-1}) = a_0^{-1} \cdot b \cdot x^{-1}.$$

This completes the proof of (b) and (c).

Sufficiency. Let $(G,\cdot)$ be a group satisfying all the conditions of the theorem. Then $G_n = G \times \{n\} \subset G \times I$ is a group and $G_n \cap G_m = \varnothing$ if $n \neq m$.

Let us define on $G \times I$ a family of $(n+1)$-ary operations o_n such that

$$(x,n)[(y_1,m)\dots(y_n,m)] = (x \cdot f_n(y_1 \cdot y_n^{-1},\dots,y_{n-1}\cdot y_n^{-1}) \cdot y_n, m)$$

for all $x, y_1,\dots,y_n \in G$ and $n,m \in I$, we obtain a group-like Menger system $\mathcal{G} = ((G_n)_{n\in I}, (o_n)_{n\in I})$. Indeed, direct but very long computations show that any of the operations o_n is superassociative. To prove that this system is group-like observe that the equation

$$(x,n)[(a_1,m)\dots(a_n,m)] = (b,m),$$

is equivalent to the equation

$$(x \cdot f_n(a_1 \cdot a_n^{-1},\dots,a_{n-1}\cdot a_n^{-1}) \cdot a_n, m) = (b,m),$$

which has a unique solution $x = b \cdot a_n^{-1} \cdot (f_n(a_1 \cdot a_n^{-1},\dots,a_{n-1}\cdot a_n^{-1}))^{-1}$.

The equation

$$(a_0,n)[(a_1,m)\dots(x_i,m)\dots(a_n,m)] = (b,m), \tag{6.1.12}$$

for $1 \leqslant i < n$, can be written in the form

$$f_n(a_1 \cdot a_n^{-1}, \ldots, x_i \cdot a_n^{-1}, \ldots, a_{n-1} \cdot a_n^{-1}) = a_0^{-1} \cdot b \cdot a_n^{-1}. \tag{6.1.13}$$

Since f_n is an $(n-1)$-ary quasigroup operation, there is only one x_i satisfying (6.1.13). So, (x_i, m) is a unique solution of (6.1.12).

For $i = n$, the equation (6.1.12) can be written in the form

$$f_n(a_1 \cdot x_n^{-1}, \ldots, a_{n-1} \cdot x_n^{-1}) = a_0^{-1} \cdot b \cdot x_n^{-1}. \tag{6.1.14}$$

According to condition (c) of the theorem, there exists only one x_n satisfying this equation. So, also in this case (6.1.12) has a unique solution. Hence $\mathcal{G}$ is a group-like Menger system.

Its diagonal semigroup H is isomorphic to the direct product of its main diagonal group G_0 and the right zero semigroup I. But, by construction, G_0 is isomorphic to G, thus $S \cong G_0 \times I \cong H$, which completes the proof. □

By *determining pair* of a Menger system $\mathcal{G} = ((G_n)_{n\in I}, (o_n)_{n\in I})$, we mean a pair $(\overline{\mathcal{E}}, \overline{W}) = ((\mathcal{E}_n)_{n\in I}, (W_n)_{n\in I})$, where $\mathcal{E}_n$ is the equivalence relation on the set G_n, W_n – the empty set or the $\mathcal{E}_n$-class of G_n satisfying for all $g \in G_n$, $x, x_1, \ldots, x_n, y_1, \ldots, y_n \in G_m$, $n, m \in I$ the following two conditions:

$$\bigwedge_{i=1}^{n} x_i \equiv y_i(\mathcal{E}_m) \longrightarrow g[x_1 \ldots x_n] \equiv g[y_1 \ldots y_n](\mathcal{E}_m),$$

$$x \in W_m \longrightarrow g[y_1 \ldots y_{i-1} x y_{i+1} \ldots y_n] \in W_m.$$

Let $\mathcal{G}^*$ be a Menger system obtained from $\mathcal{G}$ by addition of a complete collection of selectors $e_1^{(n)}, \ldots, e_n^{(n)}$ such that $e_i^{(n)} \notin G_n$ for each $n \in I$ and $i = 1, \ldots, n$. For the determining pair $(\overline{\mathcal{E}}, \overline{W})$ we denote by $(H_{a_n})_{a_n\in A_m}$ the family of the $\mathcal{E}_n$-classes which are disjoint from W_n and are uniquely indexed by elements of the set A_m. We assume that $A_n \cap A_m = \varnothing$ for $n \neq m$.

Let us consider the sets

$$E_n^i = \bigtimes_{m<n} G_m \times \{e_i^{(n)}\} \times \bigtimes_{m>n} G_m, \qquad D_n = \overline{G}^{\,n} \cup E_n^1 \times \cdots \times E_n^n,$$

$$B_n^i = \bigtimes_{m<n} A_m \times \{b_n^i\} \times \bigtimes_{m>n} A_m, \qquad \mathfrak{A}_n = \overline{A}^{\,n} \cup B_n^1 \times \cdots \times B_n^n,$$

where $\overline{A} = \bigtimes_{n\in I} A_n$, $\overline{G} = \bigtimes_{n\in I} G_n$ and $H_{b_n^i} = \{e_i^{(n)}\}$ for $b_n^i \notin A_n$. With each $g \in G_n$ we associate the n-place function $P^n_{(\overline{\mathcal{E}},\overline{W})}(g)$ defined in the following way:

$$(\bar{a}_1, \ldots, \bar{a}_n, \bar{c}) \in P^n_{(\overline{\mathcal{E}},\overline{W})}(g) \longleftrightarrow \begin{cases} (\bar{a}_1, \ldots, \bar{a}_n) \in \mathfrak{A}_n \ \text{ and} \\ (\forall k \in I)\Big(g[H_{\bar{a}_1(k)} \ldots H_{\bar{a}_n(k)}] \subset H_{\bar{c}(k)}\Big), \end{cases}$$

where $\bar{a}(k)$ is the component of the vector $\bar{a} \in \overline{A}$ which belongs to A_k.

One can prove that for all $n, m \in I$, $g \in G_n$, $g_1, \dots, g_n \in G_m$ we have

$$P^m_{(\overline{\mathcal{E}},\overline{W})}(g[g_1 \dots g_n]) = P^n_{(\overline{\mathcal{E}},\overline{W})}(g)[P^m_{(\overline{\mathcal{E}},\overline{W})}(g_1) \dots P^m_{(\overline{\mathcal{E}},\overline{W})}(g_n)].$$

This means that the family $\overline{P}_{(\overline{\mathcal{E}},\overline{W})} = \left(P^n_{(\overline{\mathcal{E}},\overline{W})}\right)_{n\in I}$ is a representation of a Menger system $\mathcal{G}$ by multiplace functions. This representation is called *simplest*.

With each simplest representation $\overline{P}_{(\overline{\mathcal{E}},\overline{W})}$ of a Menger system $\mathcal{G}$ we associate the system $\overline{\zeta}_{(\overline{\mathcal{E}},\overline{W})} = \left(\zeta^n_{(\overline{\mathcal{E}},\overline{W})}\right)_{n\in I}$ of binary relations defined on the sets of the family $(G_n)_{n\in I}$ in the following way:

$$(g_1, g_2) \in \zeta^n_{(\overline{\mathcal{E}},\overline{W})} \longleftrightarrow P^n_{(\overline{\mathcal{E}},\overline{W})}(g_1) \subset P^n_{(\overline{\mathcal{E}},\overline{W})}(g_2).$$

In other words $(g_1, g_2) \in \zeta^n_{(\overline{\mathcal{E}},\overline{W})}$ if and only if for every $(\bar{x}_1, \dots, \bar{x}_n) \in D_n$ and every $k \in I$ the following implication is valid:

$$g_1[\bar{x}_1(k) \dots \bar{x}_n(k)] \notin W_n \longrightarrow g_1[\bar{x}_1(k) \dots \bar{x}_n(k)] \equiv g_2[\bar{x}_1(k) \dots \bar{x}_n(k)](\mathcal{E}_k).$$

Now we consider the family $(T_n)_{n\in I}$ of polynomials T_n defined on $\mathcal{G}$ and satisfying the following two conditions:

- $x_n \in T_n$, where x_n is an independent variable such that $x_n \neq x_m$ for $n \neq m$,
- if $t \in T_n$, $g \in G_m$, $g_1, \dots, g_n \in G_n$, then $g[g_1 \dots g_{i-1}\, t\, g_{i+1} \dots g_m] \in T_n$ for all $i = 1, \dots, m$.

Using the polynomials T_n and the family $\mathbf{H} = (H_n)_{n\in I}$ of subsets H_n of G_n we define on $\mathcal{G}$ the family $\mathcal{E}_{\mathbf{H}} = (\mathcal{E}_{H_n})_{n\in I}$ of binary relations $\mathcal{E}_{H_n}$ on $G_n \times G_n$ and the family $W_{\mathbf{H}} = (W_{H_n})_{n\in I}$ of subsets $W_{H_n} \subset G_n$ putting

$$g_1 \equiv g_2(\mathcal{E}_{H_n}) \longleftrightarrow (\forall t \in T_n)\Big(t(g_1) \in H_n \longleftrightarrow t(g_2) \in H_n\Big),$$

$$g \in W_{H_n} \longleftrightarrow (\forall t \in T_n)\Big(t(g) \notin H_n\Big),$$

where $g, g_1, g_2 \in G_n$, $n \in I$. It is not difficult to show that $(\mathcal{E}_{\mathbf{H}}, W_{\mathbf{H}})$ is the determining pair of $\mathcal{G}$.

A system $((\Phi_n)_{n\in I}, (O_n)_{n\in I}, (\zeta_{\Phi_n})_{n\in I})$, where $((\Phi_n)_{n\in I}, (O_n)_{n\in I})$ is a Menger system of partial multiplace functions, while ζ_{Φ_n} the relation of inclusion of elements of the set Φ_n, is called a *fundamentally ordered Menger system of multiplace functions*.

Theorem 6.1.12. *A system $((G_n)_{n\in I}, (o_n)_{n\in I}, (\zeta_n)_{n\in I})$ is isomorphic to some fundamentally ordered Menger system of partial multiplace functions of rank*

I if and only if $((G_n)_{n\in I},(o_n)_{n\in I})$ is a Menger system of rank I, $(\zeta_n)_{n\in I}$ is the family of order relations on sets of the family $(G_n)_{n\in I}$, and the following conditions:

$$g_1 \leqslant_n g_2 \wedge \bigwedge_{i=1}^{n} x_i \leqslant_m y_i \longrightarrow g_1[x_1\ldots x_n] \leqslant_m g_2[y_1\ldots y_n], \qquad (6.1.15)$$

$$g_1 \leqslant_n g_2 \wedge g \leqslant_n t_1(g_1) \wedge g \leqslant_n t_2(g_2) \longrightarrow g \leqslant_n t_2(g_1) \qquad (6.1.16)$$

are satisfied for all $g_1, g_2 \in G_n$, $x_i, y_i \in G_m$, $i = 1,\ldots,n$, $t_1, t_2 \in T_n$, where $x \leqslant_n y \longleftrightarrow (x,y) \in \zeta_n$.

Proof. The necessity of these conditions can be proved without any difficulties, so we shall only prove the sufficiency. In order to do this assume that all the conditions of the theorem are satisfied and consider the representation $\overline{P} = (P_n)_{n\in I}$ of a Menger system $((G_n)_{n\in I},(o_n)_{n\in I})$, where [2]

$$P_n = \sum_{\bar{g}\in\overline{G}} P^n_{(\mathcal{E}_{\zeta\langle\bar{g}\rangle},W_{\zeta\langle\bar{g}\rangle})}, \qquad (6.1.17)$$

$\zeta\langle\bar{g}\rangle = (\zeta_n\langle\bar{g}(n)\rangle)_{n\in I}$, $\mathcal{E}_{\zeta\langle\bar{g}\rangle} = (\mathcal{E}_{\zeta_n\langle\bar{g}(n)\rangle})_{n\in I}$, $W_{\zeta\langle\bar{g}\rangle} = (W_{\zeta_n\langle\bar{g}(n)\rangle})_{n\in I}$.

Obviously, $\overline{P}$, as sum of simplest representations, is a representation by multiplace functions.

Let us show that $\zeta_n = \zeta_{P_n}$ for every $n \in I$, where

$$\zeta_{P_n} = \{(g_1,g_2) \in G_n \times G_n \mid P_n(g_1) \subset P_n(g_2)\}.$$

Let $g_1, g_2 \in G_n$ and $g_1 \leqslant_n g_2$. Then for arbitrary $(\bar{x}_1,\ldots,\bar{x}_n) \in D_n$, $k \in I$, such that

$$g_1[\bar{x}_1(k)\ldots\bar{x}_n(k)] \notin W_{\zeta_k\langle\bar{g}(k)\rangle}$$

there exists $t_1 \in T_k$ for which

$$\bar{g}(k) \leqslant_k t_1(g_1[\bar{x}_1(k)\ldots\bar{x}_n(k)]). \qquad (6.1.18)$$

If for some $t \in T_k$ we have

$$\bar{g}(k) \leqslant_k t(g_1[\bar{x}_1(k)\ldots\bar{x}_n(k)]), \qquad (6.1.19)$$

then $g_1 \leqslant_n g_2$ together with the first implication formulated in the theorem gives

$$g_1[\bar{x}_1(k)\ldots\bar{x}_n(k)] \leqslant_k g_2[\bar{x}_1(k)\ldots\bar{x}_n(k)]. \qquad (6.1.20)$$

Consequently,

$$t(g_1[\bar{x}_1(k)\ldots\bar{x}_n(k)]) \leqslant_k t(g_2[\bar{x}_1(k)\ldots\bar{x}_n(k)])$$

[2] The sum of representations of a Menger system is defined in the same way as in the case of Menger algebras (see p. 78).

for any $t \in T_k$. Thus

$$\bar{g}(k) \leqslant_k t(g_2[\bar{x}_1(k) \ldots \bar{x}_n(k)]). \tag{6.1.21}$$

So, (6.1.19) implies (6.1.21).

On the other hand, if (6.1.21) holds, then from (6.1.20), (6.1.18) and (6.1.21), by (6.1.15), we get (6.1.19). In this way, we have proved that for any $g_1, g_2 \in G_n$, $g_1 \leqslant_n g_2$ and $g_1[\bar{x}_1(k) \ldots \bar{x}_n(k)] \notin W_{\zeta_k\langle \bar{g}(k)\rangle}$ imply

$$g_1[\bar{x}_1(k) \ldots \bar{x}_n(k)] \equiv g_2[\bar{x}_1(k) \ldots \bar{x}_n(k)](\mathcal{E}_{\zeta_k\langle \bar{g}(k)\rangle}).$$

Therefore $(g_1, g_2) \in \zeta^n_{(\mathcal{E}_{\zeta\langle \bar{g}\rangle}, W_{\zeta\langle \bar{g}\rangle})}$ for every $\bar{g} \in \overline{G}$, whence $\zeta_n \subset \zeta_{P_n}$, because

$$\zeta_{P_n} = \bigcap_{\bar{g} \in \overline{G}} \zeta^n_{(\mathcal{E}_{\zeta\langle \bar{g}\rangle}, W_{\zeta\langle \bar{g}\rangle})}.$$

Conversely, if $(g_1, g_2) \in \zeta_{P_n}$, then, for every $\bar{g} \in \overline{G}$, $(\bar{x}_1, \ldots, \bar{x}_n) \in D_n$ and $k \in I$, from $g_1[\bar{x}_1(k) \ldots \bar{x}_n(k)] \notin W_{\zeta_k\langle \bar{g}(k)\rangle}$ it follows that

$$g_1[\bar{x}_1(k) \ldots \bar{x}_n(k)] \equiv g_2[\bar{x}_1(k) \ldots \bar{x}_n(k)](\mathcal{E}_{\zeta_k\langle \bar{g}(k)\rangle}).$$

This means that for $k = n$ and $(\bar{x}_1, \ldots, \bar{x}_n) \in E^1_n \times \cdots \times E^n_n$, the following implication is valid:

$$g_1[e^{(n)}_1 \ldots e^{(n)}_n] \notin W_{\zeta_n\langle \bar{g}(n)\rangle} \longrightarrow g_1[e^{(n)}_1 \ldots e^{(n)}_n] \equiv g_1[e^{(n)}_1 \ldots e^{(n)}_n](\mathcal{E}_{\zeta_n\langle \bar{g}(n)\rangle}),$$

i.e.,

$$g_1 \notin W_{\zeta_n\langle \bar{g}(n)\rangle} \longrightarrow g_1 \equiv g_2(\mathcal{E}_{\zeta_n\langle \bar{g}(n)\rangle}),$$

whence, for $\bar{g}(n) = g_1$, we obtain the implication

$$g_1 \notin W_{\zeta_n\langle g_1\rangle} \longrightarrow g_1 \equiv g_2(\mathcal{E}_{\zeta_n\langle g_1\rangle}).$$

Since $g_1 \notin W_{\zeta_n\langle g_1\rangle}$ for every $g_1 \in G_n$, the above proves $g_1 \equiv g_2(\mathcal{E}_{\zeta_n\langle g_1\rangle})$. Thus

$$(\forall t \in T_n) \Big(g_1 \leqslant_n t(g_1) \longleftrightarrow g_1 \leqslant_n t(g_2) \Big).$$

From this, replacing the polynomial t by an individual variable, we conclude $g_1 \leqslant_n g_2$, i.e., $(g_1, g_2) \in \zeta_n$. Thus $\zeta_n = \zeta_{P_n}$.

This together with the fact that ζ_n is an order relation for every $n \in I$, completes the proof that $\overline{P}$ is an isomorphism. □

6.2 Menger T-systems

In the previous section, we considered superpositions of functions, which allowed us to substitute into a function, instead of its arguments, various functions of the same arity. The given restriction did not allow us to substitute in the place of the arguments functions of different arities. This inconvenience could be avoided in many ways. One such way will be discussed in this paragraph.

Consider the family $(\mathcal{F}_n(A))_{n\in I}$ of multiplace functions on a nonempty set A. On $(\mathcal{F}_n(A))_{n\in I}$ we can define a family $(\mathcal{O}_n)_{n\in I}$ of $(n+1)$-ary operations $\mathcal{O}_n$, which from $f\in\mathcal{F}_n(A)$, $g_1\in\mathcal{F}_{m_1}(A)$, …, $g_n\in\mathcal{F}_{m_n}(A)$ form a new multiplace function $\mathcal{O}_n(f,g_1,\ldots,g_n)\in\mathcal{F}_m(A)$, where $m=\max\{m_1,\ldots,m_n\}$. This new function is denoted by $f\langle g_1\ldots g_n\rangle$ and is uniquely determined by the following equality:

$$f\langle g_1\ldots g_n\rangle(a_1,\ldots,a_m)=f(g_1(a_1,\ldots,a_{m_1}),\ldots,g_n(a_1,\ldots,a_{m_n})),$$

where $(a_1,\ldots,a_m)\in A^m$. It is assumed that the left- and right -hand sides of this equality are simultaneously defined or not defined.

It can be shown that for any $f\in\mathcal{F}_n(A)$, $g_1\in\mathcal{F}_{m_1}(A),\ldots,g_n\in\mathcal{F}_{m_n}(A)$ and $h_1\in\mathcal{F}_{l_1}(A),\ldots,h_m\in\mathcal{F}_{l_m}(A)$, where $m=\max\{m_1,\ldots,m_n\}$ the *superassociativity*

$$f\langle g_1\ldots g_n\rangle\langle h_1\ldots h_m\rangle=f\langle g_1\langle h_1\ldots h_{m_1}\rangle\ldots g_n\langle h_1\ldots h_{m_n}\rangle\rangle$$

is satisfied.

The algebraic systems $((\mathcal{T}_n(A))_{n\in I},(\mathcal{O}_n)_{n\in I})$ and $((\mathcal{F}_n(A))_{n\in I},(\mathcal{O}_n)_{n\in I})$ are called *symmetrical Menger T-systems of full multiplace functions* and *symmetrical Menger T-systems of partial multiplace functions* on the set A. Their subsystems are called *Menger T-systems of rank I of full* (*respectively: partial*) *multiplace functions.*

Let $(G_n)_{n\in I}$ be a family of pairwise disjoint nonempty sets, $(o_n)_{n\in I}$ – the family of partial $(n+1)$-ary operations on the set $G=\bigcup_{n\in I}G_n$, satisfying the following conditions:

- if $x\in G_n$, $y_1\in G_{m_1},\ldots,y_n\in G_{m_n}$, then $o_n(x,y_1^n)$ is defined and $o_n(x,y_1^n)\in G_m$, where $m=\max\{m_1,\ldots,m_n\}$,
- for all $x\in G_n$, $y_1\in G_{m_1},\ldots,y_n\in G_{m_n}$, $z_1\in G_{l_1},\ldots,z_m\in G_{l_m}$, where $m=\max\{m_1,\ldots,m_n\}$, the following *superassociativity*

$$o_m(o_n(x,y_1^n),z_1^m)=o_n(x,o_{m_1}(y_1,z_1^{m_1}),\ldots,o_{m_n}(y_n,z_1^{m_n}))$$

  is satisfied.

A system $\mathcal{G}=((G_n)_{n\in I},(o_n)_{n\in I})$ defined in such a way is called a *Menger T-system of rank I*.

A Menger T-system $\mathcal{G}$ is called *unitary* if it contains a *complete collection of selectors* of all arities, i.e., if for every $n \in I$ there are elements $e_1^{(n)}, \dots, e_n^{(n)} \in G_n$, called *selectors*, such that

$$x\langle e_1^{(n)} \dots e_n^{(n)}\rangle = x,$$
$$e_i^{(n)}\langle y_1 \dots y_n\rangle = y_i\langle e_1^{(m)} \dots e_{m_i}^{(m)}\rangle$$

for all $i = 1, \dots, n$, $x \in G_n$, $y_1 \in G_{m_1}, \dots, y_n \in G_{m_n}$, $m_1, \dots, m_n \in I$ and $m = \max\{m_1, \dots, m_n\}$. It is clear that in a symmetrical Menger T-system of full multiplace functions the role of selectors is played by projectors.

By *homomorphism* of a *Menger T-system* $((G_n)_{n\in I}, (o_n)_{n\in I})$ into a Menger T-system $((G'_n)_{n\in I}, (o'_n)_{n\in I})$, we mean a family $(P_n)_{n\in I}$ of mappings $P_n : G_n \to G'_n$ such that

$$P_m(x\langle y_1 \dots y_n\rangle) = P_n(x)\langle P_{m_1}(y_1) \dots P_{m_n}(y_n)\rangle$$

holds for all $x \in G_n$, $y_1 \in G_{m_1}, \dots, y_n \in G_{m_n}$ and $m = \max\{m_1, \dots, m_n\}$. If all P_n are one-to-one, then such homomorphism is called an *isomorphism*.

By analogy to the previous sections, the superassociativity can be written in the abbreviated form as

$$x\langle\, y_1^n\rangle\langle\, z_1^m\rangle = x\langle\, y_1\langle\, z_1^{m_1}\rangle \dots y_n\langle\, z_1^{m_n}\rangle\rangle.$$

Theorem 6.2.1. *Every Menger T-system of rank I is isomorphic to some Menger T-system* (*of the same rank*) *of full multiplace functions on some set in such a way that its selectors correspond to the projectors of this set.*

Proof. Let $\mathcal{G} = ((G_n)_{n\in I}, (o_n)_{n\in I})$ be a fixed unitary Menger T-system of rank I, $e_1^{(n)}, \dots, e_n^{(n)}$ – the selectors on G_n.

Denote by $\overline{G} = \bigtimes_{n\in I} G_n$ the Cartesian product of a family $(G_n)_{n\in I}$ and by $\bar{x}(n)$ the component of $\bar{x} \in \overline{G}$ which belongs to G_n.

For every $n \in I$ and every $g \in G_n$ we define on $\overline{G}$ the full n-place function λ_g letting:

$$\lambda_g(\bar{x}_1, \dots, \bar{x}_n) = \bar{y} \longleftrightarrow (\forall i \in I)\Big(g\langle \bar{x}_1(i) \dots \bar{x}_n(i)\rangle = \bar{y}(i)\Big),$$

where $\bar{x}_1, \dots, \bar{x}_n, \bar{y} \in \overline{G}$.

Let us show that the family $(P_n)_{n\in I}$, where $P_n : g \mapsto \lambda_g$, is an isomorphism of the system $\mathcal{G}$ onto a Menger T-system $((\mathcal{T}_n(\overline{G}))_{n\in I}, (\mathcal{O}_n)_{n\in I})$. Indeed, for $g \in G_n$, $g_1 \in G_{m_1}, \dots, g_n \in G_{m_n}$, $\bar{x}_1, \dots, \bar{x}_m, \bar{y} \in \overline{G}$ and $m = \max\{m_1, \dots, m_n\}$, the condition $\lambda_{g\langle g_1 \dots g_n\rangle}(\bar{x}_1, \dots, \bar{x}_m) = \bar{y}$ means that

$$(\forall i \in I)\Big(g\langle g_1 \dots g_n\rangle\langle \bar{x}_1(i) \dots \bar{x}_m(i)\rangle = \bar{y}(i)\Big),$$

which, by the superassociativity, can be written in the form:

$$(\forall i \in I)\Big(g\langle g_1\langle \bar{x}_1(i)\dots\bar{x}_{m_1}(i)\rangle \dots g_n\langle \bar{x}_1(i)\dots\bar{x}_{m_n}(i)\rangle\rangle = \bar{y}(i)\Big).$$

This for $\bar{z}_k(i) = g_k\langle \bar{x}_1(i)\dots\bar{x}_{m_k}(i)\rangle$, $k = 1,\dots,n$, proves that for every $i \in I$ we have $g\langle \bar{z}_1(i)\dots\bar{z}_n(i)\rangle = \bar{y}(i)$. So, $\lambda_g(\bar{z}_1,\dots,\bar{z}_n) = \bar{y}$.

Further, as

$$(\forall i \in I)\Big(g_k\langle x_1(i)\dots\bar{x}_{m_k}(i)\rangle = \bar{z}_k(i)\Big)$$

for each $k = 1,\dots,n$, we have $\lambda_{g_k}(\bar{x}_1,\dots,\bar{x}_{m_i}) = \bar{z}_k$, $k = 1,\dots,n$, whence

$$\lambda_g(\lambda_{g_1}(\bar{x}_1,\dots,\bar{x}_{m_1}),\dots,\lambda_{g_n}(\bar{x}_1,\dots,\bar{x}_{m_n})) = \bar{y}.$$

Thus $\lambda_g\langle\lambda_{g_1}\dots\lambda_{g_n}\rangle(\bar{x}_1,\dots,\bar{x}_m) = \bar{y}$, where $m = \max\{m_1,\dots,m_n\}$. So, $\lambda_{g\langle g_1\dots g_n\rangle} = \lambda_g\langle\lambda_{g_1}\dots\lambda_{g_n}\rangle$, i.e.,

$$P_m(g\langle g_1\dots g_n\rangle) = P_n(g)\langle P_{m_1}(g_1)\dots P_{m_n}(g_n)\rangle,$$

which completes the proof that $(P_n)_{n\in I}$ is a homomorphism.

To prove that $(P_n)_{n\in I}$ is an isomorphism, consider the family $(\bar{e}_k)_{k\in I}$ of elements of $\overline{G}$ defined in the following way:

$$\bar{e}_k = \begin{cases} (e_k^{(n_1)}, e_k^{(n_2)}, e_k^{(n_3)},\dots) & \text{for } k = 1,\dots,n_1, \\ (e_{n_1}^{(n_1)},\dots,e_{n_i}^{(n_i)}, e_k^{(n_{i+1})}, e_k^{(n_{i+2})},\dots) & \text{for } n_i < k \leqslant n_{i+1}, \end{cases}$$

where $n_i \in I$, $n_i < n_{i+1}$ and $i = 1,2,\dots$[3]

If $\lambda_{g_1} = \lambda_{g_2}$, then $g_1, g_2 \in G_n$ for some $n \in I$ and

$$g_1\langle \bar{x}_1(k)\dots\bar{x}_n(k)\rangle = g_2\langle \bar{x}_1(k)\dots\bar{x}_n(k)\rangle$$

for all $n \in I$, $\bar{x}_1,\dots,\bar{x}_n \in \overline{G}$, $k \in I$.

So, for every $k \in I$ we have

$$g_1\langle \bar{e}_1(k)\dots\bar{e}_n(k)\rangle = g_2\langle \bar{e}_1(k)\dots\bar{e}_n(k)\rangle,$$

whence for $k = n$ we get

$$g_1\langle \bar{e}_1(n)\dots\bar{e}_n(n)\rangle = g_2\langle \bar{e}_1(n)\dots\bar{e}_n(n)\rangle,$$

i.e., $g_1\langle e_1^{(n)}\dots e_n^{(n)}\rangle = g_2\langle e_1^{(n)}\dots e_n^{(n)}\rangle$.[4] This implies $g_1 = g_2$. Thus, $(P_n)_{n\in I}$ is an isomorphism.

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[3] For $I = \{3,4,6,8\}$ we obtain $\bar{e}_1 = (e_1^{(3)}, e_1^{(4)}, e_1^{(6)}, e_1^{(8)})$, $\bar{e}_2 = (e_2^{(3)}, e_2^{(4)}, e_2^{(6)}, e_2^{(8)})$, $\bar{e}_3 = (e_3^{(3)}, e_3^{(4)}, e_3^{(6)}, e_3^{(8)})$, $\bar{e}_4 = (e_3^{(3)}, e_4^{(4)}, e_4^{(6)}, e_4^{(8)})$, $\bar{e}_5 = (e_3^{(3)}, e_4^{(4)}, e_5^{(6)}, e_5^{(8)})$, $\bar{e}_6 = (e_3^{(3)}, e_4^{(4)}, e_6^{(6)}, e_6^{(8)})$, $\bar{e}_7 = (e_3^{(3)}, e_4^{(4)}, e_6^{(6)}, e_7^{(8)})$, $\bar{e}_8 = (e_3^{(3)}, e_4^{(4)}, e_6^{(6)}, e_8^{(8)})$.

[4] For example, if $I = \{3,4,6,8\}$ and $g_1, g_2 \in G_6$ then the above construction guarantees that $g_1\langle e_1^{(6)}e_2^{(6)}e_3^{(6)}e_4^{(6)}e_5^{(6)}e_6^{(6)}\rangle = g_2\langle e_1^{(6)}e_2^{(6)}e_3^{(6)}e_4^{(6)}e_5^{(6)}e_6^{(6)}\rangle$.

This isomorphism maps all selectors onto projectors of $\overline{G}$. Indeed, from $\bar{y} = \lambda_{e_i^{(n)}}(\bar{x}_1, \ldots, \bar{x}_n)$ it follows that $\bar{y}(k) = e_i^{(n)}\langle \bar{x}_1(k) \ldots \bar{x}_n(k)\rangle = \bar{x}_i(k)$ for all $k \in I$, whence we conclude that $\lambda_{e_i^{(n)}}(\bar{x}_1, \ldots, \bar{x}_n) = \bar{x}_i$, i.e., $\lambda_{e_i^{(n)}} = I_i^n$. □

The proofs of the next three theorems are analogous to the proofs of the corresponding theorems from the previous section, so we omit them.

Theorem 6.2.2. *Any Menger T-system of rank I is isomorphic to some Manger T-system of the same rank of partial multiplace functions.*

Theorem 6.2.3. *Any Menger T-system of rank I of partial multiplace functions is isomorphic to some Menger T-system of the same rank of full multiplace functions.*

Corollary 6.2.4. *Any Menger T-system is isomorphic to some Menger T-system of full multiplace functions.*

Theorem 6.2.5. *Any Menger T-system of full multiplace functions can be isomorphically embedded into some Menger T-system of the same rank of full multiplace functions containing a complete set of projectors.*

Corollary 6.2.6. *Any Menger T-system $\mathcal{G}$ can be isomorphically embedded into such unitary Menger T-system $\mathcal{G}^*$ of the same rank selectors of which do not belong to $\mathcal{G}$.*

By *determining pair* of a Menger T-system $\mathcal{G} = ((G_n)_{n\in I}, (o_n)_{n\in I})$, we mean the pair $(\overline{\mathcal{E}}, \overline{W}) = ((\mathcal{E}_n)_{n\in I}, (W_n)_{n\in I})$, where $\mathcal{E}_n$ is the equivalence relation on the set G_n, W_n – the empty set or the $\mathcal{E}_n$-class of G_n satisfying for all $g \in G_n$, $x_1, y_1 \in G_{m_1}, \ldots, x_n, y_n \in G_{m_n}$, $n, m_1, \ldots, m_n \in I$, and $m = \max\{m_1, \ldots, m_n\}$ the following two conditions:

$$\bigwedge_{i=1}^{n} x_i \equiv y_i(\mathcal{E}_{m_i}) \longrightarrow g\langle x_1 \ldots x_n\rangle \equiv g\langle y_1 \ldots y_n\rangle(\mathcal{E}_m),$$

$$x \in W_{m_i} \longrightarrow g\langle y_1 \ldots y_{i-1} x y_{i+1} \ldots y_n\rangle \in W_m.$$

Similarly as in the case of Menger systems, each determining pair $(\overline{\mathcal{E}}, \overline{W})$ induces for $\mathcal{G}$ the simplest representation $\overline{P}_{(\overline{\mathcal{E}},\overline{W})} = \left(P^n_{(\overline{\mathcal{E}},\overline{W})}\right)_{n\in I}$ satisfying for all $g \in G_n$, $g_1 \in G_{m_1}, \ldots, g_n \in G_{m_n}$ and $m = \max\{m_1, \ldots, m_n\}$, the following equality:

$$P^m_{(\overline{\mathcal{E}},\overline{W})}(g\langle g_1 \ldots g_n\rangle) = P^n_{(\overline{\mathcal{E}},\overline{W})}(g)\langle P^{m_1}_{(\overline{\mathcal{E}},\overline{W})}(g_1) \ldots P^{m_n}_{(\overline{\mathcal{E}},\overline{W})}(g_n)\rangle.$$

Every such representation $\overline{P}_{(\overline{\mathcal{E}},\overline{W})}$ induces on $\mathcal{G}$ two families $\overline{\zeta}_{(\overline{\mathcal{E}},\overline{W})} = (\zeta^n_{(\overline{\mathcal{E}},\overline{W})})_{n\in I}$ and $\overline{\chi}_{(\overline{\mathcal{E}},\overline{W})} = (\chi^n_{(\overline{\mathcal{E}},\overline{W})})_{n\in I}$ of binary relations defined in the following way:

$$(g_1,g_2)\in\zeta^n_{(\overline{\mathcal{E}},\overline{W})} \longleftrightarrow P^n_{(\overline{\mathcal{E}},\overline{W})}(g_1)\subset P^n_{(\overline{\mathcal{E}},\overline{W})}(g_2),$$
$$(g_1,g_2)\in\chi^n_{(\overline{\mathcal{E}},\overline{W})} \longleftrightarrow \mathrm{pr}_1\,P^n_{(\overline{\mathcal{E}},\overline{W})}(g_1)\subset \mathrm{pr}_1\,P^n_{(\overline{\mathcal{E}},\overline{W})}(g_2),$$

where $g_1,g_2\in G_n$, $n\in I$.

In other words $(g_1,g_2)\in\zeta^n_{(\overline{\mathcal{E}},\overline{W})}$ if and only if for every $(\bar{x}_1,\ldots,\bar{x}_n)\in D_n$ (see p. 298) and every $k\in I$ the following implication is valid:

$$g_1\langle\bar{x}_1(k)\ldots\bar{x}_n(k)\rangle\notin W_k \longrightarrow g_1\langle\bar{x}_1(k)\ldots\bar{x}_n(k)\rangle\equiv g_2\langle\bar{x}_1(k)\ldots\bar{x}_n(k)\rangle(\mathcal{E}_k).$$

Similarly, $(g_1,g_2)\in\chi^n_{(\overline{\mathcal{E}},\overline{W})}$ if and only if for every $(\bar{x}_1,\ldots,\bar{x}_n)\in D_n$ and every $k\in I$

$$g_1\langle\bar{x}_1(k)\ldots\bar{x}_n(k)\rangle\notin W_k \longrightarrow g_2\langle\bar{x}_1(k)\ldots\bar{x}_n(k)\rangle\notin W_k.$$

Consider a family $(T_n)_{n\in I}$ of polynomials on a Menger T-system $\mathcal{G}=((G_n)_{n\in I},(o_n)_{n\in I})$ defined in the following way:

- $x_n\in T_n$, where x_n is an individual variable such that $x_n\neq x_m$ for $n\neq m$,
- if $t\in T_{m_i}$, $1\leqslant i\leqslant n$, $g\in G_n$, $g_1\in G_{m_1},\ldots,g_n\in G_{m_n}$, then $g\langle g_1\ldots g_{i-1}\,t\,g_{i+1}\ldots g_n\rangle\in T_m$, where $m=\max\{m_1,\ldots,m_n\}$.

Now, similarly as in the previous section, for each family $\mathbf{H}=(H_n)_{n\in I}$ of subsets $H_n\subset G_n$ we can define the determining pairs $(\mathcal{E}_{\mathbf{H}},W_{\mathbf{H}})$, where $\mathcal{E}_{\mathbf{H}}=(\mathcal{E}_{H_n})_{n\in I}$ and $W_{\mathbf{H}}=(W_{H_n})_{n\in I}$.

Let $((G_n)_{n\in I},(o_n)_{n\in I})$ be a Menger T-system of rank I. The family of binary relations $(\rho_n)_{n\in I}$, where $\rho_n\subset G_n\times G_n$, is called:

- *stable*, if

$$(x,y)\in\rho_n\wedge\bigwedge_{i=1}^{n}(x_i,y_i)\in\rho_i\longrightarrow(x\langle x_1\ldots x_n\rangle,y\langle y_1\ldots y_n\rangle)\in\rho_m$$

  for any $n,m_1,\ldots,m_k\in I$, $x,y\in G_n$, $x_k,y_k\in G_{m_k}$, $k=1,\ldots,n$, where $m=\max\{m_1,\ldots,m_n\}$,

- *l-regular*, if

$$(x,y)\in\rho_n\longrightarrow(x\langle z_1\ldots z_n\rangle,y\langle z_1\ldots z_n\rangle)\in\rho_m$$

  for any $n,m_1,\ldots,m_n\in I$, $x,y\in G_n$, $z_k\in G_{m_k}$, $k=1,\ldots,n$, where $m=\max\{m_1,\ldots,m_n\}$,

- *v-negative*, if
$$(x, u\langle\bar{w}|_i y\rangle) \in \rho_m \longrightarrow (x, y) \in \rho_m$$
for any $n, m_1, \ldots, m_n \in I$, $i = 1, \ldots, n$, $m = \max\{m_1, \ldots, m_n\}$, $x, y \in G_m$, $u \in G_n$ and $\bar{w} \in G_{m_1} \times \cdots \times G_{m_n}$.

A Menger T-system $((\Phi_n)_{n\in I}, (O_n)_{n\in I})$ of partial multiplace functions with two families $(\zeta_{\Phi_n})_{n\in I}$, $(\chi_{\Phi_n})_{n\in I})$ of relations, where ζ_{Φ_n} is the inclusion relation and χ_{Φ_n} the relation of inclusion of domains of elements of Φ_n, is called (by analogy to Menger algebras, see p. 85) a *fundamentally ordered projection (f.o.p.) Menger T-system of multiplace functions.*

An abstract characterization of such Menger T-systems is given by the following theorem.

Theorem 6.2.7. *An algebraic system $((G_n)_{n\in I}, (o_n)_{n\in I}, (\zeta_n)_{n\in I}, (\chi_n)_{n\in I})$ is isomorphic to some f.o.p. Menger T-system of partial multiplace functions if and only if $((G_n)_{n\in I}, (o_n)_{n\in I})$ is a Menger T-system, $(\zeta_n)_{n\in I}$ – the family of stable orders, $(\chi_n)_{n\in I}$ – the family of l-regular and v-negative quasi-orders such that $\zeta_n \subset \chi_n$ for every $n \in I$, and for all $g, g_1, g_2 \in G_n$, $u \in G_m$, $\bar{w} \in (G_n)^n$, $n, m \in I$, the following two implication hold:*

$$g_1 \leqslant_n g \wedge g_2 \leqslant_n g \wedge g_1 \sqsubset_n g_2 \longrightarrow g_1 \leqslant_n g_2,$$
$$g_1 \leqslant_n g_2 \wedge g \sqsubset_n g_1 \wedge g \sqsubset_n u\langle\bar{w}\,|_i g_2\rangle \longrightarrow g \sqsubset_n u\langle\bar{w}\,|_i g_1\rangle,$$

where $x \leqslant_n y \longleftrightarrow (x, y) \in \zeta_n$, and $x \sqsubset_n y \longleftrightarrow (x, y) \in \chi_n$.

Proof. The necessity of these conditions can be proved in the same way as in the proof of Theorem 3.2.1. To prove the sufficiency assume that a Menger T-system $((G_n)_{n\in I}, (o_n)_{n\in I})$ with two families $(\zeta_n)_{n\in I}$, $(\chi_n)_{n\in I}$ of relations defined on $(G_n)_{n\in I}$ satisfies all the conditions given in Theorem 6.2.7. Then, similarly as in the proof of Theorem 3.2.1, we can show, that

$$g_1 \leqslant_n g_2 \wedge g \sqsubset_n t_1(g_1) \wedge g \sqsubset_n t_2(g_2) \longrightarrow g \sqsubset_n t_2(g_1), \tag{6.2.1}$$
$$g_1 \leqslant_n g_2 \wedge g \sqsubset_n t_1(g_1) \wedge g \leqslant_n t_2(g_2) \longrightarrow g \leqslant_n t_2(g_1) \tag{6.2.2}$$

for all $g_1, g_2, g \in G_n$, $t_1, t_2 \in T_n$, $n \in I$.

Consider the representation $\overline{P} = (P_n)_{n\in I}$ of a given Menger T-system $((G_n)_{n\in I}, (o_n)_{n\in I})$, where $P_n = \sum_{\bar{g}\in\overline{G}} P^n_{(\mathcal{E}_{\bar{g}}, W_{\bar{g}})}$, $\mathcal{E}_{\bar{g}} = (\mathcal{E}_{\bar{g}(n)})_{n\in I}$, $\mathcal{E}_{\bar{g}(n)} = \mathcal{E}_{\zeta_n\langle\bar{g}(n)\rangle} \cap \mathcal{E}_{\chi_n\langle\bar{g}(n)\rangle}$, $W_{\bar{g}} = (W_{\chi_n\langle\bar{g}(n)\rangle})_{n\in I}$. Obviously, $\overline{P}$, as sum of simplest representations, is a representation by multiplace functions.

Let us show that $\zeta_n = \zeta_{P_n}$ for every $n \in I$, where

$$\zeta_{P_n} = \{(g_1, g_2) \in G_n \times G_n \,|\, P_n(g_1) \subset P_n(g_2)\}.$$

Let $g_1, g_2 \in G_n$ and $(g_1, g_2) \in \zeta_{P_n}$. Then, according to the definition of ζ_{P_n}, for all $\bar{g} \in \overline{G}$, $(\bar{x}_1, \ldots, \bar{x}_n) \in D_n$ and $k \in I$, from

$$g_1\langle \bar{x}_1(k) \ldots \bar{x}_n(k)\rangle \notin W_{\chi_k\langle \bar{g}(k)\rangle}$$

it follows that

$$g_1\langle \bar{x}_1(k) \ldots \bar{x}_n(k)\rangle \equiv g_2\langle \bar{x}_1(k) \ldots \bar{x}_n(k)\rangle(\mathcal{E}_{\bar{g}(k)}),$$

which means that for $k = n$ and $(\bar{x}_1, \ldots, \bar{x}_n) \in E_n^1 \times \cdots \times E_n^n$ (see p. 298), the following implication

$$g_1\langle e_1^{(n)} \ldots e_n^{(n)}\rangle \notin W_{\chi_n\langle \bar{g}(n)\rangle} \longrightarrow g_1\langle e_1^{(n)} \ldots e_n^{(n)}\rangle \equiv g_2\langle e_1^{(n)} \ldots e_n^{(n)}\rangle(\mathcal{E}_{\bar{g}(n)}),$$

is true. Therefore

$$g_1 \notin W_{\chi_n\langle \bar{g}(n)\rangle} \longrightarrow g_1 \equiv g_2(\mathcal{E}_{\bar{g}(n)}).$$

From this, for $\bar{g}(n) = g_1$, we obtain the implication

$$g_1 \notin W_{\chi_n\langle g_1\rangle} \longrightarrow g_1 \equiv g_2(\mathcal{E}_{g_1}),$$

which is equivalent to $g_1 \equiv g_2(\mathcal{E}_{g_1})$, where $\mathcal{E}_{g_1} = \mathcal{E}_{\zeta_n\langle g_1\rangle} \cap \mathcal{E}_{\chi_n\langle g_1\rangle}$. Whence $g_1 \equiv g_2(\mathcal{E}_{\zeta_n\langle g_1\rangle})$. From this, analogously as in the case of Menger algebras (see the proof of Theorem 3.2.1), we conclude $g_1 \leqslant_n g_2$. So, $\zeta_{P_n} \subset \zeta_n$.

To prove the converse inclusion, let $g_1 \leqslant_n g_2$, $(\bar{x}_1, \ldots, \bar{x}_n) \in D_n$ and $g_1\langle \bar{x}_1(k) \ldots \bar{x}_n(k)\rangle \notin W_{\chi_k\langle \bar{g}(k)\rangle}$ for some $\bar{g} \in \overline{G}$. Then there exists a polynomial $t_1 \in T_k$ such that

$$\bar{g}(k) \sqsubset_k t_1(g_1\langle \bar{x}_1(k) \ldots \bar{x}_n(k)\rangle). \tag{6.2.3}$$

Since the relation $\leqslant_n$ is stable, $g_1 \leqslant_n g_2$ implies

$$g_1\langle \bar{x}_1(k) \ldots \bar{x}_n(k)\rangle \leqslant_k g_2\langle \bar{x}_1(k) \ldots \bar{x}_n(k)\rangle. \tag{6.2.4}$$

So, if $\bar{g}(k) \leqslant_k t(g_1\langle \bar{x}_1(k) \ldots \bar{x}_n(k)\rangle)$ for some $t \in T_k$, then also

$$\bar{g}(k) \leqslant_k t(g_2\langle \bar{x}_1(k) \ldots \bar{x}_n(k)\rangle). \tag{6.2.5}$$

Similarly, from (6.2.3) and (6.2.5), by (6.2.4) and (6.2.1), we conclude $\bar{g}(k) \leqslant_k t(g_1\langle \bar{x}_1(k) \ldots \bar{x}_n(k)\rangle)$. Thus

$$g_1\langle \bar{x}_1(k) \ldots \bar{x}_n(k)\rangle \equiv g_2\langle \bar{x}_1(k) \ldots \bar{x}_n(k)\rangle(\mathcal{E}_{\zeta_n\langle \bar{g}(k)\rangle}).$$

Now let $\bar{g}(k) \leqslant_k t(g_1\langle \bar{x}_1(k) \ldots \bar{x}_n(k)\rangle)$, $t \in T_k$. Then, by $\zeta_k \subset \chi_k$ and

$$t(g_1\langle \bar{x}_1(k) \ldots \bar{x}_n(k)\rangle) \leqslant_k t(g_2\langle \bar{x}_1(k) \ldots \bar{x}_n(k)\rangle),$$

we obtain

$$\bar{g}(k) \sqsubset_k t(g_2\langle \bar{x}_1(k)\ldots\bar{x}_n(k)\rangle),$$

which, together with (6.2.3) and (6.2.1), proves

$$\bar{g}(k) \sqsubset_k t(g_1\langle \bar{x}_1(k)\ldots\bar{x}_n(k)\rangle)\,.$$

Therefore

$$g_1\langle \bar{x}_1(k)\ldots\bar{x}_n(k)\rangle \equiv g_2\langle \bar{x}_1(k)\ldots\bar{x}_n(k)\rangle(\mathcal{E}_{\chi_n\langle\bar{g}(k)\rangle}).$$

Summarizing, in all cases $g_1 \leqslant_n g_2$ implies

$$g_1\langle \bar{x}_1(k)\ldots\bar{x}_n(k)\rangle \equiv g_2\langle \bar{x}_1(k)\ldots\bar{x}_n(k)\rangle(\mathcal{E}_{\bar{g}(k)}).$$

Thus $\zeta_n \subset \zeta_{P_n}$, and finally $\zeta_n = \zeta_{P_n}$.

To prove $\chi_{P_n} = \chi_n$, let $(g_1, g_2) \in \chi_{P_n}$. Then for all $(\bar{x}_1,\ldots,\bar{x}_n) \in D_n$, $\bar{g} \in \overline{G}$ and $k \in I$

$$g_1\langle \bar{x}_1(k)\ldots\bar{x}_n(k)\rangle \notin W_{\chi_n\langle\bar{g}(k)\rangle} \longrightarrow g_2\langle \bar{x}_1(k)\ldots\bar{x}_n(k)\rangle,$$

whence for $k = n$, $\bar{g}(n) = g_1$ and $(\bar{x}_1,\ldots,\bar{x}_n) \in E_n^1 \times\cdots\times E_n^n$ we obtain $g_2 \notin W_{\chi_n\langle g_1\rangle}$. Thus $g_1 \sqsubset_n t(g_2)$ for some $t \in T_n$, and $g_1 \sqsubset_n t(g_2) \sqsubset_n g_2$, by the v-negativity. So, $g_1 \sqsubset_n g_2$, i.e., $\chi_{P_n} \subset \chi_n$.

On the other hand, if $(g_1, g_2) \in \chi_n$, then, by l-regularity, we have

$$g_1\langle \bar{x}_1(k)\ldots\bar{x}_n(k)\rangle \sqsubset_k g_2\langle \bar{x}_1(k)\ldots\bar{x}_n(k)\rangle$$

for any $(\bar{x}_1,\ldots,\bar{x}_n) \in D_n$ and $k \in I$. If $g_1\langle \bar{x}_1(k)\ldots\bar{x}_n(k)\rangle \notin W_{\chi_k\langle\bar{g}(k)\rangle}$ for some $\bar{g} \in \overline{G}$, then $\bar{g}(k) \sqsubset_k t_1(g_1\langle \bar{x}_1(k)\ldots\bar{x}_n(k)\rangle)$ for some $t_1 \in T_k$. This, according to the v-negativity, gives $\bar{g}(k) \sqsubset_k g_1\langle \bar{x}_1(k)\ldots\bar{x}_n(k)\rangle$, whence $\bar{g}(k) \sqsubset_k g_2\langle \bar{x}_1(k)\ldots\bar{x}_n(k)\rangle$. Consequently $g_2\langle \bar{x}_1(k)\ldots\bar{x}_n(k)\rangle \notin W_{\chi_k\langle\bar{g}(k)\rangle}$, i.e., $(g_1, g_2) \in \chi_{P_n}$. Therefore $\chi_n \subset \chi_{P_n}$, which together with the previous inclusion, proves $\chi_n = \chi_{P_n}$.

Since for any $n \in I$ from $P_n(g_1) = P_n(g_2)$, it follows that $g_1 = g_2$, the representation $(P_n)_{n\in I}$ is faithful. □

Corollary 6.2.8. *An algebraic system* $((G_n)_{n\in I}, (o_n)_{n\in I}, (\chi_n)_{n\in I})$ *is isomorphic to some projection quasi-ordered Menger* T*-system of multiplace functions if and only if* $((G_n)_{n\in I}, (o_n)_{n\in I})$ *is a Menger* T*-system and* $(\chi_n)_{n\in I}$ *is a family of* l*-regular* v*-negative quasi-orders.*

Theorem 6.2.9. *An algebraic system* $((G_n)_{n\in I}, (o_n)_{n\in I}, (\zeta_n)_{n\in I})$ *is isomorphic to some fundamentally ordered Menger* T*-system of partial multiplace functions if and only if* $((G_n)_{n\in I}, (o_n)_{n\in I})$ *is a Menger* T*-system,* $(\zeta_n)_{n\in I}$ *– a family of orders and for all* $g, g_1, g_2 \in G_n$, $t_1, t_2 \in T_n$, $n \in I$ *the following condition*

$$g_1 \leqslant_n g_2 \wedge g \leqslant_n t_1(g_1) \wedge g \leqslant_n t_2(g_2) \longrightarrow g \leqslant_n t_2(g_1)$$

is satisfied.

Proof. The proof of this theorem is analogous to the proof of Theorem 6.1.12. □

6.3 Positional algebras

On the sets of multiplace functions of various arities, one often considers the binary operations of superpositions, which are called positional superpositions. Such operations are used in the theory of functional equations and in the theory of n-ary quasigroups [6]. Thus the study of positional algebras and their representations by multiplace functions has a particular interest.

A *positional algebra* is a partial algebra of the form

$$\mathfrak{G} = (G, \overset{1}{+}, \overset{2}{+}, \ldots, \overset{n}{+}, \ldots),$$

where $\overset{1}{+}, \overset{2}{+}, \ldots, \overset{n}{+}, \ldots$ are partial binary operations on a set G satisfying the Axioms $\mathbf{A_1} - \mathbf{A_5}$.

A$_1$ $\{x\} \overset{1}{+} \{y\} \neq \varnothing$ for all $x, y \in G$.

A$_2$ For every $x \in G$ there exists $n \in \mathbb{N}$ such that

$$i \leqslant n \longleftrightarrow \{x\} \overset{n}{+} \{y\} \neq \varnothing$$

for all $i \in \mathbb{N}$ and $y \in G$.

Let $\alpha \subset G \times \mathbb{N}$ be a binary relation such that $(x, n) \in \alpha$ if and only if

$$(\forall i \in \mathbb{N})\,(\forall y \in G)\ \ (\, i \leqslant n \longleftrightarrow \{x\} \overset{i}{+} \{y\} \neq \varnothing\,). \tag{6.3.1}$$

Proposition 6.3.1. *The relation α is one-valued.*

Proof. Suppose that $(x, n) \in \alpha$ and $(x, m) \in \alpha$ for some $x \in G$, $n, m \in \mathbb{N}$. Let $n \neq m$. Without loss of generality, we can assume $n < m$. Then, according to (6.3.1), we have

$$(\forall i \in \mathbb{N})\,(\forall y \in G)\ \ (\, i \leqslant n \Longleftrightarrow \{x\} \overset{i}{+} \{y\} \neq \varnothing\,), \tag{6.3.2}$$

$$(\forall i \in \mathbb{N})\,(\forall y \in G)\ \ (\, i \leqslant m \Longleftrightarrow \{x\} \overset{i}{+} \{y\} \neq \varnothing\,). \tag{6.3.3}$$

It is not difficult to see that (6.3.2) is equivalent to

$$(\forall i \in \mathbb{N})\,(\forall y \in G)\ \ (\, i > n \Longleftrightarrow \{x\} \overset{i}{+} \{y\} = \varnothing\,). \tag{6.3.4}$$

From (6.3.3), it follows that $\{x\} \overset{m}{+} \{y\} \neq \varnothing$ for all $y \in G$. Since $m > n$, then, from (6.3.4), for each $y \in G$ we obtain $\{x\} \overset{m}{+} \{y\} = \varnothing$. The obtained contradiction proves that $n = m$. □

Furthermore by arity of an element $x \in G$ we mean the value of $\alpha(x)$, which will be denoted by $|x|$. So, $|x| = \alpha(x)$. From the definition of α it follows that for $x, y \in G$ and $i \in \mathbb{N}$ the result of $x \overset{i}{+} y$ is defined if and only if $i \leqslant |x|$.

A₃ For all $x, y \in G$, $i \in \mathbb{N}$, if $i \leqslant |x|$, then

$$|x \overset{i}{+} y| = |x| + |y| - 1.$$

A₄ For $x, y, z \in G$ and $n, m \in \mathbb{N}$ such that $n \leqslant |x|$, $m \leqslant |y|$, we have

$$x \overset{n}{+} (y \overset{m}{+} z) = (x \overset{n}{+} y) \overset{n+m-1}{+} z\,.$$

A₅ For $x, y, z \in G$ and $n, m \in \mathbb{N}$ such that $m < n \leqslant |x|$, we have

$$(x \overset{n}{+} y) \overset{m}{+} z = (x \overset{m}{+} z) \overset{n+|z|-1}{+} y.$$

Let $\mathcal{T}_n(A) = \mathcal{T}(A^n, A)$ be the set of all full n-place functions on A. Instead of $f \in \mathcal{T}_n(A)$ we will write $|f| = n$. In this notation $|f|$ is, in fact, the arity of f.

For arbitrary $f, g \in \mathcal{T}(A) = \bigcup_{n\in\mathbb{N}} \mathcal{T}_n(A)$, where $|f| = n$, $|g| = m$, we define a *positional superposition* $\overset{i}{+}$ $(i \in \mathbb{N})$ putting

$$(f \overset{i}{+} g)(a_1^{n+m-1}) = f(a_1^{i-1}, g(a_i^{i+m-1}), a_{i+m}^{n+m-1}) \tag{6.3.5}$$

for all $a_1, \ldots, a_{n+m-1} \in A$. The obtained algebra $(\mathcal{T}(A), \overset{1}{+}, \overset{2}{+}, \ldots)$ will be called the *symmetrical positional algebra of operations*, its subalgebras – *positional algebras of operations*.

Let $\mathcal{F}_n(A)$ be the set of all partial n-place transformations on A and let Θ_n be the empty map from A^n into A. On the set

$$\mathcal{F}(A) = \bigcup_{n\in\mathbb{N}} \Big(\mathcal{F}_n(A) \cup \{\Theta_n\}\Big)$$

we consider the family $(\overset{i}{+})_{i\in\mathbb{N}}$ of partial binary operations $\overset{i}{+}$ $(i \in \mathbb{N})$ defined for $f \in \mathcal{F}_n(A)$, $g \in \mathcal{F}_m(A)$ and $a_1, \ldots, a_{n+m-1}, b, c \in A$, by the formula[5]

$$(a_1^{n+m-1}, c) \in f \overset{i}{+} g \longleftrightarrow (\exists b)\Big((a_i^{i+m-1}, b) \in g \wedge (a_1^{i-1}\, b\, a_{i+m}^{n+m-1}, c) \in f\Big).$$

[5] This formula is valid also for relations.

If $f \overset{i}{+} g$ is the empty transformation, then we put $f \overset{i}{+} g = \Theta_{n+m-1}$.

We assume that $\Theta_n \overset{i}{+} \Theta_m = \Theta_{n+m-1}$ for all $n, m \in \mathbb{N}$ and $i \leqslant n$. We also assume that $f \overset{i}{+} \Theta_m = \Theta_n \overset{i}{+} g = \Theta_{n+m-1}$. It is clear that the algebraic system $(\mathcal{F}(A), \overset{1}{+}, \overset{2}{+}, \ldots)$ is a positional algebra. This algebra will be called the *symmetrical positional algebra of multiplace functions*, its subalgebras – *positional algebras of multiplace functions.*

Let $\mathfrak{G}_1 = (G_1, \overset{1}{+}, \overset{2}{+}, \ldots)$ and $\mathfrak{G}_2 = (G_2, \overset{1}{+}, \overset{2}{+}, \ldots)$ be two positional algebras. A mapping $P : G_1 \to G_2$ such that

(1) $|P(g)| = |g|$ for each $g \in G_1$,

(2) $P(g_1 \overset{i}{+} g_2) = P(g_1) \overset{i}{+} P(g_2)$ for all $g_1, g_2 \in G_1$ and $i \leqslant |g_1|$,

will be called a *strong homomorphism of* $\mathfrak{G}_1$ *into* $\mathfrak{G}_2$. A strong homomorphism of a positional algebra $\mathfrak{G}$ into a symmetrical positional algebra of multiplace functions (operations) will be called a *representation of* $\mathfrak{G}$ *by multiplace functions* (*or by operations*). A representation which is an isomorphism is called *faithful* (or *isomorphic*).

Let $\mathfrak{G} = (G, \overset{1}{+}, \overset{2}{+}, \ldots)$ be a positional algebra, e – an element not belonging to G, $G^* = G \cup \{e\}$. We put $|e| = 1$, $e \overset{1}{+} e = e$, $e \overset{1}{+} g = g$ and $g \overset{i}{+} e = g$ for every $g \in G$ and $i \leqslant |g|$. It is not difficult to see that $\mathfrak{G}^* = (G^*, \overset{1}{+}, \overset{2}{+}, \ldots)$ is a positional algebra. It is called a *unitary extension* of $\mathfrak{G}$.

Theorem 6.3.2. *Every positional algebra is isomorphic to some positional algebra of operations.*

Proof. Let $\mathfrak{G}^* = (G^*, \overset{1}{+}, \overset{2}{+}, \ldots)$ be a unitary extension of a positional algebra $\mathfrak{G} = (G, \overset{1}{+}, \overset{2}{+}, \ldots)$. For every element $g \in G$ we define the $|g|$-ary operation $P(g)$ on the set G^* putting

$$P(g)(x_1, \ldots, x_n) = (\ldots((g \overset{n}{+} x_n) \overset{n-1}{+} x_{n-1}) \overset{n-2}{+} \ldots) \overset{1}{+} x_1, \tag{6.3.6}$$

where $x_1, \ldots, x_n \in G^*$, $n = |g|$. Furthermore for simplicity's sake this equation will be written without brackets

$$P(g)(x_1, \ldots, x_n) = g \overset{n}{+} x_n \overset{n-1}{+} x_{n-1} \overset{n-2}{+} \cdots \overset{1}{+} x_1$$

or, in a more abbreviated form, as

$$P(g)(x_1, \ldots, x_n) = g \underset{n}{\overset{1}{+}} x_n^1.$$

Now, let us show that $P\colon g \mapsto P(g)$ is a faithful representation of $\mathfrak{G}$ by full multiplace functions defined on the set G^*. Indeed, if $g_1, g_2 \in G$ and $n = |g_1|$, $m = |g_2|$, $i \leqslant n$, then $|g_1 \overset{i}{+} g_2| = n + m - 1$, according to the axiom $\mathbf{A}_3$. Moreover, for arbitrary $x_1, \ldots, x_{n+m-1} \in G^*$ and $i \leqslant |g_1|$ we have

$$\begin{aligned}
P(g_1 \overset{i}{+} g_2)(x_1^{n+m-1}) &= (g_1 \overset{i}{+} g_2) \underset{n+m-1}{\overset{1}{+}} x_{n+m-1}^1 \\
&\overset{\mathbf{A}_5}{=} (g_1 \overset{n}{+} x_{n+m-1}) \overset{i}{+} g_2 \underset{n+m-2}{\overset{1}{+}} x_{n+m-2}^1 \\
&\overset{\mathbf{A}_5}{=} (g_1 \overset{n}{+} x_{n+m-1} \overset{n-1}{+} x_{n+m-2}) \overset{i}{+} g_2 \underset{n+m-3}{\overset{1}{+}} x_{n+m-3} \\
&\overset{\mathbf{A}_5}{=} \cdots = \left(g_1 \underset{n}{\overset{i+1}{+}} x_{n+m-1}^{i+m}\right) \overset{i}{+} g_2 \underset{i+m-1}{\overset{1}{+}} x_{i+m-1}^1 \\
&\overset{\mathbf{A}_4}{=} \left(g_1 \underset{n}{\overset{i+1}{+}} x_{n+m-1}^{i+m}\right) \overset{i}{+} \left(g_2 \overset{m}{+} x_{i+m-1}\right) \underset{i+m-2}{\overset{1}{+}} x_{i+m-2}^1 \\
&\overset{\mathbf{A}_4}{=} \cdots = \left(g_1 \underset{n}{\overset{i+1}{+}} x_{n+m-1}^{i+m}\right) \overset{i}{+} \left(g_2 \underset{m}{\overset{1}{+}} x_{i+m-1}^{i}\right) \underset{i-1}{\overset{1}{+}} x_{i-1}^1 \\
&= \left(g_1 \underset{n}{\overset{i+1}{+}} x_{n+m-1}^{i+m}\right) \overset{i}{+} P(g_2)(x_i^{i+m-1}) \underset{i-1}{\overset{1}{+}} x_{i-1}^1 \\
&= P(g_1)(x_1^{i-1}, P(g_2)(x_i^{i+m-1}), x_{i+m}^{n+m-1}) \\
&= \left(P(g_1) \overset{i}{+} P(g_2)\right)(x_1^{n+m-1})
\end{aligned}$$

which proves that P is a representation.

To prove that it is faithful, let $P(g_1) = P(g_2)$, where $g_1, g_2 \in G$ and $|g_1| = |g_2| = n$. Then

$$P(g_1)(x_1, \ldots, x_n) = P(g_2)(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in G^*$. In particular, for $x_1 = \cdots = x_n = e$, we get $P(g_1)(e, \ldots, e) = P(g_2)(e, \ldots, e)$, i.e.,

$$g_1 \overset{n}{+} e \overset{n-1}{+} \cdots \overset{1}{+} e = g_2 \overset{n}{+} e \overset{n-1}{+} \cdots \overset{1}{+} e,$$

whence $g_1 = g_2$. This completes the proof. $\square$

Corollary 6.3.3. *Each positional algebra is isomorphic to some positional algebra of multiplace functions.*

Corollary 6.3.4. *Each positional algebra is isomorphic to some positional algebra of n-ary relations.*

Let $\mathfrak{G} = (G, \overset{1}{+}, \overset{2}{+}, \ldots)$ be a positional algebra, A – some set, $\mathbf{\Omega}(A)$ – the set of all words over A. If $\omega_1, \ldots, \omega_n \in \mathbf{\Omega}(A)$, then $\omega = \omega_1\omega_2 \ldots \omega_n$ is a new word obtained from $\omega_1, \ldots, \omega_n$ by juxtaposition. $l(\omega)$ denotes the length of the word ω. For each word $\omega \in \Omega(A)$ of length $l(\omega) = n$ by ε^ω, we denote some equivalence relation on $G_n = \{g \in G \,|\, |g| = n\}$, which corresponds to ω. So, $\varepsilon^\omega \subset G_{l(\omega)} \times G_{l(\omega)}$.

Let $\mathcal{E} = (\varepsilon^\omega)^{\omega \in B}$, where $B \subset \Omega(A)$, be some family of equivalence relations.

Definition 6.3.5. A family $\mathcal{E}$ is called *permissible* for a positional algebra $\mathfrak{G}$, if for all $g, x_1, \ldots, x_n, y_1, \ldots, y_n \in G$ and $n = |g|$

$$x_1 \equiv y_1(\varepsilon^{\omega_1}) \wedge \ldots \wedge x_n \equiv y_n(\varepsilon^{\omega_n}) \longrightarrow g \overset{1}{\underset{n}{+}} x_n^1 \equiv g \overset{1}{\underset{n}{+}} y_n^1(\varepsilon^{\omega_1 \cdots \omega_n}).$$

Definition 6.3.6. A family $\mathcal{W} = (W^\omega)^{\omega \in B}$, where W^ω is a subset of $G_{l(\omega)}$, is an *l-ideal*, if for all $g, x_k \in G$, $k \neq i$, $k, i = 1, \ldots, n$, where $|g| = n$ and $l(\omega_1) = l(\omega) + \sum\limits_{k=1, k\neq i}^{n} |x_k|$ the following implication is true:

$$h \in W^\omega \longrightarrow ((g \overset{i+1}{\underset{n}{+}} x_n^{i+1}) \overset{i}{+} h) \overset{1}{\underset{i-1}{+}} x_{i-1}^1 \in W^{\omega_1}.$$

Definition 6.3.7. By *determining pair* of a positional algebra $\mathfrak{G}$, we mean the ordered pair $(\mathcal{E}, \mathcal{W})$, where $\mathcal{E}$ is the family of equivalence relations permissible for a positional algebra $\mathfrak{G}^*$, $\mathcal{W}$ – an l-ideal of the family of subsets W^ω such that each W^ω is either empty or an ε^ω-class.

Let $(H_a^\omega)_{a \in I_\omega}$ be the family of all ε^ω-classes disjoint from W^ω, uniquely indexed by elements of some fixed set I_ω, and satisfying the following implication

$$\left.\begin{array}{l} (\ldots((g \overset{n}{+} H_{a_n}^{\omega_n}) \overset{n-1}{+} H_{a_{n-1}}^{\omega_{n-1}}) \ldots) \overset{1}{+} H_{a_1}^{\omega_1} \subset H_b^{\omega_1 \ldots \omega_n} \\ (\ldots((g \overset{n}{+} H_{a_n}^{\omega'_n}) \overset{n-1}{+} H_{a_{n-1}}^{\omega'_{n-1}}) \ldots) \overset{1}{+} H_{a_1}^{\omega'_1} \subset H_c^{\omega'_1 \ldots \omega'_n} \end{array}\right\} \longrightarrow b = c, \qquad (6.3.7)$$

where $n = |g|$. It is not difficult to see that for $I_\omega \cap I_{\omega'} = \varnothing$ this condition is satisfied.

For every $g \in G$, $|g| = n$, by $P_{(\mathcal{E},\mathcal{W})}(g)$, where $(\mathcal{E}, \mathcal{W})$ is the determining pair of $\mathfrak{G}$, we denote the partial n-place function such that

$$(a_1^n, b) \in P_{(\mathcal{E},\mathcal{W})}(g) \longleftrightarrow (\ldots((g \overset{n}{+} H_{a_n}^{\omega_n}) \overset{n-1}{+} H_{a_{n-1}}^{\omega_{n-1}}) \ldots) \overset{1}{+} H_{a_1}^{\omega_1} \subset H_b^{\omega_1 \cdots \omega_n}$$

for some $\omega_1, \ldots, \omega_n \in \Omega(A)$.

Theorem 6.3.8. *If $(\mathcal{E}, \mathcal{W})$ is the determining pair of a positional algebra $\mathfrak{G}$, then $P_{(\mathcal{E},\mathcal{W})} : g \mapsto P_{(\mathcal{E},\mathcal{W})}(g)$, where $g \in G$, is the representation of $\mathfrak{G}$ by multiplace functions.*

Proof. Let g_1, g_2 be arbitrary elements of G such that $|g_1| = n$, $|g_2| = m$. If $(a_1^{n+m-1}, c) \in P_{(\mathcal{E},\mathcal{W})}(g_1 \overset{i}{+} g_2)$ for $i \leqslant n$, then, according to the definition of $P_{(\mathcal{E},\mathcal{W})}(g)$, we obtain

$$(\ldots((g_1 \overset{i}{+} g_2) \overset{n+m-1}{+} H^{\omega_{n+m-1}}_{a_{n+m-1}}) \overset{n-1}{+} \ldots) \overset{1}{+} H^{\omega_1}_{a_1} \subset H^{\omega_1\ldots\omega_{n+m-1}}_c.$$

If $x_i \in H^{\omega_i}_{a_i}$, $i = 1, \ldots, n+m-1$, then

$$(g_1 \overset{i}{+} g_2) \underset{n+m-1}{\overset{1}{+}} x^1_{n+m-1} \in H^{\omega_1\ldots\omega_{n+m-1}}_c,$$

which, by the axioms of a positional algebra, gives

$$(g_1 \overset{i}{+} g_2) \underset{n+m-1}{\overset{1}{+}} x^1_{n+m-1} = \left(\left(g_1 \underset{n}{\overset{i+1}{+}} x^{i+m}_{n+m-1}\right) \overset{i}{+} \left(g_2 \underset{m}{\overset{1}{+}} x^i_{i+m-1}\right)\right) \underset{i-1}{\overset{1}{+}} x^1_{i-1}.$$

Therefore

$$\left(\left(g_1 \underset{n}{\overset{i+1}{+}} x^{i+m}_{n+m-1}\right) \overset{i}{+} \left(g_2 \underset{m}{\overset{1}{+}} x^i_{i+m-1}\right)\right) \underset{i-1}{\overset{1}{+}} x^1_{i-1} \in H^{\omega_1\ldots\omega_{n+m-1}}_c. \tag{6.3.8}$$

Hence

$$\left(\left(g_1 \underset{n}{\overset{i+1}{+}} x^{i+m}_{n+m-1}\right) \overset{i}{+} \left(g_2 \underset{m}{\overset{1}{+}} x^i_{i+m-1}\right)\right) \underset{i-1}{\overset{1}{+}} x^1_{i-1} \notin W^{\omega_1\ldots\omega_{n+m-1}}.$$

Since the family $\mathcal{W}$ is an l-ideal, then from the last condition we deduce $g_2 \underset{m}{\overset{1}{+}} x^i_{i+m-1} \notin W^{\omega_i\ldots\omega_{i+m-1}}$.

Suppose that

$$g_2 \underset{m}{\overset{1}{+}} x^i_{i+m-1} \in H^{\omega_i\ldots\omega_{i+m-1}}_b. \tag{6.3.9}$$

This, by the permissibility of $\mathcal{E}$, gives us

$$\left(\ldots\left(g_2 \overset{m}{+} H^{\omega_{i+m-1}}_{a_{i+m-1}}\right) \overset{m-1}{+} \ldots\right) \overset{1}{+} H^{\omega_i}_{a_i} \subset H^{\omega_i\ldots\omega_{i+m-1}}_b,$$

which means that

$$(a_i^{i+m-1}, b) \in P_{(\mathcal{E},\mathcal{W})}(g_2). \tag{6.3.10}$$

Thus (6.3.8) together with (6.3.9) proves that

$$g_1 \overset{n}{+} H^{\omega_{n+m-1}}_{a_{n+m-1}} \overset{n-1}{+} \cdots \overset{i+1}{+} H^{\omega_{i+m}}_{a_{i+m}} \overset{i}{+} H^{\omega_i\ldots\omega_{i+m-1}}_b \overset{i-1}{+} H^{\omega_{i-1}}_{a_{i-1}} \overset{i-2}{+} \cdots \overset{1}{+} H^{\omega_1}_{a_1}$$

is contained in $H^{\omega_1\ldots\omega_{n+m-1}}_c$. Hence

$$(a_1^{i-1} b\, a_{i+m}^{n+m-1}, c) \in P_{(\mathcal{E},\mathcal{W})}(g_1). \tag{6.3.11}$$

Now, comparing (6.3.10) and (6.3.11) we obtain

$$(a_1^{n+m-1}, c) \in P_{(\mathcal{E},\mathcal{W})}(g_1) \overset{i}{+} P_{(\mathcal{E},\mathcal{W})}(g_2)\,.$$

In this way, we have proved the inclusion

$$P_{(\mathcal{E},\mathcal{W})}(g_1 \overset{i}{+} g_2) \subset P_{(\mathcal{E},\mathcal{W})}(g_1) \overset{i}{+} P_{(\mathcal{E},\mathcal{W})}(g_2)\,.$$

The converse inclusion can be proved similarly. Thus

$$P_{(\mathcal{E},\mathcal{W})}(g_1 \overset{i}{+} g_2) = P_{(\mathcal{E},\mathcal{W})}(g_1) \overset{i}{+} P_{(\mathcal{E},\mathcal{W})}(g_2)$$

for all $g_1, g_2 \in G$ and $i \leqslant n$, which means that $P_{(\mathcal{E},\mathcal{W})}$ is a representation of $\mathfrak{G}$ by $(P_{(\mathcal{E},\mathcal{W})}(g))_{g\in G}$.

The fact that each $P_{(\mathcal{E},\mathcal{W})}(g)$ is a function is a consequence of the permissibility of $\mathcal{E}$. □

We say that a *representation* P of a given positional algebra $\mathfrak{G}$ *is induced by some determining pair* if there exists a determining pair $(\mathcal{E}, \mathcal{W})$ of $\mathfrak{G}$ such that $P = P_{(\mathcal{E},\mathcal{W})}$.

Theorem 6.3.9. *Every representation of a positional algebra by multiplace functions is induced by some determining pair.*

Proof. Let P be a representation of a positional algebra $\mathfrak{G}$ by multiplace functions on A. For each vector $a_1^n = (a_1, \ldots, a_n) \in A^n$ we define the binary relation $\varepsilon^{a_1^n} \subset G_n \times G_n$ and the subset $W^{a_1^n} \subset G_n$ letting

$$g_1 \equiv g_2(\varepsilon^{a_1^n}) \longleftrightarrow P(g_1)\langle a_1^n\rangle = P(g_2)\langle a_1^n\rangle\,,$$
$$g \in W^{a_1^n} \longleftrightarrow P(g)\langle a_1^n\rangle = \varnothing$$

for $g, g_1, g_2 \in G_n$. Moreover, if

$$I(G) = \{n \in \mathbb{N} \,|\, (\exists g \in G)\, n = |g|\}\,,$$
$$\mathcal{E}_P = \{\varepsilon^{a_1^n} \,|\, a_1^n \in A^n,\ n \in I(G)\}\,,$$
$$\mathcal{W}_P = \{W^{a_1^n} \,|\, a_1^n \in A^n,\ n \in I(G)\}\,,$$

then $(\mathcal{E}_P, \mathcal{W}_P)$ is the determining pair of $\mathfrak{G}$ and $P = P_{(\mathcal{E}_P,\mathcal{W}_P)}$. Indeed, each $\varepsilon^{a_1^n}$ is an equivalence relation on G_n. To prove that $\mathcal{E}_P$ is permissible for $\mathfrak{G}^*$, let $g \in G$, $|g| = n$ and

$$x_1 \equiv y_1(\varepsilon^{a_1^{m_1}}),\ \ x_2 \equiv y_2(\varepsilon^{b_1^{m_2}})\ ,\ldots,\ x_n \equiv y_n(\varepsilon^{c_1^{m_n}})\,.$$

This, by definition, implies

$$\begin{aligned} P(x_1)\langle a_1^{m_1}\rangle &= P(y_1)\langle a_1^{m_1}\rangle, \\ P(x_2)\langle b_1^{m_2}\rangle &= P(y_2)\langle b_1^{m_2}\rangle, \\ &\cdots\cdots\cdots\cdots \\ P(x_n)\langle c_1^{m_n}\rangle &= P(y_n)\langle c_1^{m_n}\rangle, \end{aligned}$$

which gives

$$\begin{aligned} &P(g)\Big(P(x_1)\langle a_1^{m_1}\rangle, P(x_2)\langle b_1^{m_2}\rangle, \ldots, P(x_n)\langle c_1^{m_n}\rangle\Big) \\ &= P(g)\Big(P(y_1)\langle a_1^{m_1}\rangle, P(y_2)\langle b_1^{m_2}\rangle, \ldots, P(y_n)\langle c_1^{m_n}\rangle\Big). \end{aligned}$$

This last statement is equivalent to

$$\begin{aligned} &\Big(P(g) \overset{n}{+} P(x_n) \overset{n-1}{+} \cdots \overset{1}{+} P(x_1)\Big)\langle a_1^{m_1} b_1^{m_2} \ldots c_1^{m_n}\rangle \\ &= \Big(P(g) \overset{n}{+} P(y_n) \overset{n-1}{+} \cdots \overset{1}{+} P(y_1)\Big)\langle a_1^{m_1} b_1^{m_2} \ldots c_1^{m_n}\rangle. \end{aligned}$$

Since P is a homomorphism, we have

$$P(g \underset{n}{\overset{1}{+}} x_n^1)\langle a_1^{m_1} \ldots c_1^{m_n}\rangle = P(g \underset{n}{\overset{1}{+}} y_n^1)\langle a_1^{m_1} \ldots c_1^{m_n}\rangle,$$

i.e.,

$$g \underset{n}{\overset{1}{+}} x_n^1 \equiv g \underset{n}{\overset{1}{+}} y_n^1(\varepsilon^{a_1^{m_1} \ldots c_1^{m_n}}).$$

So, $\mathcal{E}_P$ is permissible for $\mathfrak{G}^*$.

To prove that $\mathcal{W}_P$ is an l-ideal, consider $g, x_i \in G$, $|g| = n$, $|x_i| = m_i$, $i = 1, \ldots, n$. By definition $|g \underset{n}{\overset{1}{+}} x_n^1| = \sum_{i=1}^{n} m_i = m$.

If $g \underset{n}{\overset{1}{+}} x_n^1 \notin W^{a_1^{n_1}}$, then $P(g \underset{n}{\overset{1}{+}} x_n^1)\langle a_1^m\rangle \neq \varnothing$, whence

$$\Big(P(g) \overset{n}{+} P(x_n) \overset{n-1}{+} \cdots \overset{i}{+} P(x_i) \overset{i-1}{+} \cdots \overset{1}{+} P(x_1)\Big)\langle a_1^m\rangle \neq \varnothing.$$

Therefore

$$P(g)\Big(P(x_1)\langle a_1^{m_1}\rangle, \ldots, P(x_i)\langle a_{s_{i-1}+1}^{s_{i-1}+m_i}\rangle, \ldots, P(x_n)\langle a_{s_{n-1}}^{m}\rangle\Big) \neq \varnothing,$$

where $s_{i-1} = \sum_{k=1}^{i-1} m_k$. Hence $P(x_i)\langle a_{s_{i-1}+1}^{s_{i-1}+m_i}\rangle \neq \varnothing$, i.e., $x_i \notin W^{\omega_i^{i+m_i}}$ for each $i = 1, \ldots, n$. So, $\mathcal{W}_P$ is an l-ideal and, consequently, $(\mathcal{E}_P, \mathcal{W}_P)$ is the determining pair of $\mathfrak{G}$.

Let us show that $P = P_{(\mathcal{E}_P,\mathcal{W}_P)}$. For this we consider $a_1^n \in A^n$, $b \in A$ and

$$H_b^{a_1^n} = \{g \in G_n \mid P(g)\langle a_1^n\rangle = \{b\}\}.$$

Then

$$g \in H_b^{a_1^n} \longleftrightarrow (a_1^n, b) \in P(g)\,.$$

Obviously $e \in H_a^a$ for any $a \in A$. It is clear that $\{H_b^{a_1^n} \mid b \in A\}$ is the set of $\varepsilon^{a_1^n}$-classes disjoint from $W^{a_1^n}$.

The set of all such classes satisfies (6.3.7). Indeed, if

$$g \overset{n}{+} H_{b_n}^{c_1^{m_n}} \overset{n-1}{+} \cdots \overset{1}{+} H_{b_1}^{a_1^{m_1}} \subset H_c^{a_1^{m_1}\ldots c_1^{m_n}}$$

and

$$g \overset{n}{+} H_{b_n}^{c'^{m'_n}_1} \overset{n-1}{+} \cdots \overset{1}{+} H_{b_1}^{a'^{m'_1}_1} \subset H_d^{a'^{m'_1}_1\ldots c'^{m'_n}_1},$$

then

$$g \underset{n}{\overset{1}{+}} x_n^1 \in H_c^{a_1^{m_1}\ldots c_1^{m_n}} \quad \text{and} \quad g \underset{n}{\overset{1}{+}} y_n^1 \in H_c^{a'^{m'_1}_1\ldots c'^{m'_n}_1},$$

where $x_i \in H_{b_i}^{d_1^{m_i}}$, $y_i \in H_{b_i}^{d'^{m'_i}_1}$, $i = 1, \ldots, n$. Since P is a homomorphism

$$\begin{aligned} c &= P(g)\Big(P(x_1)(a_1^{m_1}), \ldots, P(x_n)(c_1^{m_n})\Big) = P(g)(b_1, \ldots, b_n) \\ &= P(g)\Big(P(y_1)(a'^{m'_1}_1), \ldots, P(y_n)(c'^{m'_n}_1)\Big) = d, \end{aligned}$$

i.e., $c = d$. So, condition (6.3.7) is satisfied.

Now let $(b_1^n, c) \in P(g)$, where $|g| = n$. Obviously $g \in H_c^{b_1^n}$ and $e \in H_{b_i}^{b_i}$ for every $i = 1, \ldots, n$. Thus $g = g \underset{n}{\overset{1}{+}} e$ implies

$$g \overset{n}{+} H_{b_n}^{b_n} \overset{n-1}{+} H_{b_{n-1}}^{b_{n-1}} \overset{n-1}{+} \cdots \overset{1}{+} H_{b_1}^{b_1} \subset H_c^{b_1^n},$$

which proves that $(b_1^n, c) \in P_{(\mathcal{E}_P,\mathcal{W}_P)}(g)$.

Conversely, if $(b_1^n, c) \in P_{(\mathcal{E}_P,\mathcal{W}_P)}(g)$, then for some $a_1^{m_1}, \ldots, c_1^{m_n}$ we have

$$g \overset{n}{+} H_{b_n}^{c_1^{m_n}} \overset{n-1}{+} \cdots \overset{1}{+} H_{b_1}^{a_1^{m_1}} \subset H_c^{a_1^{m_1}\ldots c_1^{m_n}}.$$

This means that $g \underset{n}{\overset{1}{+}} x_n^1 \in H_c^{a_1^{m_1}\ldots c_1^{m_n}}$ for $x_1 \in H_{b_1}^{a_1^{m_1}}$, …, $x_n \in H_{b_n}^{c_1^{m_n}}$. Thus $P(x_1)(a_1^{m_1}) = b_1$, …, $P(x_n)(c_1^{m_n}) = b_n$ and $(a_1^{m_1} \ldots c_1^{m_n}, c) \in P(g \underset{n}{\overset{1}{+}} x_n^1)$.

But P is a homomorphism, hence

$$(a_1^{m_1} \ldots c_1^{m_n}, c) \in P(g) \overset{n}{+} P(x_n) \overset{n-1}{+} \cdots \overset{1}{+} P(x_1).$$

Therefore

$$c = P(g)\Big(P(x_1)(a_1^{m_1}), \ldots, P(x_n)(c_1^{m_n})\Big) = P(g)(b_1, \ldots, b_n) = P(g)(b_1^n),$$

whence $(b_1^n, c) \in P(g)$.

So, $P(g) = P_{(\mathcal{E}_P, \mathcal{W}_P)}(g)$ for every $g \in G$, i.e., $P = P_{(\mathcal{E}_P, \mathcal{W}_P)}$. □

Theorem 6.3.10. *Let $\mathfrak{G}$ be a positional algebra, $\prec$ – a binary relation on G. For the existence of a faithful representation of $\mathfrak{G}$ by n-ary relations that maps $\prec$ onto set-theoretic inclusion it is necessary and sufficient that $\prec$ is an order satisfying the following three conditions:*

$$x \prec y \longrightarrow |x| = |y|, \tag{6.3.12}$$

$$x \prec y \longrightarrow x \overset{i}{+} z \prec y \overset{i}{+} z, \quad \textit{where} \;\; i \leqslant |x| = |y|, \tag{6.3.13}$$

$$x \prec y \longrightarrow u \overset{i}{+} x \prec u \overset{i}{+} y, \quad \textit{where} \;\; i \leqslant |u|. \tag{6.3.14}$$

Proof. The necessity of the conditions is clear and may be left to the reader to verify. To verify their sufficiency, consider a unitary extension $\mathfrak{G}^*$ of a positional algebra $\mathfrak{G}$. For every $g \in G$, $|g| = n$, we define on G^* an $(n+1)$-ary relation $P(g)$ putting

$$(x_1^n, y) \in P(g) \longleftrightarrow y \prec g \underset{n}{\overset{1}{+}} x_n^1\,,$$

where $x_1, \ldots, x_n \in G^*$, $y \in G$. Then, according to the axiom $\mathbf{A}_3$, we have $|y| = |g \underset{n}{\overset{1}{+}} x_n^1| = \sum\limits_{i=1}^{n} |x_i|$.

Let us show that $P : g \mapsto P(g)$ is a faithful representation of $\mathfrak{G}$ by $(n+1)$-ary relations which transforms $\prec$ onto $\subset$.

Indeed, let $g_1, g_2 \in G$, $n = |g_1|$, $m = |g_2|$, $x_1, \ldots, x_{n+m-1} \in G^*$, $y \in G$ and $(x_1^{n+m-1}, y) \in P(g_1 \overset{i}{+} g_2)$, where $i \leqslant n$. The last condition means that

$$y \prec (g_1 \overset{i}{+} g_2) \underset{n+m-1}{\overset{1}{+}} x_{n+m-1}^1\,.$$

Transforming this condition in the same way as in the proof of Theorem 6.3.2, we get

$$y \prec \Big(g_1 \underset{n}{\overset{i+1}{+}} x_{n+m-1}^{i+m}\Big) \overset{i}{+} \Big(g_2 \underset{m}{\overset{1}{+}} x_{i+m-1}^i\Big) \underset{i-1}{\overset{1}{+}} x_{i-1}^1\,,$$

whence

$$(x_1^{i-1}\, g_2 \underset{m}{\overset{1}{+}} x_{i+m-1}^i\, x_{i+m}^{n+m-1}, y) \in P(g_1)\,.$$

But

$$(x_i^{i+m-1}, g_2 \underset{m}{\overset{1}{+}} x_{i+m-1}^i) \in P(g_2)$$

because $g_2 \underset{m}{\overset{1}{+}} x_{i+m-1}^{i} \prec g_2 \underset{m}{\overset{1}{+}} x_{i+m-1}^{i}$ by the assumption on $\prec$. Thus

$$(x_1^{n+m-1}, y) \in P(g_1) \overset{i}{+} P(g_2)\,.$$

So,

$$P(g_1 \overset{i}{+} g_2) \subset P(g_1) \overset{i}{+} P(g_2)\,.$$

To prove the converse inclusion let $(x_1^{n+m-1}, y) \in P(g_1) \overset{i}{+} P(g_2)$. Then $(x_1^{i+m-1}, z) \in P(g_2)$ and $(x_1^{i-1} z\, x_{i+m}^{n+m-1}, y) \in P(g_1)$ for some $z \in G$. Obviously

$$|z| = \sum_{k=1}^{i+m-1} |x_k|, \qquad |y| = \sum_{k=1}^{i-1} |x_k| + |z| + \sum_{k=i+m}^{n+m-1} |x_k|\,.$$

From this, by the definition of $P(g)$, we conclude that

$$z \prec g_2 \underset{m}{\overset{1}{+}} x_{i+m-1}^{i} \quad \text{and} \quad y \prec g_1 \underset{n}{\overset{i+1}{+}} x_{n+m-1}^{i+m} \overset{i}{+} z \underset{i-1}{\overset{1}{+}} x_{i-1}^{1},$$

whence, by (6.3.13) and (6.3.14), we obtain

$$g_1 \underset{n}{\overset{i-1}{+}} x_{n+m-1}^{i+m} \overset{i}{+} z \underset{i-1}{\overset{1}{+}} x_{i-1}^{1} \prec g_1 \underset{n}{\overset{i+1}{+}} x_{n+m-1}^{i+m} \overset{i}{+} \Big(g_2 \underset{m}{\overset{1}{+}} x_{i+m-1}^{i}\Big) \underset{i-1}{\overset{1}{+}} x_{i-1}^{1}.$$

Thus

$$y \prec g_1 \underset{n}{\overset{i+1}{+}} x_{n+m-1}^{i+m} \overset{i}{+} \Big(g_1 \underset{m}{\overset{i}{+}} x_{i+m-1}^{i}\Big) \underset{i-1}{\overset{1}{+}} x_{i-1}^{1},$$

according to the transitivity of the relation $\prec$. This (in the same way as in the proof of Theorem 6.3.2) gives $y \prec (g_1 \overset{i}{+} g_2) \underset{n+m-1}{\overset{1}{+}} x_{n+m-1}^{1}$, which means that $(x_1^{n+m-1}, y) \in P(g_1 \overset{i}{+} g_2)$. So, $P(g_1) \overset{i}{+} P(g_2) \subset P(g_1 \overset{i}{+} g_2)$. Thus $P(g_1 \overset{i}{+} g_2) = P(g_1) \overset{i}{+} P(g_2)$ for all $g_1, g_2 \in G$, i.e., P is a representation of $\mathfrak{G}$ by $(n+1)$-ary relations.

Now, if $g_1 \prec g_2$, then, according to (6.3.13), for $x_1, \ldots, x_n \in G^*$, where $n = |g_1| = |g_2|$, we have $g_1 \underset{n}{\overset{1}{+}} x_n^1 \prec g_2 \underset{n}{\overset{1}{+}} x_n^1$. If $(x_1^n, y) \in P(g_1)$, then $y \prec g_1 \underset{n}{\overset{1}{+}} x_n^1$, whence $y \prec g_2 \underset{n}{\overset{1}{+}} x_n^1$, i.e., $(x_1^n, y) \in P(g_2)$. So, $P(g_1) \subset P(g_2)$. This proves that P maps $\prec$ into $\subset$. In fact, the converse statement is also true. Namely, if $P(g_1) \subset P(g_2)$ for some $g_1, g_2 \in G$, then

$$(\forall x_1^n)(\forall y)\Big((x_1^n, y) \in P(g_1) \longrightarrow (x_1^n, y) \in P(g_2)\Big),$$

i.e., $y \prec g_1 \overset{1}{\underset{n}{+}} x_n^1 \longrightarrow y \prec g_2 \overset{1}{\underset{n}{+}} x_n^1$ for every $x_1, \ldots, x_n, y \in G^*$. In particular, for $x_1 = x_2 = \ldots = x_n = e$ and $y = g_1$, we have

$$g_1 \prec g_1 \overset{1}{\underset{n}{+}} e^n \longrightarrow g_1 \prec g_2 \overset{1}{\underset{n}{+}} e^n.$$

Thus $g_1 \prec g_1 \longrightarrow g_1 \prec g_2$. So, $g_1 \prec g_2$. Summarizing:

$$g_1 \prec g_2 \longleftrightarrow P(g_1) \subset P(g_2).$$

From this equivalence it follows that the representation P is faithful. □

6.4 Mal'cev-Post iterative algebras

Following A. I. Mal'cev (cf. [106], [108]) let us consider on the set $\mathcal{T}(A)$ of all full multiplace functions on A four unary operations $\zeta, \tau, \Delta, \nabla$ and one binary operation $*$ defined in the following way:

$$\zeta f(a_1^n) = f(a_2^n, a_1), \tag{6.4.1}$$

$$\tau f(a_1^n) = f(a_2, a_1, a_3^n), \tag{6.4.2}$$

$$\Delta f(a_1^{n-1}) = f(a_1, a_1^{n-1}), \tag{6.4.3}$$

$$\nabla f(a_1^{n+1}) = f(a_2^{n+1}), \tag{6.4.4}$$

$$(f * g)(a_1^{n+m-1}) = f(g(a_1^m), a_{m+1}^{n+m-1}) \tag{6.4.5}$$

for all $a_1, \ldots, a_{n+m-1} \in A$, $f \in \mathcal{T}_n(A)$, $g \in \mathcal{T}_m(A)$ and $n, m \in \mathbb{N}$. Additionally we put $\zeta f = \tau f = \Delta f = f$ for every $f \in \mathcal{T}_1(A)$. Since the operation $*$ is associative, $(\mathcal{T}(A), *)$ is called the *semigroup of multiplace functions* or the *semigroup of operations*. On the set $\mathcal{T}(A)$, we shall consider the countable set of partial binary operations $\overset{i}{+}$, defined by (6.3.5), where $i \in \mathbb{N}$. It is clear that the operations $\overset{1}{+}$ and $*$ coincide. Also it is not difficult to see that for any $f \in \mathcal{T}_n(A)$, $g \in \mathcal{T}(A)$ and $i = 1, \ldots, n$, we have

$$f \overset{i}{+} g = \zeta^{i-1}((\zeta^{n-i+1} f) * g).$$

Let I_i^n be the i-th n-ary projector on A, i.e., $I_i^n(a_1^n) = a_i$ for $a_1, \ldots, a_n \in A$. It is evident that $I_1^1 = \Delta I_1^2$, $I_2^2 = \tau I_2^2 = \nabla I_1^1$, $I_n^n = (I_2^2)^{n-1} = \nabla^{n-1} I_1^1$, $I_i^n = \zeta^i I_n^n = \zeta^i(\nabla^{n-1} I_1^1)$. If $|f|$ denotes the arity of the operation f, then $|I_i^n| = n$, $|\zeta f| = |\tau f| = |f|$, $|\Delta f| = |f| - 1$ for $|f| \geqslant 2$, and $|\nabla f| = |f| + 1$ for all $f, g \in \mathcal{T}(A)$.

An algebra of the form $(\mathcal{T}(A), \zeta, \tau, \Delta, \nabla, *)$ is called an *iterative Mal'cev-Post algebras of operations*, an algebra of the form $(\mathcal{T}(A), \zeta, \tau, \Delta, *)$ – a *pre-iterative*

Mal'ev-Post algebra of operations. Iterative (pre-iterative) Mal'cev-Post algebras of operations containing the projector I_1^1 (respectively I_2^2) in the signature, are called *central iterative* (*pre-iterative*) *Mal'cev-Post algebras*.[6] Each subalgebra of an iterative (pre-iterative) Mal'cev-Post algebra of operations will be called for simplicity's sake an *iterative* (*pre-iterative*) *operation algebra*.

By D_n^m, we denote the composition $H_0H_1\ldots H_{m-1}$ of unary operations H_k, where $H_k = \zeta^{m-1}(\Delta\zeta(\tau\zeta)^k)^{n-1}$ for $0 \leqslant k \leqslant m-1$. It is not difficult to verify that

$$(D_n^m f)(a_1^m) = f(\underbrace{a_1^m, a_1^m, \ldots, a_1^m}_{n \text{ times}}) \tag{6.4.6}$$

for all $a_1, a_2, \ldots, a_m \in A$ and $f \in \mathcal{T}(A)$ such that $|f| = nm$. Indeed, for every $0 \leqslant k \leqslant m-1$, we have

$$\begin{aligned}
& f(a_{m-k}^m, a_1^m, \underbrace{a_{m-k}^m, \ldots, a_{m-k}^m}_{n-2}) \\
&= ((\tau\zeta)^k f)(a_{m-k}^m, a_{m-k+1}^m, a_1^m, \underbrace{a_{m-k}^m, \ldots, a_{m-k}^m}_{n-3}, a_{m-k}) \\
&= (\Delta\zeta(\tau\zeta)^k f)(a_{m-k}^m, a_{m-k+1}^m, a_1^m, \underbrace{a_{m-k}^m, \ldots, a_{m-k}^m}_{n-3}) \\
&= ((\Delta\zeta(\tau\zeta)^k)^{n-1} f)(a_{m-k}^m, \underbrace{a_{m-k+1}^m, \ldots, a_{m-k+1}^m}_{n-1}, a_1^{m-k+1}) \\
&= (\zeta^{m-1}(\Delta\zeta(\tau\zeta)^k)^{n-1} f)(a_{m-k+1}^m, a_1^m, \underbrace{a_{m-k+1}^m, \ldots, a_{m-k+1}^m}_{n-2}).
\end{aligned}$$

Thus, we have showed that

$$f(a_{m-k}^m, a_1^m, \underbrace{a_{m-k}^m, \ldots, a_{m-k}^m}_{n-2}) = (H_k f)(a_{m-k+1}^m, a_1^m, \underbrace{a_{m-k+1}^m, \ldots, a_{m-k+1}^m}_{n-2})$$

for $0 \leqslant k \leqslant m-1$. Consequently

$$\begin{aligned}
f(a_1^m, a_1^m, \ldots, a_1^m) &= (H_{m-1} f)(a_2^m, a_1^m, a_2^m, \ldots, a_2^m) \\
&= (H_{m-2}H_{m-1} f)(a_3^m, a_1^m, a_3^m, \ldots, a_3^m) \\
&= \cdots = (H_1 \ldots H_{m-1} f)(a_m, a_1^m, a_m \ldots, a_m) \\
&= (D_n^m f)(a_1^m),
\end{aligned}$$

which completes the proof of (6.4.6).

[6] In fact every such algebra is unitary, i.e., it contains all projectors because for all $i = 1, \ldots, n-1$, $n \geqslant 2$, we have $I_n^n = (I_2^2)^{n-1}$, $I_i^n = \zeta^i I_n^n$ and $I_1^1 = \Delta I_1^2$, $I_2^2 = \nabla I_1^1$. But for the simplicity of proofs and axioms we assume only the existence of one projector.

Proposition 6.4.1. *Let $f, g, h \in \mathcal{T}(A)$ and $n, m \in \mathbb{N}$. Then*

(a) $D_1^m f = f$, *if* $|f| = m$,

(b) $D_n^m(\zeta^m f) = D_n^m f$, *if* $|f| = nm$,

(c) $D_n^m(((\tau f) \overset{2}{+} g) * h) = D_n^m((f \overset{2}{+} h) * g)$, *if* $|f| = (n-2)m + 2$ *and* $|g| = |h| = m$,

(d) $D_n^m((\Delta f) * g) = D_{n+1}^m((f \overset{2}{+} h) * g)$, *if* $|f| = (n-1)m + 2$, $|g| = m$,

(e) $D_n^m((\nabla f) * g) = D_{n-1}^m f$, *if* $|f| = (n-1)m$, $|g| = m$,

(f) $D_{l+n-1}^m(f * g) = D_l^m(f * D_n^m g)$, *if* $|g| = nm$ *and* $|f| = (l-1)m + 1$.

Proof. We limit ourselves to the proof of (c) and (f). The proof of the other conditions is very similar.

Let $|f| = (n-2)m + 2$, $|g| = |h| = m$. Then $|((\tau f) \overset{2}{+} g) * h| = nm$ and

$$\begin{aligned} D_n^m(((\tau f) \overset{2}{+} g) * h)(a_1^m) &= (((\tau f) \overset{2}{+} g) * h)(a_1^m, a_1^m, \dots, a_1^m) \\ &= ((\tau f) \overset{2}{+} g)(h(a_1^m), a_1^m, \dots, a_1^m) \\ &= (\tau f)(h(a_1^m), g(a_1^m), a_1^m, \dots, a_1^m) \\ &= f(g(a_1^m), h(a_1^m), a_1^m, \dots, a_1^m) \\ &= D_n^m((f \overset{2}{+} h) * g)(a_1^m) \end{aligned}$$

for all $a_1, \dots, a_m \in A$, which proves (c).

Now let $|g| = nm$, $|f| = (l-1)m + 1$. Then $|f * g| = (l+n-1)m$ and

$$\begin{aligned} (D_{l+n-1}^m(f * g))(a_1^m) &= (f * g)(\underbrace{a_1^m, a_1^m, \dots, a_1^m}_{l+n-1}) \\ &= f(g(\underbrace{a_1^m, \dots, a_1^m}_{n}), a_1^m, \dots, a_1^m) \\ &= f(D_n^m g(a_1^m), \underbrace{a_1^m, \dots, a_1^m}_{l-1}) \\ &= (f * D_n^m g)(\underbrace{a_1^m, \dots, a_1^m}_{l}) = D_l^m(f * D_n^m g)(a_1^m), \end{aligned}$$

for any $a_1, \dots, a_m \in A$. This proves (f). □

Let $(G, \zeta, \tau, \Delta, \nabla, *)$ be an abstract algebra of type $(1,1,1,1,2)$. For each $g \in G$ let αg denote the smallest positive integer n such that $(\nabla g) * g = \nabla^n g$.

The existence and uniqueness of αg is guaranteed by the axioms. By definition we put

$$g_1 \overset{i}{+} g_2 = \zeta^{i-1}((\zeta^{n-i+1}g_1) * g_2)$$

for $g_1, g_2 \in G$, $1 \leqslant i \leqslant n = \alpha g_1$ and $D_n^m = H_0 H_1 \ldots H_{m-1}$, where $H_k = \zeta^{m-1}(\Delta\zeta(\tau\zeta)^k)^{n-1}$ for $0 \leqslant k \leqslant m-1$, $n, m \in \mathbb{N}$.

Theorem 6.4.2. *The algebra $(G, e, \zeta, \tau, \Delta, \nabla, *)$ of type $(0,1,1,1,1,2)$ is isomorphic to a central iterative operation algebra if and only if for $g, g_1, g_2, g_3 \in G$ and $i, j, l, n, m \in \mathbb{N}$ the following conditions are valid:*

1.1. $(\nabla g) * g = \nabla^k g$ *for some* $k \in \mathbb{N}$,

1.2. $\nabla^n g = \nabla^m g$ *implies* $n = m$,

1.3. $\nabla(g_1 * g_2) = g_1 * \nabla g_2$,

1.4. $(\nabla g_1) * g_2 = \nabla^n g_1$ *for* $n = \alpha g_2$,

1.5. $\alpha\zeta g = \alpha\tau g = \alpha g$ *and* $\alpha\Delta g = \alpha g - 1$ *for* $\alpha g \geqslant 2$,

1.6. $g * e = e * g = g$,

1.7. $\zeta g = \tau g = \Delta g = g$ *for* $\alpha g = 1$,

1.8. $\zeta^n g = g$ *for* $n = \alpha g$,

1.9. $(\tau g_1) \overset{i}{+} g_2 = \tau(g_1 \overset{i}{+} g_2)$ *for* $3 \leqslant i \leqslant \alpha g_1$,

1.10. $(\Delta g_1) \overset{i}{+} g_2 = \Delta(g_1 \overset{i+1}{+} g_2)$ *for* $2 \leqslant i \leqslant \alpha g_1 - 1$,

1.11. $(\nabla g_1) \overset{i}{+} g_2 = \nabla(g_1 \overset{i-1}{+} g_2)$ *for* $2 \leqslant i \leqslant \alpha g_1 + 1$,

1.12. $g_1 \overset{i}{+} (g_2 \overset{j}{+} g_3) = (g_1 \overset{i}{+} g_2) \overset{i+j-1}{+} g_3$ *for* $i \leqslant \alpha g_1$, $j \leqslant \alpha g_2$,

1.13. $(g_1 \overset{i}{+} g_2) \overset{j}{+} g_3 = (g_1 \overset{j}{+} g_3) \overset{i+n-1}{+} g_2$ *for* $j < i \leqslant \alpha g_1$, $n = \alpha g_3$,

1.14. $D_n^m(\zeta^m g) = D_n^m g$ *for* $\alpha g = nm$,

1.15. $D_n^m(((\tau g_1) \overset{2}{+} g_2) * g_3) = D_n^m((g_1 \overset{2}{+} g_3) * g_2)$, *where* $\alpha g_1 = (n-2)m + 2$, $\alpha g_2 = \alpha g_3 = m$,

1.16. $D_n^m((\Delta g_1) * g_2) = D_{n+1}^m((g_1 \overset{2}{+} g_2) * g_2)$, *where* $\alpha g_1 = (n-1)m + 2$, $\alpha g_2 = m$,

1.17. $D_n^m((\nabla g_1) * g_2) = D_{n-1}^m g_1$, *where* $\alpha g_1 = (n-1)m$, $\alpha g_2 = m$,

1.18. $D_{l+n-1}^m(g_1 * g_2) = D_l^m(g_1 * D_n^m g_2)$, *where* $\alpha g_1 = (l-1)m + 1$ *and* $\alpha g_2 = nm$,

1.19. $D_n^n(((\ldots((g \overset{n}{+} e_n^{(n)}) \overset{n-1}{+} e_{n-1}^{(n)}) \ldots) \overset{2}{+} e_2^{(n)}) * e_1^{(n)}) = g$, *where* $n = \alpha g$, $e_i^{(n)} = \zeta^i \nabla^{n-1} e$ *for all* $n \in \mathbb{N}$ *and* $1 \leqslant i \leqslant n$.

Proof. Necessity. Let $(\Phi, I_1^1, \zeta, \tau, \Delta, \nabla, *)$, where $\Phi \subset \mathcal{T}(A)$, be an iterative operation algebra with a distinguished projector I_1^1. From Proposition 6.4.1, it follows that this algebra satisfies conditions 1.14–1.18. By the results obtained in the previous sections 1.12 and 1.13 are also satisfied. Conditions 1.5–1.7 are obvious. The other conditions can be deduced directly from the definition of the operations $\zeta, \tau, \Delta, \nabla, *$. For example, to prove 1.9, let $f, g \in \Phi$, $3 \leqslant i \leqslant \alpha f = |f|$ and $a_1, \dots, a_n \in A$, where $n = |f \overset{i}{+} g|$, $m = |g|$. Then

$$\begin{aligned}((\tau f) \overset{i}{+} g)(a_1^n) &= \tau f(a_1, a_2, a_3^{i-1}, g(a_i^{i+m-1}), a_{i+m}^n) \\ &= f(a_2, a_1, a_3^{i-1}, g(a_i^{i+m-1}), a_{i+m}^n) = (f \overset{i}{+} g)(a_2, a_1, a_3^n) \\ &= \tau(f \overset{i}{+} g)(a_1^n).\end{aligned}$$

To prove 1.19, observe that for $f \in \Phi$, $n = |f|$, $a_1, \dots, a_n \in A$ we have

$$\begin{aligned}&D_n^n((\dots((f \overset{n}{+} I_n^n) \overset{n-1}{+} I_{n-1}^n)\dots) \overset{2}{+} I_2^n) * I_1^n(a_1^n) \\ &= ((\dots((f \overset{n}{+} I_n^n) \overset{n-1}{+} I_{n-1}^n)\dots) \overset{2}{+} I_2^n) * I_1^n(\underbrace{a_1^n, \dots, a_1^n}_{n}) \\ &= f(I_1^n(a_1^n), I_2^n(a_1^n), \dots, I_{n-1}^n(a_1^n), I_n^n(a_1^n)) = f(a_1^n),\end{aligned}$$

which completes the proof of 1.19.

Sufficiency. Let the algebra $\mathcal{G} = (G, e, \zeta, \tau, \Delta, \nabla, *)$ satisfy all the conditions of the theorem. Directly from the conditions 1.1 and 1.2, it follows that for every $g \in G$ there exists a uniquely determined positive integer αg such that $(\nabla g) * g = \nabla^{\alpha g} g$. Moreover, by 1.3 and 1.4, we obtain

$$(\nabla^2 g) * \nabla g = \nabla((\nabla^2 g) * g) = \nabla^{\alpha g + 2} g = \nabla^{\alpha g + 1}(\nabla g),$$

which proves $\alpha(\nabla g) = \alpha g + 1$. Similarly $g_1 \overset{1}{+} g_2 = (\zeta^{\alpha g_1}) * g_2$, by 1.8, gives $g_1 \overset{1}{+} g_2 = g_1 * g_2$. Now putting $i = j = 1$ in 1.12 we obtain the associativity of the operation $*$. So,

$$\begin{aligned}\nabla(g_1 * g_2) * g_1 * g_2 &= g_1 * (\nabla g_2) * g_1 * g_2 = g_1 * (\nabla^{\alpha g_1} g_2) \\ &= g_1 * (\nabla(\nabla^{\alpha g_1 - 1} g_2)) * g_2 = g_1 * \nabla^{\alpha g_2}(\nabla^{\alpha g_1 - 1} g_2) \\ &= g_1 * \nabla^{\alpha g_1 + \alpha g_2 - 1} g_2 = \nabla^{\alpha g_1 + \alpha g_2 - 1}(g_1 * g_2).\end{aligned}$$

Thus $\alpha(g_1 * g_2) = \alpha g_1 + \alpha g_2 - 1$, whence, according to 1.5, we deduce $\alpha(g_1 \overset{i}{+} g_2) = \alpha g_1 + \alpha g_2 - 1$ for every $1 \leqslant i \leqslant \alpha g_1$. But $(\nabla e) * e = \nabla e$, by 1.6, consequently $\alpha e = 1$. Let us consider the element $D_n^m g$ for some $n, m \in \mathbb{N}$ and $g \in G$. Since D_n^m contains the operation Δ precisely $(n-1)m$ times, 1.5 and

1.7 imply $\alpha D_n^m g = \alpha g - (n-1)m$ for $\alpha g \geqslant (n-1)m+1$. For $\alpha g < (n-1)m$ we have $\alpha D_n^m g = 1$. So, $\alpha D_n^m g = m$ for $\alpha g = nm$.

Let us show that for $g_1, g_2 \in G$ the following conditions are valid:

$$(\zeta g_1) * g_2 = \zeta^m (g_1 \overset{n}{+} g_2) \ \ \textit{for} \ \ n = \alpha g_1, \ m = \alpha g_2, \tag{6.4.7}$$

$$(\zeta g_1) \overset{i}{+} g_2 = \zeta (g_1 \overset{i-1}{+} g_2) \ \ \textit{for} \ \ 2 \leqslant i \leqslant \alpha g_1, \tag{6.4.8}$$

$$D_1^m g = g \ \ \textit{for} \ \ m = \alpha g. \tag{6.4.9}$$

Indeed, $\zeta^m(g_1 \overset{i}{+} g_2) = \zeta^m \zeta^{n-1}((\zeta g_1) * g_2) = \zeta^{n+m-1}((\zeta g_1) * g_2)$. Since $\alpha(\zeta g_1) * g_2 = n+m-1$, then, according to 1.8, $\zeta^m(g_1 \overset{n}{+} g_2) = (\zeta g_1) * g_2$. This proves (6.4.7). Furthermore, for $2 \leqslant i \leqslant \alpha g_1$, we have

$$\begin{aligned}(\zeta g_1) \overset{i}{+} g_2 &= \zeta^{i-1}((\zeta^{\alpha g_1 - i + 2} g_1) * g_2) \\ &= \zeta(\zeta^{i-2}((\zeta^{\alpha g_1 - i+2} g_1) * g_2)) = \zeta(g_1 \overset{i-1}{+} g_2),\end{aligned}$$

which proves (6.4.8). By definition $D_1^m = (\zeta^{m-1})^m = (\zeta^m)^{m-1}$. This, by 1.8, implies $D_1^m g = (\zeta^m)^{m-1} g = g$ for $m = \alpha g$. So, (6.4.9) is true.

Consider now the family $(G_n)_{n\in\mathbb{N}}$ of sets $G_n = \{g \in G \,|\, \alpha g = n\}$. It is clear that $G = \bigcup_{n\in\mathbb{N}} G_n$ and $G_n \cap G_m = \varnothing$ for $n \neq m$. Obviously each G_n is nonempty because $e_i^{(n)} \in G_n$ for all $i = 1, \ldots, n$. Let $\overline{G}$ be the Cartesian product of the family $(G_n)_{n\in\mathbb{N}}$ and let $\bar{x}(k)$ be a component of $\bar{x} \in \overline{G}$ which belongs to G_k. For every $g \in G$, where $\alpha g = n$, we define an n-ary operation f_g on the set $\overline{G}$ putting $f_g(\bar{x}_1, \ldots, \bar{x}_n) = \bar{y}$ if and only if

$$\bar{y}(k) = D_n^k((...((g \overset{n}{+} \bar{x}_n(k)) \overset{n-1}{+} \bar{x}_{n-1}(k)) \overset{n-2}{+} \cdots) \overset{1}{+} \bar{x}_1(k)) \tag{6.4.10}$$

for every $k \in \mathbb{N}$.

Let us show that $P : g \mapsto f_g$ is an isomorphism of $\mathcal{G}$ into the central iterative Mal'cev-Post operation algebra $(\mathcal{T}(\overline{G}), I_1^1, \zeta, \tau, \Delta, \nabla, *)$ with a distinguished projector I_1^1. Since $\alpha e = 1$, for every $\bar{x} \in \overline{G}$ and $k \in \mathbb{N}$, by 1.6 and (6.4.9), we have

$$\bar{y}(k) = D_1^k(e \overset{1}{+} \bar{x}(k)) = D_1^k(e * \bar{x}(k)) = D_1^k(\bar{x}(k)) = \bar{x}(k),$$

which implies $f_e(\bar{x}) = \bar{x}$ for every $\bar{x} \in \overline{G}$. Thus $P(e) = I_1^1$.

Now let $g \in G$ and $n = \alpha g$. Then, applying (6.4.7) and (6.4.8), for all $\bar{x}_1, \ldots, \bar{x}_n \in \overline{G}$ and $k \in \mathbb{N}$ we get

$$\begin{aligned}&D_n^k((...((\zeta g) \overset{n}{+} \bar{x}_n(k)) \overset{n-1}{+} \cdots) \overset{1}{+} \bar{x}_1(k)) \\ &= D_n^k((...(\zeta(g \overset{n-1}{+} \bar{x}_n(k)) \overset{n-1}{+} \bar{x}_{n-1}(k)) \overset{n+2}{+} \cdots) \overset{1}{+} \bar{x}_1(k)) \\ &= \cdots = D_n^k(\zeta(...(g \overset{n-1}{+} \bar{x}_n(k)) \overset{n-2}{+} \cdots) \overset{1}{+} \bar{x}_1(k)) \\ &= D_n^k \zeta^k((...(g \overset{n-1}{+} \bar{x}_n(k)) \overset{n-2}{+} \cdots) \overset{s}{+} \bar{x}_1(k)),\end{aligned}$$

where $s = (n-1)k + 1$. Applying to this last expression $n-1$ times condition 1.13 we obtain

$$D_n^k \zeta^k((...((g \overset{n}{+} \bar{x}_1(k)) \overset{n-1}{+} \bar{x}_n(k)) \overset{n-2}{+} \cdots) \overset{1}{+} \bar{x}_2(k)),$$

whence, by 1.14, we conclude

$$D_n^k((...((g \overset{n}{+} \bar{x}_1(k)) \overset{n-1}{+} \bar{x}_n(k)) \overset{n-2}{+} \cdots) \overset{1}{+} \bar{x}_2(k)).$$

So,

$$\begin{aligned} &D_n^k((...((\zeta g) \overset{n}{+} \bar{x}_n(k)) \overset{n-1}{+} \cdots) \overset{1}{+} \bar{x}_1(k)) \\ &\qquad = D_n^k((...((g \overset{n}{+} \bar{x}_1(k)) \overset{n-1}{+} \bar{x}_n(k)) \overset{n-2}{+} \cdots) \overset{1}{+} \bar{x}_2(k)), \end{aligned}$$

i.e.,

$$f_{\zeta g}(\bar{x}_1, \ldots, \bar{x}_n) = f_g(\bar{x}_2, \ldots, \bar{x}_n, \bar{x}_1) = \zeta f_g(\bar{x}_1, \ldots, \bar{x}_n).$$

Thus, $P(\zeta g) = \zeta P(g)$ for every $g \in G$. Analogously, using respectively 1.9 and 1.15, 1.10 and 1.16, 1.11 and 1.17, we prove that $P(\tau g) = \tau P(g)$, $P(\Delta g) = \Delta P(g)$ and $P(\nabla g) = \nabla P(g)$ for all $g \in G$.

Now let $g_1, g_2 \in G$, $n = \alpha g_1$, $m = \alpha g_2$, $s = n + m - 1$. Then for all $\bar{x}_1, \ldots, \bar{x}_s \in \overline{G}$, $k \in \mathbb{N}$, according to 1.12, 1.13 and 1.18 we get

$$\begin{aligned} &D_s^k((...((g_1 * g_2) \overset{s}{+} \bar{x}_s(k))...) \overset{1}{+} \bar{x}_1(k)) = \\ &D_s^k((...((g_1 \overset{n}{+} \bar{x}_s(k)) * g_2)...) \overset{1}{+} \bar{x}_1(k)) = \cdots = \\ &D_s^k((...((((...(g_1 \overset{n}{+} \bar{x}_s(k))...) \overset{m+1}{+} \bar{x}_{m+1}(k)) * g_2) \overset{m}{+} \bar{x}_m(k)) \ldots) \overset{1}{+} \bar{x}_1(k)) = \\ &D_s^k((...(((...(g_1 \overset{n}{+} \bar{x}_s(k))...) \overset{m+1}{+} \bar{x}_{m+1}(k)) \\ &\qquad\qquad * (g_2 \overset{m}{+} \bar{x}_m(k)) \overset{m-1}{+} \bar{x}_{m-1}(k))...) \overset{1}{+} \bar{x}_1(k)) = \cdots = \\ &D_s^k(((...(g_1 \overset{n}{+} \bar{x}_s(k))...) \overset{m+1}{+} \bar{x}_{m+1}(k)) * ((...(g_2 \overset{m}{+} \bar{x}_m(k))...) \overset{1}{+} \bar{x}_1(k))) = \\ &D_n^k(((...(g_1 \overset{n}{+} \bar{x}_s(k))...) \overset{m+1}{+} \bar{x}_{m+1}(k)) * D_m^k((...(g_2 \overset{m}{+} \bar{x}_m(k))...) \overset{1}{+} \bar{x}_1(k))). \end{aligned}$$

This means that

$$f_{g_1 * g_2}(\bar{x}_1, \ldots, \bar{x}_s) = f_{g_1}(f_{g_2}(\bar{x}_1, \ldots, \bar{x}_m), \bar{x}_{m+1}, \ldots, \bar{x}_s),$$

i.e.,

$$f_{g_1 * g_2}(\bar{x}_1, \ldots, \bar{x}_s) = (f_{g_1} * f_{g_2})(\bar{x}_1, \ldots, \bar{x}_s).$$

So, $P(g_1 * g_2) = P(g_1) * P(g_2)$ for all $g_1, g_2 \in G$. In this way we have proved that P is a homomorphism.

To prove that P is an isomorphism we need the elements $\bar{e}_k \in \overline{G}$, $k \in \mathbb{N}$, where
$$\bar{e}_k = (e_1^{(1)}, e_2^{(2)}, \ldots, e_k^{(k)}, e_k^{(k+1)}, e_k^{(k+2)}, \ldots).$$
If $P(g_1) = P(g_2)$ for some $g_1, g_2 \in G$, then $f_{g_1}(\bar{x}_1, \ldots, \bar{x}_n) = f_{g_2}(\bar{x}_1, \ldots, \bar{x}_n)$ for all $\bar{x}_1, \ldots, \bar{x}_n \in \overline{G}$, where $n = \alpha g_1 = \alpha g_2$. In particular, $f_{g_1}(\bar{e}_1, \ldots, \bar{e}_n) = f_{g_2}(\bar{e}_1, \ldots, \bar{e}_n)$, whence we obtain

$$D_n^n((...(g_1 \overset{n}{+} \bar{e}_n(n)) \overset{n-1}{+} ...) \overset{1}{+} \bar{e}_1(n)) = D_n^n((...(g_2 \overset{n}{+} \bar{e}_n(n)) \overset{n-1}{+} ...) \overset{1}{+} \bar{e}_1(n)),$$

i.e.,

$$D_n^n((...(g_1 \overset{n}{+} e_n^{(n)}) \overset{n-1}{+} ...) \overset{1}{+} e_1^{(n)}) = D_n^n((...(g_2 \overset{n}{+} e_n^{(n)}) \overset{n-1}{+} ...) \overset{1}{+} e_1^{(n)}).$$

This, by 1.19, implies $g_1 = g_2$. So, P is an isomorphism. □

Let $\mathcal{G} = (G, \zeta, \tau, \Delta, \nabla, *)$ be an algebra of type $(1,1,1,1,2)$. For all $g \in G$, $n, m, i, k \in \mathbb{N}$, where $1 \leqslant i \leqslant m$, $1 \leqslant k \leqslant n$, $n = \alpha g$, we will use the following abbreviated notions

$$g \overset{k}{+} \varepsilon_i^{(m)} := \zeta^{k-1}\nabla^{i-1}\zeta\nabla^{m-i}\zeta^{n-k}g \quad \textit{and} \quad \varepsilon_i^{(m)} \overset{i}{+} g := \nabla^{i-1}\zeta^n\nabla^{m-i}g,$$

where $\varepsilon_i^{(m)}$ do not belongs to G, and $\varepsilon_i^{(m)} \neq \varepsilon_j^{(t)}$ for $m \neq t$ or $i \neq j$.

Theorem 6.4.3. *An algebra $(G, \zeta, \tau, \Delta, \nabla, *)$ of type $(1,1,1,1,2)$ is isomorphic to an iterative operation algebra if and only if for all elements $g, g_1, g_2, g_3 \in G$ and $i, j, k, l, n, m, r \in \mathbb{N}$ the following conditions are valid:*

2.1. $(\nabla g) * g = \nabla^s g$ *for some* $s \in \mathbb{N}$,

2.2. $\nabla^n g = \nabla^m g$ *implies* $n = m$,

2.3. $\nabla(g_1 \overset{1}{+} g_2) = g_1 \overset{1}{+} \nabla g_2$, *where* $g_1 \in G \cup \{\varepsilon_1^{(m)}\}$,

2.4. $(\nabla g_1) \overset{1}{+} g_2 = \nabla^m g_1$ *for* $m = \alpha g_2$, $g_2 \in G \cup \{\varepsilon_i^{(m)}\}$, $i \leqslant m$,

2.5. $\alpha\zeta g = \alpha\tau g = \alpha g$,

2.6. $\alpha\Delta g = \alpha g - 1$ *for* $\alpha g \geqslant 2$,

2.7. $\zeta g = \tau g = \Delta g = g$ *for* $\alpha g = 1$,

2.8. *(a)* $\zeta^n g = g$ *for* $n = \alpha g$,
(b) $\zeta^{n+m}\nabla^{r+m} g = \nabla^m \zeta^n \nabla^r g$ *for* $n = \alpha g$,

2.9. *(a)* $(\tau g_1) \overset{k}{+} g_2 = \tau(g_1 \overset{k}{+} g_2)$ *for* $3 \leqslant k \leqslant \alpha g_1$, $g_2 \in G \cup \{\varepsilon_i^{(m)}\}$, $i \leqslant m$,
(b) $\tau\nabla^k g = \nabla^k g$ *for* $k \geqslant 2$

2.10. *(a)* $(\Delta g_1) \overset{k}{+} g_2 = \Delta(g_1 \overset{k+1}{+} g_2)$ *for* $2 \leqslant k \leqslant \alpha g_1 - 1$, $g_2 \in G \cup \{\varepsilon_i^{(m)}\}$, $i \leqslant m$,

(b) $\Delta\nabla g = g$,

2.11. $(\nabla g_1) \overset{k}{+} g_2 = \nabla(g_1 \overset{k-1}{+} g_2)$ *for* $2 \leqslant k \leqslant \alpha g_1 + 1$, $g_2 \in G \cup \{\varepsilon_i^{(m)}\}$, $i \leqslant m$,

2.12. *(a)* $g_1 \overset{k}{+} (g_2 \overset{l}{+} g_3) = (g_1 \overset{k}{+} g_2) \overset{k+l-1}{+} g_3$ *for* $k \leqslant \alpha g_1$, $l \leqslant \alpha g_2$ *and* $g_1 \in G \cup \{\varepsilon_k^{(m)}\}$, $g_3 \in G \cup \{\varepsilon_j^{(r)}\}$, $k \leqslant m$, $j \leqslant r$,

(b) *if* $l \geqslant k$, *then*

$$(g \overset{k}{+} \varepsilon_i^{(m)}) \overset{l}{+} \varepsilon_j^{(r)} = \begin{cases} g \overset{k}{+} \varepsilon_i^{(m+r-1)} & \text{for } i < l-k+1, \\ g \overset{k}{+} \varepsilon_{i+j-1}^{(m+r-1)} & \text{for } i = l-k+1, \\ g \overset{k}{+} \varepsilon_{m+i-1}^{(m+r-1)} & \text{for } i > l-k+1, \end{cases}$$

where $k \leqslant \alpha g$, $i \leqslant m$, $j \leqslant r$, $l-k+1 \leqslant m$,

(c) *if* $k \leqslant \alpha g_1$, $i \leqslant m$, $r = \alpha g_2$, *then*

$$(g_1 \overset{k}{+} \varepsilon_i^{(m)}) \overset{k+j-1}{+} g_2 = \begin{cases} g_1 \overset{k}{+} \varepsilon_i^{(m+r-1)} & \text{for } i < j, \\ g_1 \overset{k}{+} (\varepsilon_j^{(m)} \overset{j}{+} g_2) & \text{for } i = j, \\ g_1 \overset{k}{+} \varepsilon_{i+r-1}^{(m+r-1)} & \text{for } i > j, \end{cases}$$

2.13. *(a)* $(g_1 \overset{k}{+} g_2) \overset{l}{+} g_3 = (g_1 \overset{l}{+} g_3) \overset{k+r-1}{+} g_2$ *for* $l < k \leqslant n = \alpha g_1$, $r = \alpha g_3$, *where* $g_1 \in G$, $g_2 \in G \cup \{\varepsilon_i^{(m)}\}$, $g_3 \in G \cup \{\varepsilon_j^{(r)}\}$ *or* $g_1 \in \{\varepsilon_k^{(n)}, \varepsilon_l^{(n)}\}$, $g_2, g_3 \in G$, *for* $i \leqslant m$, $j \leqslant r$,

(b) $\varepsilon_l^{(n+n-1)} \overset{l}{+} g = (\varepsilon_l^{(n)} \overset{l}{+} g) \overset{k+r-1}{+} \varepsilon_i^{(m)}$, *where* $l < k \leqslant n$, $r = \alpha g$, $i \leqslant m$,

(c) $(\varepsilon_k^{(n)} \overset{k}{+} g) \overset{l}{+} \varepsilon_i^{(m)} = \nabla^{m-1}(\varepsilon_k^{(n)} \overset{k}{+} g)$, *where* $l < k \leqslant n$, $i \leqslant m$,

2.14. $D_n^m(\zeta^m g) = D_n^m g$, *where* $\alpha g = nm$,

2.15. *(a)* $D_n^m(((\tau g_1) \overset{2}{+} g_2) \overset{1}{+} g_3) = D_n^m((g_1 \overset{2}{+} g_3) \overset{1}{+} g_2)$, *where* $m = \alpha g_2 = \alpha g_3$, $\alpha g_1 = (n-2)m + 2$, $g_2 \in G \cup \{\varepsilon_i^{(m)}\}$, $g_3 \in G \cup \{\varepsilon_j^{(m)}\}$, $i, j \leqslant m$,

(b) $D_n^m(\zeta^m \nabla^{(n-1)m} g) = D_n^m(\nabla^m \zeta^m \nabla^{(n-2)m} g)$, *where* $m = \alpha g$,

2.16. *(a)* $D_n^m((\Delta g_1) \overset{1}{+} g_2) = D_{n+1}^m((g_1 \overset{2}{+} g_2) \overset{1}{+} g_2)$ *for* $\alpha g_1 = (n-1)m + 2$, $\alpha g_2 = m$, $g_2 \in G \cup \{\varepsilon_i^{(m)}\}$, *(b)* $D_n^m g = D_{n+1}^m \nabla^m g$, *where* $\alpha g = mn$, *(c)* $D_n^m \nabla^m g = D_n^m \zeta^m \nabla^m g$, *where* $\alpha g = m(n-1)$,

2.17. $D_n^m((\nabla g_1) \overset{1}{+} g_2) = D_{n-1}^m g_1$, *where* $\alpha g_1 = (n-1)m$, $\alpha g_2 = m$, $g_2 \in G \cup \{\varepsilon_i^{(m)}\}$, $i \leqslant m$,

2.18. *(a)* $D_{l+n-1}^m (g_1 \overset{1}{+} g_2) = D_l^m (g_1 \overset{1}{+} D_n^m g_2)$, *where* $\alpha g_1 = (l-1)m+1$, $\alpha g_2 = nm$,
(b) $D_{l+n-1}^m(\zeta^{mn}\nabla^{(l-1)m} g) = D_l^m(\zeta^n \nabla^{(l-1)m} D_n^m g)$, *where* $\alpha g = mn$,

2.19. $D_n^n\Big(\Big(\prod\limits_{i=1}^{n} \zeta^{i-1}\nabla^{i-1}\zeta\nabla^{n-i}\zeta^{(n-i)n}\Big) g\Big) = g$, *where* $n = \alpha g$.

Proof. Necessity. Let $(\mathcal{T}(A), \zeta, \tau, \Delta, \nabla, *)$ be an iterative Mal'cev-Post operation algebra with n-place projectors I_i^n, $n \in \mathbb{N}$, $1 \leqslant i \leqslant n$. In the beginning, we shall verify that for each $f \in \mathcal{T}_n(A)$ the equalities

$$f \overset{k}{+} I_i^m = f \overset{k}{+} \varepsilon_i^{(m)} \quad \text{and} \quad I_i^m \overset{i}{+} f = \varepsilon_i^{(m)} \overset{i}{+} f,$$

where $1 \leqslant k \leqslant n$, $1 \leqslant i \leqslant m$, are true. Indeed, since $|f \overset{k}{+} I_i^m| = n+m-1$, then for all $a_1, a_2, \ldots, a_{n+m-1} \in A$ we have

$$\begin{aligned}
(f \overset{k}{+} I_i^m)(a_1^{n+m-1}) &= f(a_1^{k-1}, I_i^m(a_k^{k+m-1}), a_{k+m}^{n+m-1}) \\
&= f(a_1^{k-1}, a_{k+i-1}, a_{k+m}^{n+m-1}) \\
&= \zeta^{n-k} f(a_{k+m}^{n+m-1}, a_1^{k-1}, a_{k+i-1}) \\
&= \nabla^{m-i}\zeta^{n-k} f(a_{k+i}^{n+m-1}, a_1^{k_1}, a_{k+i-1}) \\
&= \zeta\nabla^{m-i}\zeta^{n-k} f(a_{k+i-1}^{n+m-1}, a_1^{k-1}) \\
&= \nabla^{i-1}\zeta\nabla^{m-i}\zeta^{n-k} f(a_k^{n+m-1}, a_1^{k-1}) \\
&= \zeta^{k-1}\nabla^{i-1}\zeta\nabla^{m-i}\zeta^{n-k} f(a_1^{n+m-1}) \\
&= (f \overset{k}{+} \varepsilon_i^m)(a_1^{n+m-1}),
\end{aligned}$$

which proves the first equality. The proof of the second equality is similar. Replacing in the conditions 2.1–2.19 the elements g, g_1, g_2, g_3 by operations from $\mathcal{T}(A)$ and $\varepsilon_i^{(m)}$ by projectors I_i^m, we can verify that these conditions are valid. For example, to verify that condition 2.12 *b* is true in the case $i < l-k+1$, observe that for $s = l-k+1$, we have $i < s$ and $I_i^m \overset{s}{+} I_j^r = I_i^{m+r-1}$, whence

$$g \overset{k}{+} (I_i^m \overset{s}{+} I_j^r) = (g \overset{k}{+} I_i^m) \overset{k+s-1}{+} I_j^r.$$

The last equality is valid for all operations $g \in \mathcal{T}(A)$ and $k \leqslant \alpha g$.

Sufficiency. Let all the conditions of the theorem be satisfied by an algebra $\mathcal{G} = (G, \zeta, \tau, \Delta, \nabla, *)$. Let us show that $\mathcal{G}$ is isomorphic to some iterative operation algebra. For this, we consider the set $G' = G \cup E$, where $E = \{e_i^{(m)} \mid m \in \mathbb{N}, 1 \leqslant i \leqslant m\}$ is the set of pairwise distinct elements not belonging to G, and define on G' the operations $\zeta, \tau, \Delta, \nabla, *$ putting

(a) $\zeta e_i^{(m)} = e_{i+1}^{(m)}$ for $1 \leqslant i < m$ and $\zeta e_m^{(m)} = e_1^{(m)}$,

(b) $\tau e_i^{(m)} = e_i^{(m)}$ for $2 < i \leqslant m$ or $i = m = 1$, as well $\tau e_1^{(m)} = e_2^{(m)}$, $\tau e_2^{(m)} = e_1^{(m)}$, $m \neq 1$,

(c) $\Delta e_i^{(m)} = e_{i-1}^{(m-1)}$ for $2 \leqslant i \leqslant m$, $\Delta e_1^{(m)} = e_1^{(m-1)}$ for $m \geqslant 2$ and $\Delta e_1^{(1)} = e_1^{(1)}$,

(d) $\nabla e_i^{(m)} = e_{i+1}^{(m+1)}$,

(e) $e_1^{(m)} * g = \zeta^n \nabla^{m-1} g$, $e_i^{(m)} * g = e_{n+i-1}^{(n+m-1)}$ for $i \geqslant 2$, $g * e_i^{(m)} = \nabla^{i-1} \zeta \nabla^{m-i} \zeta^{n-1} g$ for every $g \in G$, where $n = \alpha g$, and also $e_1^{(n)} * e_i^{(m)} = e_i^{(n+m-1)}$, $e_j^{(n)} * e_i^{(m)} = e_{m+j-1}^{(n+m-1)}$ for $j \geqslant 2$.

From (a), it follows that $e_i^{(m)} = \zeta^i \nabla^{m-1} e_1^{(1)}$ for all $m \in \mathbb{N}$, $1 \leqslant i \leqslant m$. Applying 2.8 (a) to (e) we obtain $e_1^{(1)} * g = g * e_1^{(1)} = g$ for any $g \in G$ and $e_1^{(1)} * e_i^{(m)} = e_i^{(m)} * e_1^{(1)} = e_i^{(m)}$ for all $m \in \mathbb{N}$, $1 \leqslant i \leqslant m$. Thus, $e_1^{(1)}$ is a neutral element of the groupoid $(G', *)$. From (d) it follows that

$$(\nabla e_i^{(m)}) * e_i^{(m)} = e_{i+1}^{(m+1)} * e_i^{(m)} = e_{m+i}^{(2m)} = \nabla^m e_i^{(m)},$$

which implies $\alpha e_i^{(m)} = m$ for any $m \in \mathbb{N}$, $1 \leqslant i \leqslant m$.

Using the formula

$$g_1 \overset{k}{+} g_2 = \zeta^{k-1}((\zeta^{n-k+1} g_1) * g_2),$$

where $n = \alpha g_1$, we can extend the operation $\overset{k}{+}$ to the whole set G'. At first we show that

$$e_j^{(n)} \overset{k}{+} e_i^{(m)} = \begin{cases} e_j^{(n+m-1)} & \text{for } j < k, \\ e_{k+i-1}^{(n+m-1)} & \text{for } j = k, \\ e_{m+j-1}^{(n+m-1)} & \text{for } j > k. \end{cases}$$

Indeed, if $j < k$, then from $1 \leqslant j < k \leqslant n$, we get $0 \leqslant k - j - 1 \leqslant n - 2$, whence $n - (k - j - 1) \geqslant 2$. Thus

$$\begin{aligned} e_j^{(n)} \overset{k}{+} e_i^{(m)} &= \zeta^{k-1}((\zeta^{n-k+1} e_j^{(n)}) * e_i^{(m)}) = \zeta^{k-1}(e_{n-(k-j-1)}^{(k)} * e_i^{(m)}) \\ &= \zeta^{k-1} e_{m+n-(k_j-1)-1}^{(n+m-1)} = e_{(n+m-1)+j(\mathrm{mod}\,(n+m-1))}^{(n+m-1)} = e_j^{(n+m-1)}. \end{aligned}$$

The other two cases can be verified in a similar way.

Further, for each $g \in G$ we have $g \overset{k}{+} e_i^{(m)} = g \overset{k}{+} \varepsilon_i^{(m)}$, where $1 \leqslant k \leqslant \alpha g$, $1 \leqslant i \leqslant m$, and $e_k^{(m)} \overset{k}{+} g = \varepsilon_k^{(m)} \overset{k}{+} g$, where $1 \leqslant k \leqslant n$. Let us show the second equality. The proof of the first is very similar. According to the definition we have

$$\begin{aligned} e_k^{(m)} \overset{k}{+} g &= \zeta^{k-1}((\zeta^{m-k+1} e_k^{(m)}) * g) = \zeta^{k-1}(e_{m+1(\mathrm{mod}\, m)}^{(m)} * g) \\ &= \zeta^{k-1}(e_1^{(m)} * g) = \zeta^{k-1}(\zeta^n \nabla^{m-1} g) = \zeta^{n+k-1} \nabla^{m-1} g, \end{aligned}$$

where $n = \alpha g$. Now, using the condition 2.8 b, we get

$$\zeta^{n+k-1} \nabla^{m-1} g = \zeta^{n+k-1} \nabla^{m-k+(k-1)} g = \nabla^{k-1} \zeta^n \nabla^{m-k} g = \varepsilon_k^{(m)} \overset{k}{+} g,$$

which was to be proved.

Using the same method we can show that $e_i^{(m)} \overset{k}{+} g = e_i^{(n+m-1)}$ for $i < k$ and $e_i^{(m)} \overset{k}{+} g = e_{n+i-1}^{(n+m-1)}$ for $i > k$, where $n = \alpha g$.

Further on we will use the equality

$$D_n^m e_i^{(mn)} = e_{i-km}^{(m)} \tag{6.4.11}$$

which holds for all $m, n \in \mathbb{N}$, $1 \leqslant i \leqslant mn$ and some $k \in \{0, 1, 2, \ldots, n-1\}$ such that $km < i \leqslant (k+1)m$. Let us prove it.

It is clear that the set E is closed under the operations ζ, τ, Δ. So, $\mathcal{E} = (E, \zeta, \tau, \Delta)$ is an algebra. Let $\mathcal{I} = \{I_i^m \,|\, m \in \mathbb{N}, 1 \leqslant i \leqslant m\}$ be the set of all n-place projectors defined on the set of all natural numbers. It is not difficult to see that $\mathcal{I}$ is closed under the operations ζ^*, τ^*, Δ^*, which are defined by formulas (6.4.1), (6.4.2) and (6.4.3). Moreover, the mapping $P : I_i^m \mapsto e_i^{(m)}$ from the set $\mathcal{I}$ onto E is an isomorphism between algebras $\mathcal{E}^* = (\mathcal{I}, \zeta^*, \tau^*, \Delta^*)$ and $\mathcal{E}$. Put $D_n^{*m} = H_0^* H_1^* \ldots H_{m-1}^*$, where $H_k^* = (\zeta^*)^{m-1}(\Delta^* \zeta^* (\tau^* \zeta^*)^k)^{n-1}$ for every $0 \leqslant k \leqslant m-1$. Now let $n, m \in \mathbb{N}$, $1 \leqslant i \leqslant nm$ and $km < i \leqslant (k+1)m$ for some $k \in \{0, 1, 2, \ldots, n-1\}$. Then, according to the formula (6.4.6) we have

$$(D_n^{*m} I_i^{nm})(n_1^m) = I_i^{nm}(\underbrace{n_1^m}_{1}, \ldots, \underbrace{n_1^m}_{k+1}, \ldots, \underbrace{n_1^m}_{n}) = n_{i-km} = I_{i-km}^m(n_1^m)$$

for all $n_1, \ldots, n_m \in \mathbb{N}$. Thus $D_n^{*m} I_i^{nm} = I_{i-km}^m$. Since P is an isomorphism, $P(D_n^{*m} I_i^{nm}) = P(I_{i-km}^m) = e_{i-km}^{(m)}$. But $P(D_n^{*m} I_i^{nm}) = D_n^m e_i^{(nm)}$, so, $D_n^m e_i^{nm} = e_{i-km}^{(m)}$, which was to be proved.

Now let us consider the algebra $\mathcal{G}' = (G', e_1^{(1)}, \zeta, \tau, \Delta, \nabla, *)$. We will show that it satisfies all the conditions of Theorem 6.4.2. If the elements g, g_1, g_2, g_3

belong to G, then, obviously, all the conditions $1.1 - 1.19$ are satisfied. So, it is necessary to consider the case when at least one of these elements is in $E = G' \setminus G$. If $g = e_i^{(m)}$ in condition 2.1, then we have $(\nabla e_i^{(m)}) * e_i^{(m)} = e_{i+1}^{(m+1)} * e_i^{(m)} = e_{m+i}^{(2m)} = \nabla^m e_i^{(m)}$, that is $\mathcal{G}'$ satisfies 1.1. Further, if $\nabla^n e_i^{(r)} = \nabla^m e_i^{(r)}$, then $e_{i+n}^{(r+n)} = e_{i+m}^{(r+m)}$, whence $r + n = r + m$, consequently $n = m$, i.e., $\mathcal{G}'$ satisfies 1.2. Now let us consider condition 2.3. Here the following three cases are possible:

$$(i)\ g_1, g_2 \in E, \quad (ii)\ g_1 \in E,\ g_2 \in G, \quad (iii)\ g_1 \in G,\ g_2 \in E.$$

Let $g_1, g_2 \in E$, i.e., let $g_1 = e_i^{(m)}$, $g_2 = e_j^{(r)}$. Then for $i = 1$ we have

$$\nabla(e_i^{(m)} * e_j^{(r)}) = \nabla e_j^{(m+r-1)} = e_{j+1}^{(m+r)} = e_1^{(m)} * e_{j+1}^{(r+1)} = e_1^{(m)} * \nabla e_j^{(r)}.$$

For $i \geqslant 2$ we get $\nabla(e_i^{(m)} * e_j^{(r)}) = e_i^{(m)} * e_{j+1}^{(r+1)} = e_i^{(m)} * \nabla e_j^{(r)}$. Thus, in the case (i) the condition 1.3 is satisfied. Let now $g_1 = e_i^{(m)}$, $g_2 \in G$. Then, according to 2.3, for $i = 1$ we have

$$\begin{aligned}\nabla(e_1^{(m)} * g_2) &= \nabla(\zeta^n \nabla^{m-1} g_2) = \nabla(\varepsilon_1^{(m)} \overset{1}{+} g_2) = \varepsilon_1^{(m)} \overset{1}{+} \nabla g_2 \\ &= \zeta^n \nabla^m g_2 = \zeta^n \nabla^{m-1}(\nabla g_2) = e_1^{(m)} * \nabla g_2.\end{aligned}$$

Similarly $\nabla(e_i^{(m)} * g_2) = \nabla e_{n+i-1}^{(n+m-1)} = e_{n+i}^{(n+m)} = e_i^{(m)} * \nabla g_2$ for $i \geqslant 2$, where $n = \alpha g_2$. This proves 1.3 in the case (ii). Now let $g_1 \in G,\ g_2 = e_i^{(m)}$. Then

$$\begin{aligned}\nabla(g_1 * e_i^{(m)}) &= \nabla(\nabla^{i-1} \zeta \nabla^{m-i} \zeta^{n-1} g_1) \\ &= \nabla^i \zeta \nabla^{m-i} \zeta^{n-1} g_1 = g_1 * e_{i+1}^{(m+1)} = g_1 * \nabla e_i^{(m)},\end{aligned}$$

where $n = \alpha g_1$, which proves 1.3 in the third case. So, 1.3 holds in $\mathcal{G}'$ in any case.

To verify 1.4 let $g_1 = e_i^{(m)},\ g_2 = e_j^{(r)}$. Then $(\nabla e_i^{(m)}) * e_j^{(r)} = e_{i+1}^{(m+1)} * e_j^{(r)} = e_{r+i}^{(r+m)} = \nabla^r e_i^{(m)}$. In the same manner, for $g_1 = e_i^{(m)},\ g_2 \in G$, $n = \alpha g_2$, we have $(\nabla e_i^{(m)}) * g_2 = e_{i+1}^{(m+1)} * g_2 = e_{n+i}^{(n+m)} = \nabla^n e_i^{(m)}$. For $g_1 \in G$, $n = \alpha g_1$ and $g_2 = e_i^{(m)}$, according to 2.4, we obtain

$$(\nabla g_1) * e_i^{(m)} = \nabla^{i-1} \zeta \nabla^{m-i} \zeta^n (\nabla g_1) = (\nabla g_1) \overset{1}{+} \varepsilon_i^{(m)} = \nabla^m g_1.$$

In this way we have showed that 1.4 holds in $\mathcal{G}'$. $\mathcal{G}'$ satisfies also 1.5 because $\alpha e_i^{(m)} = m$. Since $e_1^{(1)}$ is the identity of $(G', *)$, $\mathcal{G}'$, 1.6 is also satisfied. Condition 1.7 is a consequence of the definitions of the operations ζ, τ, Δ on E; 1.8 follows directly from $2.8\,a$.

In each of the conditions 1.9 − 1.11, it is necessary to consider following three cases:

$$\text{(a) } g_1 = e_j^{(m)},\ g_2 = e_k^{(r)}, \quad \text{(b) } g_1 \in G,\ g_2 = e_j^{(m)}, \quad \text{(c) } g_1 = e_j^{(m)},\ g_2 \in G.$$

At first, we verify 1.9. Let $3 \leqslant i \leqslant \alpha g_1$. In the case (a) for $j \geqslant 3$ we have $(\tau e_j^{(m)}) \overset{i}{+} e_k^{(r)} = e_j^{(m)} \overset{i}{+} e_k^{(r)}$. Whence for $j < i$ we obtain

$$e_j^{(m)} \overset{i}{+} e_k^{(r)} = e_j^{(m+r-1)} = \tau(e_j^{(m+r-1)}) = \tau(e_j^{(m)} \overset{i}{+} e_k^{(r)}).$$

For $j = i$, we have

$$e_j^{(m)} \overset{i}{+} e_k^{(r)} = e_{i+k-1}^{(m+r-1)} = \tau(e_{i+k-1}^{(m+r-1)}) = \tau(e_i^{(m)} \overset{i}{+} e_k^{(r)}).$$

Analogously

$$e_k^{(m)} \overset{i}{+} e_k^{(r)} = e_{r+k-1}^{(m+r-1)} = \tau(e_{r+j-1}^{(m+r-1)}) = \tau(e_j^{(m)} \overset{i}{+} e_k^{(r)})$$

for $j > i$. In a similar way we can verify this condition for $j = 1$ and $j = 2$. This completes the verification of 1.9 in the case (a). In the case (b), $n = \alpha g_1$ and

$$(\tau g_1) \overset{i}{+} e_j^{(m)} = (\tau g_1) \overset{i}{+} \varepsilon_j^{(m)} = \tau(g_1 \overset{i}{+} \varepsilon_j^{(m)}) = \tau(g_1 \overset{i}{+} \varepsilon_j^{(m)}) = \tau(g_1 \overset{i}{+} e_j^{(m)}),$$

according to 2.9 a. In the case (c) subcases $j = 1$, $j = 2$, and $j \geqslant 3$ can be checked by simple transformations. For $j = i \geqslant 3$, according to 2.9 b, we have:

$$\begin{aligned}(\tau e_i^{(m)}) \overset{i}{+} g_2 = e_i^{(m)} \overset{i}{+} g_2 = \varepsilon_i^{(m)} \overset{i}{+} g_2 = \nabla^{i-1}(\zeta^n \nabla^{m-i} g_2) \\ = \tau \nabla^{i-1}(\zeta^n \nabla^{m-i} g_2) = \tau(\varepsilon_i^{(m)} \overset{i}{+} g_2) = \tau(e_i^{(m)} \overset{i}{+} g_2),\end{aligned}$$

where $n = \alpha g_2$. So, 1.9 is valid in any case. Condition 1.10 in the case (a) is obvious, in the case (b) it follows from 2.10 a, in the case (c) — from 2.10 b. Condition 1.11 can be deduced from 2.11.

To prove 1.12 and 1.13, we must separately consider the following cases:
(a) $g_1 \in E,\ g_2, g_3 \in G$, (b) $g_2 \in E,\ g_1, g_3 \in G$, (c) $g_3 \in E,\ g_1, g_2 \in G$,
(d) $g_1, g_2 \in E,\ g_3 \in G$, (e) $g_1, g_3 \in E,\ g_2 \in G$, (f) $g_2, g_3 \in E,\ g_1 \in G$,
(g) $g_1, g_2, g_3 \in E$.

We prove 1.12 in the case (d). In the other cases the proof is similar.

Let $g_1 = e_j^{(n)}$, $g_2 = e_k^{(m)}$. Then the left side of equation 1.12 has the form $h = e_j^{(n)} \overset{i}{+} (e_k^{(m)} \overset{l}{+} g_3)$, where $1 \leqslant i \leqslant n$, $1 \leqslant l \leqslant m$. The proof that the right side of 1.12 is equal to h can be reduced to the consideration of the following eight subcases: (1) $i = k$, $j = l$, (2) $i = k$, $j < l$, (3) $i = k$, $j > l$, (4) $i < k$,

$j = l$, (5) $i < k$, $j < l$, (6) $i < k$, $j > l$, (7) $i > k$, $j = l$, (8) $i > k$, $j < l$, (9) $i > k$, $j > l$. In all these cases the proof is similar. As an example we prove (1) and (6). In the case (1), we have

$$h = e_k^{(n)} \overset{k}{+} (e_l^{(m)} \overset{l}{+} g_3) = \varepsilon_k^{(n)} \overset{k}{+} (\varepsilon_l^{(m)} \overset{l}{+} g_3),$$

whence, according to 2.12 d, we get

$$h = \varepsilon_{k+l-1}^{(n+m-1)} \overset{k+l-1}{+} g_3 = e_{k+l-1}^{(n+m-1)} \overset{k+l-1}{+} g_3 = (e_k^{(n)} \overset{k}{+} e_l^{(m)}) \overset{k+l-1}{+} g_3.$$

In the case (6), $h = e_i^{(n)} \overset{k}{+} (e_j^{(m)} \overset{l}{+} g_3) = e_i^{(n)} \overset{k}{+} e_{r+j-1}^{(r+m-1)}$, where $r = \alpha g_3$. But $i < k$, so, $h = e_i^{(n+r+m-2)}$. On the other hand

$$(e_i^{(n)} \overset{k}{+} e_j^{(m)}) \overset{k+l-1}{+} g_3 = e_i^{(n+m-1)} \overset{k+l-1}{+} g_3 = e_i^{(r+n+m-2)}.$$

Therefore $h = (e_i^{(n)} \overset{k}{+} e_j^{(m)}) \overset{k+l-1}{+} g_3$, which completes the proof of (6).

The proof of 1.13 is analogous. Also in the proof of the conditions 1.14–1.19 we must consider several cases. The proof in each of these cases is similar. For this reason we prove only 1.16. We select three cases: (a) $g_1, g_2 \in E$, (b) $g_1 \in E$, $g_2 \in G$, (c) $g_1 \in G$, $g_2 \in E$.

According to the assumption in the case (a) we have $g_1 = e_i^{((n-1)m+2)}$, $g_2 = e_j^{(m)}$. Let $h = D_n^m((\Delta e_i^{((n-1)m+2)}) \overset{1}{+} e_j^{(m)})$. If $i > 2$, then $h = D_n^m(e_{i-1}^{((n-1)m+1)} * e_j^{(m)}) = D_n^m e_{m+i-2}^{(nm)}$. But $km < m+i-2 \leqslant (k+1)m$ for some $k \in \{1, 2, \ldots, n-1\}$, so, applying (6.4.11), we obtain $h = e_{m+i-2-km}^{(m)}$. On the other hand

$$D_{n+1}^m((e_i^{((n-1)m+2)} \overset{2}{+} e_j^{(m)}) \overset{1}{+} e_j^{(m)}) = D_n^m(e_{m+i-1}^{(nm+1)} * e_j^{(m)}) = D_{n+1}^m e_{2m+i-2}^{((n+1)m)},$$

where $(k+1)m < 2m + i - 2 \leqslant (k+2)m$. Thus, by (6.4.11), we have

$$D_{n+1}^m e_{2m+i-2}^{((n+1)m)} = e_{2m+i-2-(k+1)m}^{(m)} = e_{m+i-2-km}^{(m)}.$$

This completes the proof of 1.16 in the case $i > 2$. For $i = 1$ and $i = 2$ the proof is analogous.

Now let us consider the case (b). Then $g_1 = e_i^{((n-1)m+2)}$, where $m = \alpha g_2$ and $g_2 \in G$. Let $h = D_n^m((\Delta e_i^{((n-1)m+2)}) \overset{1}{+} g_2)$. In the same way as in (a), for $i > 2$ we obtain $h = D_{n+1}^m((e_i^{((n-1)m+2)} \overset{2}{+} g_2) \overset{1}{+} g_2)$, which means that in this case the condition 1.16 is satisfied.

For $i = 1$ we have $h = D_n^m(e_1^{((n-1)m+1)} * g_2) = D_n^{(m)}(\zeta^m \nabla^{(n-1)m} g_2)$. But on the other hand

$$\begin{aligned} D_{n+1}^m((e_1^{((n-1)m+2)} \overset{2}{+} g_2) \overset{1}{+} g_2) &= D_{n+1}^m(e_1^{(nm+1)} * g_2) = D_{n+1}^m(\zeta^m \nabla^{nm} g_2) \\ &= D_{n+1}^m(\zeta^m \nabla^m(\nabla^{(n-1)m} g_2)) = h_1, \end{aligned}$$

whence, by 2.16, we obtain

$$h_1 = D^m_{n+1}(\nabla^m(\nabla^{(n-1)m} g_2)) = D^m_n(\nabla^{(n-1)m} g_2) = D^m_n(\zeta^m \nabla^{(n-1)m} g_2).$$

Thus $h = h_1$.

For $i = 2$, similarly as for $i = 1$, the left side of 1.16 can be written as $h = D^m_n(\zeta^m \nabla^{(n-1)m} g_2)$. The right side has the form

$$D^m_n((e_2^{((n-1)m+2)} \overset{2}{+} g_2) \overset{1}{+} g_2) = D^m_{n+1}((\nabla\zeta^m \nabla^{(n-1)m} g_2) * g_2) = h_2.$$

But $\nabla(\zeta^m \nabla^{(n-1)m} g_2) * g_2 = \nabla^m \zeta^m \nabla^{(n-1)m} g_2$, by 2.4, so, according to 2.16 b

$$h_2 = D^m_{n+1}(\nabla^m \zeta^m \nabla^{(n-1)m} g_2) = D^m_n(\zeta^m \nabla^{(n-1)m} g_2) = h.$$

This completes the proof of (b). The proof of (c) is analogous.

In this way we have proved that $\mathcal{G}'$ satisfies 1.16.

The proof of 1.19 is obvious if we rewrite 2.19 in an equivalent form as

$$D^n_n((...((g \overset{n}{+} \varepsilon^{(n)}_n) \overset{n-1}{+} \varepsilon^{(n)}_{n-1}) \ldots) \overset{1}{+} \varepsilon^{(n)}_1) = g.$$

So, we have shown that the algebra $\mathcal{G}' = (G', e^{(1)}_1, \zeta, \tau, \Delta, \nabla, *)$ satisfies all the conditions of Theorem 6.4.2, that is, it is isomorphic to some central iterative operation algebra, which completes the proof of Theorem 6.4.3. □

Denote by $\mathfrak{M}$ the class of all algebras $\mathcal{G} = (G, \zeta, \tau, \Delta, \nabla, *)$ of type $(1, 1, 1, 1, 2)$ satisfying the conditions 1.1 – 1.5 and 1.7 – 1.13 from Theorem 6.4.2, as well as the identities:

$$\Delta^{n-1}(\zeta g) = \Delta^{n-1} g, \tag{6.4.12}$$

$$\Delta(g_1 * g_2) = g_1 * \Delta g_2, \tag{6.4.13}$$

where $n = \alpha g$ is the smallest positive integer n such that $(\nabla g) * g = \nabla^n g$. We shall say that the subset H of G is a *v-subsystem* of an algebra $\mathcal{G} \in \mathfrak{M}$ if and only if for any $h, h_1, \ldots, h_n \in H$, $g \in G$, where $n = \alpha g$, the following conditions are satisfied:

$$\alpha h = 1, \tag{6.4.14}$$

$$h * h_1 = h, \tag{6.4.15}$$

$$\Delta^{n-1}((...((g \overset{n}{+} h_n) \overset{n-1}{+} h_{n-1}) \ldots) \overset{1}{+} h_1) \in H, \tag{6.4.16}$$

$$\Delta^{n-1}(((\tau g) \overset{2}{+} h_2) \overset{1}{+} h_1) = \Delta^{n-1}((g \overset{2}{+} h_1) \overset{1}{+} h_2), \tag{6.4.17}$$

$$\Delta^{n-2}((\Delta g) * h) = \Delta^{n-1}((g \overset{2}{+} h) \overset{1}{+} h), \tag{6.4.18}$$

$$\Delta^n((\nabla g) * h) = \Delta^{n-1} g. \tag{6.4.19}$$

A v-subsystem H of $\mathcal{G}$ is called a *dense v-subsystem* of $\mathcal{G}$ in the class $\mathfrak{M}$, if:

(a) any homomorphism from $\mathcal{G}$ onto another algebra $\mathcal{G}' \in \mathfrak{M}$ which is not an isomorphism, induces on H a homomorphism but not an isomorphism,

(b) if an algebra $\mathcal{G}' \in \mathfrak{M}$ contains $\mathcal{G}$ as a subalgebra, $\mathcal{G} \neq \mathcal{G}'$ and H is a v-subsystem of $\mathcal{G}'$, then there exists a homomorphism from $\mathcal{G}'$ onto some algebra from $\mathfrak{M}$ which is not an isomorphism, but induces an isomorphism on H.

Let H be a v-subsystem of $\mathcal{G} \in \mathfrak{M}$. For every $g \in G$ we define on H an n-ary operation λ_g, where $n = \alpha g$, putting

$$\lambda_g(h_1^n) = \Delta^{n-1}((\ldots((g \overset{n}{+} h_n) \overset{n-1}{+} h_{n-1})\ldots) \overset{1}{+} h_1). \tag{6.4.20}$$

Proposition 6.4.4. *The mapping $P : g \mapsto \lambda_g$ is a homomorphism of $\mathcal{G}$ into an iterative Mal'cev-Post operation algebra $(\mathcal{T}(H), \zeta, \tau, \Delta, \nabla, *)$.*

Proof. Let $g \in G$, $n = \alpha g$. Then, according to (6.4.7) and (6.4.8), for all $h_1, \ldots, h_n \in H$ we have

$$\begin{aligned}\lambda_{\zeta g}(h_1^n) &= \Delta^{n-1}((\ldots(((\zeta g) \overset{n}{+} h_n) \overset{n-1}{+} h_{n-1})\ldots) \overset{1}{+} h_1)\\ &= \Delta^{n-1}((\ldots(\zeta(g \overset{n-1}{+} h_n) \overset{n-1}{+} h_{n-1})\ldots) \overset{1}{+} h_1)\\ &= \Delta^{n-1}\zeta((\ldots(g \overset{n-1}{+} h_n)\ldots) \overset{n}{+} h_1).\end{aligned}$$

Applying $n-1$ times condition 1.13 to the last expression and using (6.4.12), we obtain

$$\begin{aligned}\Delta^{n-1}\zeta((\ldots(g \overset{n-1}{+} h_n)\ldots) \overset{n}{+} h_1) &= \Delta^{n-1}\zeta((\ldots((g \overset{n}{+} h_1) \overset{n-1}{+} h_n)\ldots) \overset{1}{+} h_2)\\ &= \Delta^{n-1}((\ldots((g \overset{n}{+} h_1) \overset{n-1}{+} h_n)\ldots) \overset{1}{+} h_2)\\ &= \lambda_g(h_2^n, h_1).\end{aligned}$$

So, $\lambda_{\zeta g}(h_1, h_2^n) = \lambda_g(h_2^n, h_1) = \zeta\lambda_g(h_1, h_2^n)$, i.e., $P(\zeta g) = \zeta P(g)$.

In the same manner using 1.9 and (6.4.17), 1.10 and (6.4.18), 1.11 and (6.4.19), we prove that $P(\tau g) = \tau P(g)$, $P(\Delta g) = \Delta P(g)$ and $P(\nabla g) = \nabla P(g)$ for every $g \in G$.

Similarly for $g_1, g_2 \in G$, $n = \alpha g_1$, $m = \alpha g_2$ and $s = n + m - 1$, using 1.12, 1.13 and (6.4.13), we have

$$\begin{aligned}\lambda_{g_1 * g_2} &= \Delta^{s-1}((\ldots(((g_1 * g_2) \overset{s}{+} h_s) \overset{s-1}{+} h_{s-1})\ldots) \overset{1}{+} h_1)\\ &= \Delta^{s-1}(((\ldots(g_1 \overset{n}{+} h_s)\ldots) \overset{m+1}{+} h_{m+1}) * ((\ldots(g_2 \overset{m}{+} h_m)\ldots) \overset{1}{+} h_1))\\ &= \Delta^{n-1}(((\ldots(g_1 \overset{n}{+} h_s)\cdots) \overset{m+1}{+} h_{m+1}) \overset{1}{+} \Delta^{m-1}((\ldots(g_1 \overset{m}{+} h_m)\ldots) \overset{1}{+} h_1))\\ &= \lambda_{g_1}(\lambda_{g_2}(h_1^m), h_{m+1}^s) = (\lambda_{g_1} * \lambda_{g_2})(h_1^s),\end{aligned}$$

which proves $P(g_1 * g_2) = P(g_1) * P(g_2)$. So, P is a homomorphism of $\mathcal{G}$ into $(\mathcal{T}(H), \zeta, \tau, \Delta, \nabla, *)$. □

Now let us consider the iterative Mal'cev-Post operation algebra $\mathbf{T} = (\mathcal{T}(A), \zeta, \tau, \Delta, \nabla, *)$ and the set of all constant one-place functions defined on A, i.e., the set

$$C_A = \{f_a \,|\, f_a(x) = a \in A \text{ for all } x \in A\}.$$

Proposition 6.4.5. *$\boldsymbol{T} \in \mathfrak{M}$ and C_A is a dense v-subsystem of $\boldsymbol{T}$ in $\mathfrak{M}$.*

Proof. The fact that the algebra T satisfies the conditions 1.1–1.5 and 1.7 –1.13 follows from Theorem 6.4.2. Let $f \in \mathcal{T}(A)$, $n = |f| = \alpha f$. Then $\Delta^{n-1}(\zeta f)(a) = \zeta f(a, \ldots, a) = f(a, \ldots, a) = \Delta^{n-1} f(a)$ for every $a \in A$. This proves (6.4.12). Analogously we can prove (6.4.13)–(6.4.19). So, T $\in \mathfrak{M}$ and C_A is a v-subsystem of T.

Let P be a homomorphism of T onto some algebra from $\mathfrak{M}$, which is not an isomorphism. Then there are $f, g \in \mathcal{T}(A)$ such that $P(f) = P(g)$ and $f \neq g$. Obviously $(\nabla f) * f = \nabla^n f$, $(\nabla g) * g = \nabla^m g$, where $n = |f|$, $m = |g|$. But P is a homomorphism and $P(f) = P(g)$, therefore $\nabla^n P(f) = \nabla^m P(g)$, which implies $n = m$, i.e., $|f| = |g|$. Since $f \neq g$, we have $f(a_1^n) \neq g(a_1^n)$ for some $a_1, \ldots, a_n \in A$. Thus, $f_{b_1} \neq f_{b_2}$ for $b_1 = f(a_1^n)$, $b_2 = g(a_1^n)$, and

$$f_{b_1} = \Delta^{n-1}((\ldots((f \overset{n}{+} f_{a_n}) \overset{n-1}{+} f_{a_{n-1}}) \ldots) \overset{1}{+} f_{a_1}),$$
$$f_{b_2} = \Delta^{n-1}((\ldots((g \overset{n}{+} f_{a_n}) \overset{n-1}{+} f_{a_{n-1}}) \ldots) \overset{1}{+} f_{a_1}),$$

whence, applying P and using the fact that $P(f) = P(g)$, we get $P(f_{b_1}) = P(f_{b_2})$. So, P induces on C_A a homomorphism which is not an isomorphism.

Now, let $(G, \zeta, \tau, \Delta, \nabla, *)$ be an arbitrary algebra from $\mathfrak{M}$ for which T is a proper subalgebra and C_A is its v-subsystem. From Proposition 6.4.4 it follows that the mapping $P : G \mapsto \mathcal{T}(A)$ defined for every $g \in G$ in the following way:

$$P(g)(a_1^n) = b \longleftrightarrow \Delta^{n-1}((\ldots((g \overset{n}{+} f_{a_n}) \overset{n-1}{+} f_{a_{n-1}}) \ldots) \overset{1}{+} f_{a_1}) = f_b,$$

where $a_1, \ldots, a_n, b \in A$, $n = \alpha g$, is a homomorphism from $\mathcal{G}$ to T. For all $f \in \mathcal{T}(A)$, $n = |f|$ and $a_1, \ldots, a_n, b \in A$ we also have

$$f(a_1^n) = b \longleftrightarrow \Delta^{n-1}((\ldots((f \overset{n}{+} f_{a_n}) \overset{n-1}{+} f_{a_{n-1}}) \ldots) \overset{1}{+} f_{a_1}) = f_b.$$

Thus $P(f)(a_1^n) = b$, i.e., $P(f) = f$. This means that P restricted to T is the identity isomorphism. Obviously, $P(g) \in \mathcal{T}(A)$ for every $g \in G$, whence, for $g \in G \setminus \mathcal{T}(A)$ we have $P(g) \neq g$ and $P(P(g)) = P(g)$. So, P is not an isomorphism. Therefore C_A is a dense v-subsystem of T. □

Theorem 6.4.6. *An algebra* $(G, \zeta, \tau, \Delta, \nabla, *)$ *of type* $(1, 1, 1, 1, 2)$ *is isomorphic to some iterative Mal'cev-Post operation algebra if and only if it belongs to the class* $\mathfrak{M}$ *and contains a dense v-subsystem in the class* $\mathfrak{M}$.

Proof. The necessity of these conditions follows from Proposition 6.4.5, therefore we shall only prove their sufficiency. Assume that $(G, \zeta, \tau, \Delta, \nabla, *)$ belongs to $\mathfrak{M}$ and H is its dense v-subsystem. Clearly, $(H, *)$ is a left zero semigroup. By (6.4.15), it is isomorphic to a semigroup $(C_H, *)$ of all constant one-place functions on H. Let λ_g be an n-ary operation defined by (6.4.20). Then, according to Proposition 6.4.4, $P\colon g \mapsto \lambda_g$ is a homomorphism from $\mathcal{G}$ into an iterative Mal'cev-Post operation algebra $\mathrm{T} = (\mathcal{T}(A), \zeta, \tau, \Delta, \nabla, *)$ and, as it is easy to see, $P(h) = f_h$ for every $h \in H$. So, P restricted to H is an isomorphism. Comparing this fact with condition (a) of the definition of a dense v-subsystem we deduce that P is an isomorphism. Since H is a dense v-subsystem of $\mathcal{G}$ and $(H, *)$ isomorphic to $(C_H, *)$, C_H is a dense v-subsystem of $(P(G), \zeta, \tau, \Delta, \nabla, *) \in \mathfrak{M}$. By Proposition 6.4.5, C_H is also a dense v-subsystem of T, which together with $P(G) \subset \mathcal{T}(H)$ and the condition (b) of the definition of a dense v-subsystem imply $P(G) = \mathcal{T}(H)$. Therefore $\mathcal{G}$ and T are isomorphic. □

Corollary 6.4.7. *An algebra* $(G, e, \zeta, \tau, \Delta, \nabla, *)$ *of type* $(0, 1, 1, 1, 1, 2)$ *is isomorphic to some central iterative Mal'cev-Post algebra if and only if* e *is the neutral element of a semigroup* $(G, *)$ *and* $(G, \zeta, \tau, \Delta, \nabla, *)$ *satisfies all the conditions of the previous theorem.*

Now, we shall characterize central pre-iterative operation algebras.

Theorem 6.4.8. *An algebra* $(G, e, \zeta, \tau, \Delta, *)$ *of type* $(0, 1, 1, 1, 2)$ *is isomorphic to some central pre-iterative operation algebra if and only if it satisfies the conditions* 1.7 – 1.10, 1.12 – 1.16, 1.18, 1.19, *where* $e_1^{(1)} = \Delta e$, $e_1^{(2)} = \tau e$, $e_2^{(2)} = e$, $e_n^{(n)} = e^{n-1}$ *and* $e_i^{(n)} = \zeta^i e_n^{(n)}$ *for* $i = 1, 2, \ldots, n-1$,[7] *and for all* $g, g_1, g_2 \in G$ *we have*

(i) $e * g = e^n$ *for some natural* $n \in \mathbb{N}$,

(ii) $e^n = e^m$ *implies* $n = m$,

(iii) $e * g = e * \zeta g = e * \tau g = e^2 * \Delta g$,

(iv) $(\Delta e) * g = g * \Delta e = g$,

(v) $g * e = e \overset{2}{+} g$,

(vi) $g_1 * e = g_2 * e$ *implies* $g_1 = g_2$,

(vii) $D_n^m[(e \overset{2}{+} g_2) * g_1] = D_{n-1}^m g_2$ *for* $m = \alpha g_1$, $\alpha g_2 = (n-1)m$, *where* αg *denotes a positive integer satisfying the equality* $e * g = e^{\alpha g}$.

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[7] Note that e^{n-1} is the power of e in a semigroup $(G, *)$.

Proof. The necessity of these conditions can be checked without any difficulties, therefore we shall prove only their sufficiency. First of all, let us notice that from (i) and (ii) follows the existence of a uniquely determined natural number αg, where $g \in G$, such that $e * g = e^{\alpha g}$. In particular, we have $\alpha e = 2$. Moreover, as it is easy to see, $\alpha(g_1 * g_2) = \alpha g_1 + \alpha g_2 - 1$ for all $g_1, g_2 \in G$.

We shall consider a new unary operation ∇ on G putting $\nabla g = g * e$ for every $g \in G$. We shall prove that the algebra $\mathcal{G}$ with this newly added operation and a distinguished element Δe satisfies all the conditions of Theorem 6.4.2. For this it is enough to check conditions 1.1–1.5, 1.11, and 1.17. We have

$$(\nabla g) * g = (g * e) * g = g * (e * g) = g * e^{\alpha g} = \nabla^{\alpha g} g,$$

i.e., condition 1.1 is satisfied. Now let $\nabla^n g = \nabla^m g$, i.e., $g * e^n = g * e^m$. For $n < m$, according to (vi), from the last equality we get $g = g * e^{m-n}$. Thus $\alpha g = \alpha g + m - n + 1$, which is impossible. The case $n > m$ also is impossible. Therefore $n = m$. This proves 1.2. Furthermore,

$$\nabla(g_1 * g_2) = (g_1 * g_2) * e = g_1 * (g_2 * e) = g_1 * \nabla g_2,$$

which proves 1.3. Similarly

$$(\nabla g_1) * g_2 = (g_1 * e) * g_2 = g_1 * (e * g_2) = g_1 * e^{\alpha g_2} = \nabla^{\alpha g_2} g_1$$

proves 1.4. The condition 1.5 is a consequence of (iii). For all $i = 2, 3, \ldots, \alpha g_1 + 1$ we also have

$$\nabla(g_1 \overset{i-1}{+} g_2) = (g_1 \overset{i-1}{+} g_2) * e = (g_1 * e) \overset{i}{+} g_2 = (\nabla g_1) \overset{i}{+} g_2.$$

This implies 1.11. Now let $m = \alpha g_2$ and $\alpha g_1 = (n-1)m$. Then, according to (v) and (vii), we get

$$D_n^m((\nabla g_1) * g_2) = D_n^m((g_1 * e) * g_2) = D_n^m((e \overset{2}{+} g_1) * g_2) = D_{n-1}^m g_1,$$

i.e., 1.17 is valid too.

So, the algebra $(G, \Delta e, \zeta, \tau, \Delta, \nabla, *)$ satisfies all the conditions of Theorem 6.4.2, hence it is isomorphic to some central iterative operation algebra. Consequently, the algebra $\mathcal{G}$ is isomorphic to some central pre-iterative operation algebra. □

Let us consider the class $\mathfrak{N}$ of all algebras $\mathcal{G} = (G, e, \zeta, \tau, \Delta, *)$ of type $(0, 1, 1, 1, 2)$ satisfying conditions 1.7–1.10, 1.12 and 1.13 of Theorem 6.4.2, the conditions (i)–(vi) of Theorem 6.4.8, as well the identities (6.4.12) and (6.4.13), where the arity αg of $g \in G$ is defined as a positive integer such that

$e * g = e^{\alpha g}$. A nonempty subset H of G is called a *v'-subsystem* of $\mathcal{G} \in \mathfrak{N}$ if the conditions (6.4.14)–(6.4.18) and

$$\Delta((e \overset{2}{+} h_2) * h_1) = h_2$$

are valid for all $h, h_1, h_2 \in H$ and $g \in G$.

Theorem 6.4.9. *An algebra* $(G, e, \zeta, \tau, \Delta, *)$ *of type* $(0, 1, 1, 1, 2)$ *is isomorphic to some central pre-iterative Mal'cev-Post algebra of operations if and only if it belongs to the class* $\mathfrak{N}$ *and contains a* v'*-subsystem which is dense in this class.*

Proof. Let $(G, e, \zeta, \tau, \Delta, *)$ be an algebra satisfying all the conditions of the theorem and let H be its dense v'-subsystem. Putting $\nabla g = g * e$ for all $g \in G$ we define the new unary operation ∇ on G. For all $g \in G$, $n = \alpha g$, and $h \in H$ we have

$$\Delta^n((\nabla g) * h) = \Delta^n((g * e) * h) = \Delta^{n-1}\Delta((e \overset{2}{+} g) * h) = \Delta^{n-1} g,$$

which means that H is a v-subsystem of $(G, \zeta, \tau, \Delta, \nabla, *)$. Now, using the same argumentation as in the proof of Theorem 6.4.8, we can see that $(G, \Delta e, \zeta, \tau, \Delta, \nabla, *)$ satisfies all the conditions of Corollary 6.4.7, i.e., it is isomorphic to some central iterative operation algebra. Consequently $(G, e, \zeta, \tau, \Delta, *)$ is isomorphic to some central pre-iterative operation algebra on the set H. □

Corollary 6.4.10. *An algebra* $(G, \zeta, \tau, \Delta, *)$ *of type* $(1, 1, 1, 2)$ *is isomorphic to some pre-iterative Mal'cev-Post algebra of operations if and only if it contains an element* $e \in G$ *such that the algebra* $(G, e, \zeta, \tau, \Delta, *)$ *satisfies all the conditions of Theorem* 6.4.9.

6.5 Semigroups of functions

Let A be some set. By $\mathcal{T}(A) = \bigcup_{n \in \mathbb{N}} \mathcal{T}(A^n, A)$ we denote the set of all operations on A, by $\mathcal{F}(A) = \bigcup_{n \in \mathbb{N}} \mathcal{F}(A^n, A)$ the set of all partial multiplace functions and by $\mathfrak{Rel}(A) = \bigcup_{n \in \mathbb{N}} \mathfrak{P}(A^n \times A)$ the set of all relations. $|\rho|$ denotes the arity of ρ. Obviously $\mathcal{T}(A) \subset \mathcal{F}(A) \subset \mathfrak{Rel}(A)$.

For any $\rho \in \mathfrak{P}(A^n \times A)$ and $\sigma \in \mathfrak{P}(A^m \times A)$ we define a binary composition $*$ putting:

$$(\rho * \sigma)\langle a_1^{n+m-1} \rangle = \rho(\sigma\langle a_1^m \rangle, a_{m+1}^{n+m-1}).$$

A composition defined in such a way is an associative operation which coincides with the operation $\overset{1}{+}$ considered in the previous sections. If Φ is closed under this operation, then we say that $(\Phi, *)$ is a *semigroup of operations*, if $\Phi \subset \mathcal{T}(A)$;

a *semigroup of partial multiplace functions*, if $\mathbf{\Phi} \subset \mathcal{F}(A)$ and a *semigroup of relations*, if $\mathbf{\Phi} \subset \mathfrak{Rel}(A)$. The set

$$I_{\mathbf{\Phi}} = \{n \in \mathbb{N} \,|\, (\exists \rho)\, (\rho \in \mathbf{\Phi} \wedge n = |\rho|)\}$$

is called the *range* of a semigroup $(\mathbf{\Phi}, *)$.

The set $\mathbb{N}$ of all natural numbers with the operation $\oplus$ defined by the formula:

$$n \oplus m = n + m - 1$$

is a semigroup. Any subsemigroup of $(\mathbb{N}, \oplus)$ is called a *$\oplus$-semigroup of natural numbers*. The semigroup $(\{1\}, \oplus)$ is called *trivial*. It is not difficult to see that if $(\mathbf{\Phi}, *)$ is a semigroup of relations, then $(I_{\mathbf{\Phi}}, \oplus)$ is a $\oplus$-semigroup of natural numbers.

Theorem 6.5.1. *A semigroup $(G, \cdot)$ is isomorphic to some semigroup of operations if and only if it is homomorphic to a $\oplus$-semigroup of natural numbers $(I, \oplus)$.*

Proof. Let a semigroup $(G, \cdot)$ be isomorphic to a semigroup of operations $(\mathbf{\Phi}, *)$. Then the mapping $\varphi : \mathbf{\Phi} \to I_{\mathbf{\Phi}}$ defined by

$$\varphi(f) = n \longleftrightarrow |f| = n$$

is an epimorphism from $(\mathbf{\Phi}, *)$ onto $(I_{\mathbf{\Phi}}, \oplus)$. Indeed, for any $f, g \in \mathbf{\Phi}$ we have $|f * g| = |f| + |g| - 1$, i.e., $|f * g| = |f| \oplus |g|$, whence,

$$\varphi(f * g) = \varphi(f) \oplus \varphi(g).$$

The latest equality means that $(I_{\mathbf{\Phi}}, \oplus)$ is a homomorphic image of $(G, \cdot)$.

Conversely, let $\psi : G \to I$ be an epimorphism from $(G, \cdot)$ onto $(I, \oplus)$. Let $(G \cup \{e\}, \cdot)$ be a semigroup with joined unit $e \notin G$. For every $g \in G$ we define on $G \cup \{e\}$ the operation f_g of arity $\psi(g)$ by putting:

$$f_g(x_1, \ldots, x_{\psi(g)}) = g x_1.$$

The mapping $P : g \to f_g$ is an isomorphism of $(G, \cdot)$ onto $(\mathbf{\Phi}, *)$, where $\mathbf{\Phi} = \{f_g \,|\, g \in G\}$. Indeed, if $g_1, g_2 \in G$ and $n = \psi(g_1)$, $m = \psi(g_2)$, then $\psi(g_1 g_2) = \psi(g_1) \oplus \psi(g_2) = n \oplus m = n + m - 1$. Moreover,

$$\begin{aligned} f_{g_1 g_2}(x_1^{n+m-1}) &= (g_1 g_2) x_1 = g_1(g_2 x_1) = g_1 \cdot f_{g_2}(x_1^m) \\ &= f_{g_1}(f_{g_2}(x_1^m), x_{m+1}^{n+m-1}) = f_{g_1} * f_{g_2}(x_1^{n+m-1}) \end{aligned}$$

for all $x_1, \ldots, x_{n+m-1} \in G^*$. Thus, $f_{g_1 g_2} = f_{g_1} * f_{g_2}$, i.e., P is a homomorphism. This homomorphism is one-to-one because $f_{g_1} = f_{g_2}$ implies $g_1 e = f_{g_1}(e, \ldots, e) = f_{g_2}(e, \ldots, e) = g_2 e$, whence we deduce $g_1 = g_2$. So, P is an isomorphism. □

Corollary 6.5.2. *A semigroup* $(G,\cdot)$ *is isomorphic to some semigroup of partial multiplace functions* (*relations*) *of rank* I *if and only if it is homomorphic to a* $\oplus$*-semigroup* $(I,\oplus)$.

Corollary 6.5.3. *Each semigroup is isomorphic to some semigroup of transformations* (*partial transformations, binary relations*) *on some set.*

In fact, the first corollary is a consequence of the following inclusions $\mathcal{T}(A)\subset\mathcal{F}(A)\subset\mathfrak{Rel}(A)$; the second is explained by the fact that any semigroup is isomorphic to trivial $\oplus$-semigroup.

A system $(G,\cdot,\psi)$, where $\psi: G\to\mathbb{N}$ is a homomorphism of a semigroup $(G,\cdot)$ into a semigroup $(\mathbb{N},\oplus)$, is called the *normed semigroup*, the set $I=\psi(G)$ – the *range* of this semigroup, the number $\psi(g)$ – the *norm* of $g\in G$.

By a homomorphism of normed semigroups $(G_1,\cdot,\psi_1)$ and $(G_2,\circ,\psi_2)$, we mean a mapping $P: G_1\longrightarrow G_2$ satisfying the following two identities:

$$P(g_1\cdot g_2)=P(g_1)\circ P(g_2)\quad\text{and}\quad\psi_1(g)=\psi_2(P(g)).$$

A one-to-one homomorphism is called an *isomorphism.* Since homomorphic normed semigroups have the same rank each normed semigroup will be identified with the corresponding semigroups of rank I, where I denotes the set of norms of all elements of a given semigroup. From Theorem 6.5.1 it follows that each semigroup of rank I is isomorphic to some semigroup of operations (relations, partial multiplace functions) of the same rank.

Theorem 6.5.4. *A semigroup* $(G,\cdot)$ *of rank* I *with a binary relation* $\prec$ *defined on* G *is isomorphic to a semigroup of relations of the same rank* I *in this way that* $\prec$ *corresponds to the set-theoretic inclusion if and only if* $\prec$ *is a stable order on* $(G,\cdot)$ *and*

$$g_1\prec g_2\longrightarrow |g_1|=|g_2| \tag{6.5.1}$$

holds for all $g_1,g_2\in G$.

Proof. The necessity of these conditions is obvious, so we shall just prove their sufficiency. To do this, for each element $g\in G$ we define on a semigroup $(G\cup\{e\},\cdot)$ with joined unit $e\notin G$, a relation ρ_g of arity $|g|+1$ putting:

$$(x_1,\ldots,x_n,y)\in\rho_g\longleftrightarrow y\prec gx_1 \tag{6.5.2}$$

for all $y\in G$ and $x_1,\ldots,x_n\in G\cup\{e\}$, where $n=|g|$.

Let us show that $P: g\mapsto\rho_g$ is a faithful representation of a semigroup $(G,\cdot)$ by the above relations. Since for $g_1,g_2\in G$ we have

$$|\rho_{g_1g_2}|=|g_1g_2|=|g_1|\oplus|g_2|=|g_1|+|g_2|-1,$$

there are $y \in G$ and $x_1, \ldots, x_{n+m-1} \in G \cup \{e\}$, where $n = |g_1|$, $m = |g_2|$, such that

$$(x_1, \ldots, x_{n+m-1}, y) \in \rho_{g_1 g_2}.$$

Thus $y \prec (g_1 g_2) x_1 = g_1(g_2 x_1)$. But $g_2 x_1 \prec g_2 x_1$, so

$$(x_1, \ldots, x_m, g_2 x_1) \in \rho_{g_2}. \tag{6.5.3}$$

Analogously, $y \prec g_1(g_2 x_1)$ implies

$$(g_2 x_1, x_{m+1}, \ldots, x_{n+m-1}, y) \in \rho_{g_1}. \tag{6.5.4}$$

From (6.5.3) and (6.5.4) we get

$$(x_1, \ldots, x_{n+m-1}, y) \in \rho_{g_1} * \rho_{g_2}, \tag{6.5.5}$$

which proves the inclusion $\rho_{g_1 g_2} \subset \rho_{g_1} * \rho_{g_2}$.

To prove the converse inclusion assume that (6.5.5) is satisfied. Then there exists $z \in G$ such that

$$(x_1, \ldots, x_m, z) \in \rho_{g_2} \quad \text{and} \quad (z, x_{m+1}, \ldots, x_{n+m-1}, y) \in \rho_{g_1},$$

i.e., $z \prec g_2 x_1$ and $y \prec g_1 z$. By the stability of $\prec$ from $z \prec g_2 x_1$ we obtain $g_1 z \prec g_1(g_2 x)$. Therefore

$$y \prec g_1(g_2 x_1) = (g_1 g_2) x_1,$$

i.e., $(x_1, \ldots, x_{n+m-1}, y) \in \rho_{g_1 g_2}$. So, $\rho_{g_1} * \rho_{g_2} \subset \rho_{g_1 g_2}$. Thus $\rho_{g_1 g_2} = \rho_{g_1} * \rho_{g_2}$, which gives $P(g_1 \cdot g_2) = P(g_1) * P(g_2)$ for all $g_1, g_2 \in G$. This means that P is a representation of a semigroup $(G, \cdot)$ by relations.

Now let $g_1 \prec g_2$ and $(x_1, \ldots, x_n, y) \in \rho_{g_1}$, where $n = |g_1|$. Then $|g_1| = |g_2|$ and $y \prec g_1 x_1 \prec g_2 x_1$. Consequently, $(x_1, \ldots, x_n, y) \in \rho_{g_2}$, i.e., $\rho_{g_1} \subset \rho_{g_2}$. Conversely, if the last inclusion is true, then

$$y \prec g_1 x_1 \longrightarrow y \prec g_2 x_1,$$

for every $x_1 \in G \cup \{e\}$ and $y \in G$. Whence, for $x_1 = e$ and $y = g_1$ we obtain $g_1 \prec g_2$. This completes the proof that

$$g_1 \prec g_2 \longleftrightarrow P(g_1) \subset P(g_2)$$

for all $g_1, g_2 \in G$. So, $\prec$ corresponds to $\subset$.

The representation P is faithful because $P(g_1) = P(g_2)$ means that $P(g_1) \subset P(g_2)$ and $P(g_2) \subset P(g_1)$, that is, $g_1 \prec g_2$ and $g_2 \prec g_1$, which according to the assumption on $\prec$ implies $g_1 = g_2$. □

Theorem 6.5.5. *A semigroup $(G,\cdot)$ of rank I with a binary relation $\prec$ defined on G is isomorphic to a semigroup of partial multiplace functions of the same rank I in such a way that $\prec$ corresponds to the set-theoretic inclusion if and only if $\prec$ is a stable order on $(G,\cdot)$ such that* (6.5.1) *and*

$$g_1 \prec g_2 \wedge g \prec xg_1 \wedge g \prec yg_2 \longrightarrow g \prec yg_1 \tag{6.5.6}$$

are satisfied for all $g, g_1, g_2, x, y \in G$, where x, y may be empty symbols.

Proof. Necessity. Let $(G,\cdot,\prec)$ be isomorphic to some semigroup $(\mathbf{\Phi}, *, \subset)$ of partial multiplace functions of rank I. In view of Theorem 6.5.4 we can restrict our proof to the proof of (6.5.6). For this assume that

$$f_1 \subset f_2, \quad f \subset h * f_1, \quad f \subset p * f_2$$

for some functions $f, f_1, f_2, h, p \in \mathbf{\Phi}$. Then $|f_1| = |f_2|$, $|f| = |h| + |f_1| - 1$, $|f| = |p| + |f_2| - 1$. Consequently $|h| = |p|$. If $(x_1^{n+m-1}, y) \in f$ for some $x_1, \ldots, x_{n+m-1}$, where $n = |f_1| = |f_2|$, $m = |h| = |p|$, then $(x_1^{n+m-1}, y) \in h * f_1$, i.e., $y = h(f_1(x_1^n), x_{n+1}^{n+m-1})$. Since for $z = f_1(x_1^n)$, from $f_1 \subset f_2$, it follows $z = f_2(x_1^n)$, from $(x_1^{n+m-1}, y) \in f$ we deduce $(x_1^{n+m-1}, y) \in p * f_2$. Thus

$$\begin{aligned} y = p * f_2(x_1^{n+m-1}) &= p(f_2(x_1^n), x_{n+1}^{n+m-1}) = p(z, x_{n+1}^{n+m-1}) \\ &= p(f_1(x_1^n), x_{n+1}^{n+m-1}) = p * f_1(x_1^{n+m-1}), \end{aligned}$$

i.e., $(x_1^{n+m-1}, y) \in p * f_1$. So, $f \subset p * f_1$. This completes the proof of (6.5.6).

Sufficiency. Assume that a partially ordered semigroup $(G,\cdot,\prec)$ satisfies all the conditions of the theorem. For $n = 1$ from Theorem 3.2.3 it follows that $(G,\cdot,\prec)$ is isomorphic to some semigroup $(F, \circ, \subset)$ of partial transformations of some set A. Let $\varphi : G \to F$ be this isomorphism. Using this isomorphism, we define on A the partial n-place function $P(g)$ putting

$$P(g)(a_1, a_2, \ldots, a_n) = \varphi(g)(a_1),$$

where $g \in G$, $|g| = n$.

Let us show that the mapping $P : G \to \mathbf{\Phi}$, where $\mathbf{\Phi} = \{P(g) \,|\, g \in G\}$, is an isomorphism between semigroups $(G,\cdot)$ and $(\mathbf{\Phi}, *)$ and $\prec$ corresponds to $\subset$. Indeed, for all $g_1, g_2 \in G$, $|g_1| = n, |g_2| = m$ and $a_1, \ldots, a_{n+m_1} \in A$ we have

$$\begin{aligned} P(g_1g_2)(a_1^{n+m-1}) &= \varphi(g_1g_2)(a_1) = (\varphi(g_1) \circ \varphi(g_2))(a_1) \\ &= \varphi(g_1)(\varphi(g_2)(a_1)) = \varphi(g_1)(P(g_2)(a_1^m)) \\ &= P(g_1)(P(g_2)(a_1^m), a_{m+1}^{n+m-1}) = P(g_1) * P(g_2)(a_1^{n+m-1}). \end{aligned}$$

This means that P is a homomorphism.

If $P(g_1) = P(g_2)$, then $P(g_1)(a_1^n) = P(g_2)(a_1^n)$ for all $a_1, \ldots, a_n \in A$, where $n = |g_1| = |g_2|$. Whence, according to the definition of $P(g)$, we obtain $\varphi(g_1)(a_1) = \varphi(g_2)(a_1)$ for every $a_1 \in A$. Thus $\varphi(g_1) = \varphi(g_2)$, and consequently $g_1 = g_2$. So, P is an isomorphism.

Now let $g_1 \prec g_2$. Then $|g_1| = |g_2| = n$ and $\varphi(g_1) \subset \varphi(g_2)$, i.e.,

$$(a_1, b) \in \varphi(g_1) \longrightarrow (a_1, b) \in \varphi(g_2),$$

for every $a_1 \in A$ and $b = \varphi(g_1)(a_1) = \varphi(g_2)(a_1)$.

Since $\varphi(g_i)(a_1) = P(g_i)(a_1^n)$, $i = 1, 2$, the above implication means also that

$$(a_1^n, b) \in P(g_1) \longrightarrow (a_1^n, b) \in P(g_2)$$

for all $a_1, \ldots, a_n \in A$. Thus $P(g_1) \subset P(g_2)$. Therefore,

$$g_1 \prec g_2 \longleftrightarrow P(g_1) \subset P(g_2),$$

which completes the proof of this theorem. □

Besides the relation of inclusion on a semigroup $(\Phi, *)$ of partial multiplace functions (or relations), we can also consider other relations such as, for example, the relation of inclusion of domains of functions of different arity. The last relation, denoted by $\mathcal{X}_\Phi$, is defined in the following way:

$$(f, g) \in \mathcal{X}_\Phi \longleftrightarrow |f| \geqslant |g| \wedge (\forall x_1^n)\left(f\langle x_1^n\rangle \neq \varnothing \longrightarrow g\langle x_1^m\rangle \neq \varnothing\right),$$

where $f, g \in \Phi$, $|f| = n$, $|g| = m$. It is clear that $\mathcal{X}_\Phi$ is a quasi-order such that $\zeta_\Phi \subset \mathcal{X}_\Phi$, where

$$\zeta_\Phi = \{(f, g) \in \Phi \times \Phi \,|\, f \subset g\},$$

and satisfies the following two conditions:

$$f \sqsubset_\Phi g \longrightarrow f * h \sqsubset_\Phi g * h, \tag{6.5.7}$$

$$f * g \sqsubset_\Phi g\,, \tag{6.5.8}$$

where $f \sqsubset_\Phi g$ means $(f, g) \in \mathcal{X}_\Phi$. Each algebraic system of the form $(\Phi, *, \zeta_\Phi, \mathcal{X}_\Phi)$ is called a *fundamentally ordered projection semigroup of multiplace functions* (*relations*).

Theorem 6.5.6. *An algebraic system $(G, \cdot, \zeta, \mathcal{X})$, where $(G, \cdot)$ is a semigroup of rank I, ζ and $\mathcal{X}$ are binary relations, is isomorphic to some fundamentally ordered projection semigroup of relations of the same rank as $(G, \cdot)$, if and only if ζ is a stable order such that $\zeta \subset \mathcal{X}$ and*

$$x \prec y \longrightarrow |x| = |y|, \tag{6.5.9}$$

$$x \sqsubset y \longrightarrow |x| \geqslant |y|, \tag{6.5.10}$$

$$x \sqsubset y \longrightarrow xz \sqsubset yz, \tag{6.5.11}$$

$$xy \sqsubset y \tag{6.5.12}$$

for all $x, y, z \in G$, *where* $x \prec y \longleftrightarrow (x,y) \in \zeta$ *and* $x \sqsubset y \longleftrightarrow (x,y) \in \mathcal{X}$.

Proof. The necessity of the above conditions is obvious. To prove their sufficiency we select some element $a \in G$ and consider a one-element extension $(G^*, \cdot)$ of a semigroup $(G, \cdot)$ such that this additional element $e \notin G$ is the unity of the semigroup $(G^*, \cdot)$. Next, for every $g \in G$, where $|g| = n$, we define the relation $P_a(g) \subset (G^*)^n \times G$ putting

$$(x_1^n, y) \in P_a(g) \longleftrightarrow a \sqsubset y \prec gx_1.$$

Let us show that P_a is a homomorphism. Indeed, if $(x_1^{n+m-1}, y) \in P_a(g_1g_2)$, where $n = |g_1|$, $m = |g_2|$, then

$$a \sqsubset y \prec (g_1g_2)x_1 = g_1(g_2x_1), \tag{6.5.13}$$

whence, in view of $\zeta \subset \mathcal{X}$ and (6.5.12), we have $a \sqsubset g_1(g_2x_1) \sqsubset g_2x_1$. But $g_2x_1 \prec g_2x_1$, so, $a \sqsubset g_2x_1 \prec g_2x_1$, which gives us

$$(x_1^m, g_2x_1) \in P_a(g_2). \tag{6.5.14}$$

From (6.5.13) we also obtain $(g_2x_1, x_{m+1}^{n+m-1}, y) \in P_a(g_1)$. Therefore

$$(x_1^{n+m-1}, y) \in P_a(g_1) * P_a(g_2).$$

This proves the inclusion

$$P_a(g_1g_2) \subset P_a(g_1) * P_a(g_2).$$

Conversely, if $(x_1^{n+m-1}, y) \in P_a(g_1) * P_a(g_2)$, then

$$(x_1^m, z) \in P_a(g_2) \quad \textit{and} \quad (zx_{m+1}^{n+m-1}, y) \in P_a(g_1)$$

for some $z \in G$. Thus

$$a \sqsubset z \prec g_2x_1 \quad \textit{and} \quad a \sqsubset y \prec g_1z.$$

Whence, applying (6.5.9) and (6.5.10), we obtain

$$|a| \geqslant |z| = |g_2| + |x_1| - 1, \quad |a| \geqslant |y| = |g_1| + |z| - 1,$$

which implies

$$|a| \geqslant |y| = |g_1| + |g_2| + |x_1| - 2 = |(g_1g_2)x_1|.$$

Further, applying the stability of the relation $\prec$ to $z \prec g_2x_1$, we get

$$g_1z \prec g_1(g_2x_1) = (g_1g_2)x_1.$$

Thus $a \sqsubset y \prec (g_1g_2)x_1$, i.e., $(x_1^{n+m-1}, y) \in P_a(g_1g_2)$. So, $P_a(g_1) * P_a(g_2) \subset P_a(g_1g_2)$. Thus,

$$P_a(g_1g_2) = P_a(g_1) * P_a(g_2)$$

for any $g_1, g_2 \in G$. This proves that P_a is a representation of $(G, \cdot)$ by relations. Clearly the sum $P = \sum P_a$ of all representations $(P_a)_{a\in G}$ is also a representation of $(G, \cdot)$ by relations.

Let us show that P satisfies the relations ζ and $\mathcal{X}$, i.e.,

$$g_1 \prec g_2 \longleftrightarrow P(g_1) \subset P(g_2) \tag{6.5.15}$$

$$g_1 \sqsubset g_2 \longleftrightarrow P(g_1) \sqsubset_{\Phi} P(g_2) \tag{6.5.16}$$

for all $g_1, g_2 \in G$, where $\Phi = P(G)$.

If $g_1 \prec g_2$ for some $g_1, g_2 \in G$ then $|g_1| = |g_2| = n$. For every $(x_1^n, y) \in P(g_1)$ there exists such $a \in G$ for which $(x_1^n, y) \in P_a(g_1)$, i.e., $a \sqsubset y \prec g_1x_1$. Since $g_1 \prec g_2$, by the stability of $\prec$, implies $g_1x_1 \prec g_2x_1$, we also have $a \sqsubset y \prec g_2x_1$. Thus $(x_1^n, y) \in P_a(g_2) \subset P(g_2)$. So, $P(g_1) \subset P(g_2)$.

Conversely, $P(g_1) \subset P(g_2)$ means that $|g_1| = |g_2| = n$ and

$$(\forall x_1^n \in G^*)(\forall y \in G)(\forall a \in G)(a \sqsubset y \prec g_1x_1 \longrightarrow a \sqsubset y \prec g_2x_1).$$

In view of $|a| \geqslant |y| = |g_1x_1| = |g_2x_1|$, the last condition is also true for $a = y = g_1x_1$. In this case it has the form

$$(\forall x_1 \in G^*)(g_1x_1 \sqsubset g_1x_1 \prec g_1x_1 \longrightarrow g_1x_1 \sqsubset g_1x_1 \prec g_2x_1),$$

which, in fact, means that $g_1x_1 \prec g_2x_1$ for every $x_1 \in G^*$. Whence, for $x_1 = e$, we get $g_1 \prec g_2$. This completes the proof of (6.5.15).

Now let $g_1 \sqsubset g_2$. Then $n = |g_1| \geqslant |g_2| = m$. For any $P(g_1)\langle x_1^n\rangle \neq \varnothing$, there are $a, y \in G$ such that $(x_1^n, y) \in P_a(g_1)$, i.e., $a \sqsubset y \prec g_1x_1$. Since $\zeta \subset \mathcal{X}$, the last condition gives $a \sqsubset y \sqsubset g_1x_1$ and $a \sqsubset y \sqsubset g_2x_1$, because $g_1 \sqsubset g_2$, according to (6.5.11), implies $g_1x_1 \sqsubset g_2x_1$. So, $a \sqsubset g_2x_1 \prec g_2x_1$. Thus $(x_1^m, g_2x_1) \in P_a(g_2) \subset P(g_2)$. Therefore $P(g_2)\langle x_1^m\rangle \neq \varnothing$. This proves that $g_1 \sqsubset g_2$ implies $P(g_1) \sqsubset_{\Phi} P(g_2)$ for $\Phi = P(G)$.

Conversely, if $P(g_1) \sqsubset_{\Phi} P(g_2)$ holds for $\Phi = P(G)$ and $g_1, g_2 \in G$, then

$$(\forall a \in G)(\forall x_1^n \in G^*)\left((\exists y \in G) a \sqsubset y \prec g_1x_1 \longrightarrow (\exists z \in G) a \sqsubset z \prec g_2x_1\right).$$

This condition means that

$$(\forall a \in G)(\forall x_1^n \in G^*)(\forall y \in G)\left(a \sqsubset y \prec g_1x_1 \longrightarrow (\exists z \in G) a \sqsubset z \prec g_2x_1\right).$$

Putting $a = y = g_1x_1$, we see that for every $x_1 \in G^*$ there exists $z \in G$ such that $g_1x_1 \sqsubset z \sqsubset g_2x_1$. In particular, for every $x_1 \in G^*$ we have $g_1x_1 \sqsubset g_2x_1$.

Whence, for $x_1 = e$, we obtain $g_1 \sqsubset g_2$. So, $P(g_1) \sqsubset_\Phi P(g_2)$ implies $g_1 \sqsubset g_2$. This proves (6.5.16).

Now, in a similar way as in the proof of Theorem 6.5.4, we can verify that the representation P is faithful using (6.5.15). □

Corollary 6.5.7. *An algebraic system* $(G, \cdot, \mathcal{X})$, *where* $(G, \cdot)$ *is a semigroup of range* I *and* $\mathcal{X}$ *is a quasi-order, is isomorphic to some semigroup of relations of the form* $(\Phi, *, \mathcal{X}_\Phi)$ *on the same rank as* $(G, \cdot)$, *if and only if the conditions* (6.5.10), (6.5.11), (6.5.12) *are satisfied.*

Theorem 6.5.8. *An algebraic system* $(G, \cdot, \zeta, \mathcal{X})$, *where* $(G, \cdot)$ *is a semigroup of rank* I, ζ *and* $\mathcal{X}$ *are binary relations, is isomorphic to some fundamentally ordered projection semigroup of partial functions of the same rank as* $(G, \cdot)$ *if and only if all the conditions of Theorem* 6.5.6 *are satisfied and*

$$g_1 \prec g \,\wedge\, g_2 \prec g \,\wedge\, g_1 \sqsubset g_2 \longrightarrow g_1 \prec g_2, \tag{6.5.17}$$

$$g_1 \prec g_2 \,\wedge\, g \sqsubset g_1 \,\wedge\, g \sqsubset ug_2 \longrightarrow g \sqsubset ug_1 \tag{6.5.18}$$

for all $g, g_1, g_2, u \in G$.

Proof. The proof is analogous to the proof of Theorem 6.5.5 and follows from Theorem 3.2.1. □

6.6 Central semigroups of operations

An algebra $(\mathcal{T}(A), *)$, where $\mathcal{T}(A)$ is the set of all operations of arbitrary arity defined on the set A, is called the *semigroup of operations*. The semigroup of operation with a distinguished projector I_2^2, i.e., the algebra $(\mathcal{T}(A), *, I_2^2)$ is called the *central semigroup of operations*.

The projector I_2^2 plays an important role in these semigroups. For example, the arity of $f \in \mathcal{T}(A)$ can be characterized by I_2^2. Namely, $f \in \mathcal{T}(A)$ is an n-ary operation if and only if $I_2^2 * f = (I_2^2)^n$. Indeed, if $n = |f|$, then $|I_2^2 * f| = n+1$ and $(I_2^2 * f)(a_1^{n+1}) = I_2^2(f(a_1^n), a_{n+1}) = a_{n+1}$ for arbitrary $a_1, \ldots, a_{n+1} \in A$. On the other hand $(I_2^2)^n(a_1^{n+1}) = I_2^2((I_2^2)^{n-1}(a_1^n), a_{n+1}) = a_{n+1}$. Thus, $(I_2^2)^n(a_1^{n+1}) = (I_2^2 * f)(a_1^{n+1})$, i.e., $I_2^2 * f = (I_2^2)^n$. The reverse statement is obvious.

Let C_A be the set of all unary operations $\varphi_a \in \mathcal{T}(A, A)$, $a \in A$, such that $\varphi_a(x) = a$. Since $\varphi_{a_1} * \varphi_{a_2} = \varphi_{a_1}$, $(C_A, *)$ is a left zero semigroup.

For every $a \in A$ and every n-ary operation $f \in \mathcal{T}(A)$ we define a new $(n-1)$-ary operation f^a putting

$$f^a(x_1, x_2, \ldots, x_{n-1}) = f(a, x_1, x_2, \ldots, x_{n-1})$$

for all $x_1, \ldots, x_n \in A$. It is not difficult to see that

$$f * \varphi_a = f^a * I_2^2.$$

Let $\mathfrak{M}(H)$, where H is a nonempty set, be the class of all *central semigroups* $(G, \cdot, e)$, i.e., abstract semigroups $(G, \cdot)$ with distinguished element $e \in G$, such that $H \subset G$, $e \notin H$, in which the following conditions are satisfied:

$$(\forall g \in G)(\exists n \in N)\,(eg = e^n), \tag{6.6.1}$$

$$(\forall h \in H)\,(eh = e), \tag{6.6.2}$$

$$(\forall n \in N)(\forall m \in N)\,(e^n = e^m \longrightarrow n = m), \tag{6.6.3}$$

$$(\forall g \in G)(\forall h \in H)\,(eg = e \longrightarrow gh \in H), \tag{6.6.4}$$

$$(\forall g \in G)(\forall h \in H)(\exists x \in G)\,(eg \neq e \longrightarrow gh = xe), \tag{6.6.5}$$

$$(\forall g_1 \in G)(\forall g_2 \in G)\,(g_1 e = g_2 e \longrightarrow g_1 = g_2), \tag{6.6.6}$$

$$(\exists x \in G)(\forall g \in G)\,(gx = xg = g). \tag{6.6.7}$$

It is clear that $(\mathcal{T}(A), *, I_2^2) \in \mathfrak{M}(C_A)$.

Let $(G, \cdot, e) \in \mathfrak{M}(H)$. From (6.6.1) and (6.6.3), it follows that for every $g \in G$ there exists a uniquely determined natural number n such that $eg = e^n$. This number is denoted by αg. We have $\alpha e = 2$ and $\alpha h = 1$ for every $h \in H$. Note that the element x, whose existence is postulated in (6.6.5), is uniquely determined. Indeed, if $gh = x_1 e$ and $hg = x_2 e$, then $x_1 e = x_2 e$, whence, according to (6.6.6), we have $x_1 = x_2$. Moreover, if $\alpha(g_1 g_2) = n$, $\alpha g_1 = m_1$, $\alpha g_2 = m_2$ for some $g_1, g_2 \in G$, then $eg_1 g_2 = e^n$, $eg_1 = e^{m_1}$, $eg_2 = e^{m_2}$. Whence $e^n = eg_1 g_2 = e^{m_1} g_2 = e^{m_1 - 1} \cdot eg_2 = e^{m_1 - 1} \cdot e^{m_2} = e^{m_1 + m_2 - 1}$. So, $\alpha(g_1 g_2) = \alpha g_1 + \alpha g_2 - 1$. In particular, $h \in H$ and $gh = ex$ imply $\alpha x = \alpha g - 1$.

Now let us consider the mapping $P : G \longrightarrow \mathcal{T}(H)$ defined in the following way:

(a) $|P(g)| = n \longleftrightarrow \alpha g = n$,

(b) $P(g)(h_1^n) = h$ for $h_1, \ldots, h_n, h \in H$, where $n = \alpha g$, if and only if there exist $x_1, x_2, \ldots, x_{n-1} \in G$ such that

$$gh_1 = x_1 e \wedge \bigwedge_{i=1}^{n-2} (x_i h_{i+1} = x_{i+1} e) \wedge x_{n-1} h_n = h. \tag{6.6.8}$$

Proposition 6.6.1. *P is a homomorphism of $(G, \cdot, e)$ into $(\mathcal{T}(H), *, I_2^2)$.*

Proof. From (6.6.7), follows that any semigroup $(G, \cdot, e) \in \mathfrak{M}(H)$ has a unit. Denote this unit by e'. According to (6.6.2), for all $h_1, h_2 \in H$ we have $eh_1 = e = e'e$ and $e'h_2 = h_2$, whence, by (6.6.8), we get $P(e)(h_1, h_2) = h_2$. So, $P(e) = I_2^2$. Now let $P(g_1 g_2)(h_1^n) = h$, where $n = n_1 + n_2 - 1$, $n_1 = \alpha g_1$, $n_2 = \alpha g_2$, $n = \alpha(g_1 g_2)$. Then, one can find $x_1, x_2, \ldots, x_{n-1} \in G$ such that

$$g_1 g_2 h = x_1 e \wedge \bigwedge_{i=1}^{n-2} (x_i h_{i+1} = x_{i+1} e) \wedge x_{n-1} h_n = h. \tag{6.6.9}$$

Further, (6.6.4) and (6.6.5) guarantee the existence of $y_1, y_2, \ldots, y_{n_2-1} \in G$ such that

$$g_2 h_1 = y_1 e \wedge \bigwedge_{i=1}^{n_2-2} (y_i h_{i+1} = y_{i+1} e). \qquad (6.6.10)$$

Thus $\alpha y_{n_2-1} = 1$, which, by (6.6.4), implies $y_{n_2-1} h_{n_2} \in H$. This, according to the definition of $P(g)$, means $P(g_2)(h_1^{n_2}) = h'$ for $h' = y_{n_1-1} h_{n_2}$.

Now, multiplying all equalities of (6.6.10) by g_1, we get

$$g_1 g_2 h_1 = g_1 y_1 e \wedge \bigwedge_{i=1}^{n_2-2} (g_1 y_i h_{i+1} = g_1 y_{i+1} e). \qquad (6.6.11)$$

Comparing (6.6.10) and (6.6.11), we obtain $x_1 e = g_1 y_1 e$. So, $x_1 = g_1 y_1$ and $x_1 h_2 = g_1 y_1 h_2$. Thus $x_2 e = g_1 y_2 e$, whence $x_2 = g_1 y_2$. Continuing this procedure we get $x_{n_2-1} = g_1 y_{n_2-1}$. Consequently, $x_{n_2-1} h_{n_2} = g_1 y_{n_2-1} h_{n_2} = g_1 h'$. But $x_{n_2-1} h_{n_2} = x_{n_2} e$, so, $g_1 h = x_1 e$. Therefore,

$$g_1 h' = x_{n_2} e \wedge \bigwedge_{i=0}^{n_1-3} (x_{n_2+i} h_{n_2+i+1} = x_{n_2+i+1} e) \wedge x_{n-1} h_n = h,$$

whence we obtain $P(g_1)(h', h_{n_2+1}^n) = h$, i.e., $P(g_1)(P(g_2)(h_1^{n_2}), h_{n_2+1}^n) = h$. This gives $P(g_1) * P(g_2)(h_1^n) = h$. In this way we have proved that $P(g_1 g_2)(h_1^n) = h$ implies $P(g_1) * P(g_2)(h_1^n) = h$. The converse implication can be proved similarly. So, $P(g_1 g_2) = P(g_1) * P(g_2)$. □

Theorem 6.6.2. *In the class* $\mathfrak{M}(C_A)$ $(C_A, *)$ *is a dense subsemigroup of* $(\mathcal{T}(A), *, I_2^2)$.

Proof. Let P be a homomorphism, which is not an isomorphism, from $(\mathcal{T}(A), *, I_2^2)$ onto some central semigroup $(G, \cdot, e) \in \mathfrak{M}(C_A)$. Then $P(I_2^2) = e$. Since P is not an isomorphism there are $f_1, f_2 \in \mathcal{T}(A)$ such that $f_1 \neq f_2$ and $P(f_1) = P(f_2)$. Thus $P(I_2^2)P(f_1) = P(I_2^2)P(f_2)$, i.e., $P(I_2^2 * f_1) = P(I_2^2 * f_2)$, whence $P((I_2^2)^{|f_1|}) = P((I_2^2)^{|f_2|})$. Consequently, $e^{|f_1|} = e^{|f_2|}$. So, $|f_1| = |f_2|$. Thus $b_1 = f_1(a_1^n) \neq f_2(a_1^n) = b_2$ for some $a_1, \ldots, a_n \in A$, where $n = |f_1| = |f_2|$. Therefore

$$f_i * \varphi_{a_1} = f_i^{a_1} * I_2^2 \wedge \bigwedge_{k=n-1}^{2} \left(f_i^{a_1^{n-k}} * \varphi_{a_{n-k+1}} = f_i^{a_1^{n-k+1}} * I_2^2 \right) \wedge f_i^{a_1^{n-1}} * \varphi_{a_n} = \varphi_{b_i}$$

for $i = 1, 2$. This implies $P(f_i)P(\varphi_{a_1}) = P(f_i^{a_1})e$, $P(f_i^{a_1^{n-k}})P(\varphi_{a_{n-k+1}}) = P(f_i^{a_1^{n-k+1}})e$ for $k = 2, \ldots, n-2$, and $P(f_i^{a_1^{n-1}})P(\varphi_{a_n}) = P(\varphi_{b_i})$. Since $P(f_1) = P(f_2)$, we have $P(f_1)P(\varphi_{a_1}) = P(f_2)P(\varphi_{a_1})$. Thus $P(f_1^{a_1})e = P(f_2^{a_1})e$ and

$P(f_1^{a_1}) = P(f_2^{a_1})$, by (6.6.6). Repeating this argumentation we obtain $P(\varphi_{b_1}) = P(\varphi_{b_2})$, but $\varphi_{b_1} \neq \varphi_{b_2}$, so, P induces on $(C_A, *)$ a homomorphism which is not an isomorphism.

Let $(G, *, I_2^2) \in \mathfrak{M}(C_A)$ be a central semigroup containing $(\mathcal{T}(A), *, I_2^2)$ as a proper subsemigroup. Let us consider the mapping $P : G \longrightarrow \mathcal{T}(A)$ which for every $g \in G$ defines on A the operation $P(g)$ of arity $n = \alpha g$ such that $P(g)(a_1^n) = b$ if and only if there exist elements $x_1, \ldots, x_{n-1} \in G$ for which we have

$$g * \varphi_{a_1} = x_1 * I_2^2 \;\wedge\; \bigwedge_{i=1}^{n-2} (x_i * \varphi_{a_{i+1}} = x_{i+1} * I_2^2) \;\wedge\; x_{n-1} * \varphi_{a_n} = \varphi_b.$$

By Proposition 6.6.1, P defined in such a way is a homomorphism. Directly from the definition of P it follows that $P(f) = f$ for $f \in \mathcal{T}(A)$ and $P(g) \neq g$ for $g \in G \setminus \mathcal{T}(A)$. Thus P is not an isomorphism, but on $\mathcal{T}(A)$, and consequently on C_A, it is an isomorphism. □

Theorem 6.6.3. *A central semigroup $(G, \cdot, e)$ is isomorphic to some central semigroup of operations if and only if there exists a left zero semigroup $(H, \cdot)$ such that $(G, \cdot, e) \in \mathfrak{M}(H)$ and $(H, \cdot)$ is a dense subsemigroup of $(G, \cdot, e)$.*

Proof. The necessity of these conditions follows form the previous theorem. To prove their sufficiency assume that a central semigroup $(G, \cdot, e)$ satisfies all these conditions. Consider the homomorphism P from $(G, \cdot, e)$ into $(\mathcal{T}(H), *, I_2^2)$ defined according to the formula (6.6.8). Since $hh_1 = h$ for every $h, h_1 \in H$, we must have $P(h) = \varphi_h$, i.e., P restricted to $(H, \cdot)$ must be an isomorphism. So, $(H, \cdot)$ and $(C_H, *)$ are isomorphic. Identifying these two semigroups, it is easy to show that $(P(G), *, I_2^2) \in \mathfrak{M}(H)$. Since in the class $\mathfrak{M}(H)$ the semigroup $(H, \cdot)$ is a dense subsemigroup of $(G, \cdot, e)$, the mapping P is a one-to-one embedding of $(G, \cdot, e)$ into $(\mathcal{T}(H), *, I_2^2)$. So, $(H, \cdot)$, and consequently $(C_H, *)$, can be considered in the class $\mathfrak{M}(C_H)$ as a dense subsemigroup of $(P(G), *, I_2^2)$. By Theorem 6.6.2, in the class $\mathfrak{M}(C_H)$ a semigroup $(C_H, *)$ is a dense subsemigroup of $(\mathcal{T}(H), *, I_2^2)$. Therefore $P(G) = \mathcal{T}(H)$. Thus, $(G, \cdot, e)$ is isomorphic to $(\mathcal{T}(H), *, I_2^2)$. □

Corollary 6.6.4. *A semigroup $(G, \cdot)$ is isomorphic to some semigroup of operations if and only if there exists a left zero semigroup $(H, \cdot)$ and an element $e \in G$ such that $(G, \cdot, e) \in \mathfrak{M}(H)$ and $(H, \cdot)$ is a dense subsemigroup of $(G, \cdot, e)$.*

6.7 Algebras of vector-valued functions

Any mapping $f \colon A^n \longrightarrow A^m$, where n, m are natural numbers and A is a nonempty set, is called a *multiplace vector-valued function* (or simply a *vector-valued function*) *of degree* n and *rank* m. The degree of f is denoted by αf, the rank by βf. The *index* of f is the difference $\gamma f = \alpha f - \beta f$. The set of all vector-valued functions of degree n and rank m defined on A is denoted by $\mathcal{T}(A^n, A^m)$.

On the set $\mathcal{T}(A) = \bigcup_{n,m\in\mathbb{N}} \mathcal{T}(A^n, A^m)$ we can consider two binary compositions: serial $\circ$ and parallel $\star$, which were considered by B. Schweizer and A. Sklar in [191–193]. These compositions are motivated by the algebraic theory of automaton. As it is known any automaton with n entrances and m exits can be characterized by some vector-valued function of degree n and rank m. The serial composition of automatons can be characterized by serial composition of the corresponding vector-valued functions. Similarly, the parallel composition of automatons can be characterized by serial composition of vector-valued functions (cf. [50] or [145]).

Definition 6.7.1. The *serial composition* of two vector-valued functions $f, g \in \mathcal{T}(A)$ is defined by

$$(f \circ g)(a_1, \ldots, a_d) = f(b_1, \ldots, b_{\alpha f}) b_{\alpha f+1} \ldots b_{d-\gamma g}\,, \tag{6.7.1}$$

where $a_1, a_2, \ldots, a_d \in A$, $d = \max\{\alpha f + \gamma g, \alpha g\}$, $b_1, b_2, \ldots, b_{d-\gamma g} \in A$ and $b_1 \ldots b_{d-\gamma g} = g(a_1, \ldots, a_{\alpha g}) a_{\alpha g+1} \ldots a_d$.

Definition 6.7.2. The *parallel composition* of two vector-valued functions $f, g \in \mathcal{T}(A)$ is the vector-valued function $f \star g$ defined by

$$(f \star g)(a_1, \ldots, a_d) = f(a_1, \ldots, a_{\alpha f}) g(a_1, \ldots, a_{\alpha g})\,, \tag{6.7.2}$$

where $a_1, \ldots, a_d \in A$ and $d = \max\{\alpha f, \alpha g\}$.

It is easy to see that these two compositions are associative operations on $\mathcal{T}(A)$. Putting $\Delta(x) = I_1^1(x) = x$ and $F(x, y) = I_2^2(x, y) = y$, we can verify that

$$I_i^n = (F \circ (F \star \Delta))^{n-i} \circ F^{i-1}$$

for any $n \in \mathbb{N}$, $1 \leqslant i \leqslant n$, where $f^0 = \Delta$ and $f^{n+1} = f \circ f^n$ for any $f \in \mathcal{T}(A)$.

Any subset $\mathbf{\Phi}$ of $\mathcal{T}(A)$ containing Δ, F and closed under serial and parallel compositions, i.e., the system $(\mathbf{\Phi}, \circ, \star, \Delta, F)$, is called an *algebra of vector-valued functions*. In the case $\mathbf{\Phi} = \mathcal{T}(A)$ we say that this system is *symmetrical*.

First, we shall consider the abstract algebra $(G, \circ, \star, e, f)$ of type $(2, 2, 0, 0)$ satisfying the following six axioms:

Axiom 1. *$(G, \circ)$ and $(G, \star)$ are semigroups and e is the unit of $(G, \circ)$.*

Let e_i^p denote the expression $(f \circ (f \star e))^{p-i} \circ f^{i-1}$, where $p \in \mathbb{N}$, $i = 1, \ldots, p$ and $(f \circ (f \star e))^0 = f^0 = e$.

Axiom 2. *For each $g \in G$ there exist $m, n \in \mathbb{N}$ such that*

$$g \circ (e_1^p \star \cdots \star e_p^p) = g\,, \quad (e_1^q \star \cdots \star e_q^q) \circ g = g$$

for all natural $p \leqslant n,\ q \leqslant m,\ p, q \in \mathbb{N}$ *and*

$$g \circ (e_1^p \star \cdots \star e_p^p) \neq g\,, \quad (e_1^q \star \cdots \star e_q^q) \circ g \neq g$$

for all $p > n,\ q > m$.

The number n is called the *degree* of g and is denoted by αg. The number m, denoted by βg, is called the *rank* of g.

Axiom 3. *For any* $g_1, g_2 \in G$ *the following conditions are true:*
(a) $\alpha e = \beta e = \beta f = 1,\ \alpha f = 2,$
(b) $\alpha(g_1 \star g_2) = \max\{\alpha g_1, \alpha g_2\}, \quad \beta(g_1 \star g_2) = \beta g_1 + \beta g_2,$
(c) $\alpha(g_1 \circ g_2) = \max\{\alpha g_1 + \gamma g_2, \alpha g_2\}, \quad \beta(g_1 \circ g_2) = \max\{\beta g_1, \beta g_2 - \gamma g_1\},$
where $\gamma g = \alpha g - \beta g$.

Axiom 4. *For all* $g_1, g_2 \in G$ *such that* $\alpha g_1 = \alpha g_2$ *and* $\beta g_1 = \beta g_2 = 1$ *we have* $f \circ (g_1 \star g_2) = g_2$.

Axiom 5. *For all* $g_1, g_2, g_3 \in G$
(a) $g_1 \circ (g_2 \star g_3) = (g_1 \circ g_2) \star g_3,$ *if* $\alpha g_1 \leqslant \beta g_2,$
(b) $(g_1 \star g_2) \circ g_3 = (g_1 \circ g_3) \star (g_2 \circ g_3),$ *if* $\beta g_3 \leqslant \min\{\alpha g_1, \alpha g_2\}$.

Axiom 6. *For all* $g_1, g_2, g_3, g_4 \in G$
(a) $(g_1 \star g_2) \circ (g_3 \star g_4) = (g_1 \circ g_3) \star (g_2 \circ (g_3 \star g_4)),$ *if* $\alpha g_1 < \alpha g_2,$ $\alpha g_1 = \beta g_3,\quad \alpha g_2 = \beta(g_3 \star g_4),$
(b) $(g_1 \star g_2) \circ (g_3 \star g_4) = (g_1 \circ (g_3 \star g_4)) \star (g_2 \circ g_3),$ *if* $\alpha g_1 > \alpha g_2,$ $\alpha g_2 = \beta g_3,\quad \alpha g_1 = \beta(g_3 \star g_4).$

Any abstract algebra $(G, \circ, \star, e, f)$ of type $(2, 2, 0, 0)$ satisfying the above six axioms is called a *v-algebra*.

Proposition 6.7.3. *In any v-algebra* $(G, \circ, \star, e, f)$ *the following identities take place:*
(a) $\gamma(g_1 \circ g_2) = \gamma g_1 + \gamma g_2,$
(b) $\gamma(g_1 \star g_2) = \gamma g_1 + \gamma g_2 - \min\{\alpha g_1, \alpha g_2\}.$

Proposition 6.7.4. *For each* $n \in \mathbb{N}$ *and all* $i = 1, \ldots, n$ *we have* $\alpha e_i^n = n,$ $\beta e_i^n = 1$.

Proof. Indeed, let $g \in G$ be such that $\beta g = 1$. Then, by Axiom 3(c) we have $\alpha g^n = n\alpha g - n + 1$ and $\beta g^n = 1$. Further

$$\alpha e_i^n = \alpha((f \circ (f \star e))^{n-i} \circ f^{i-1}) = \max\{\alpha((f \circ (f \star e))^{n-i}) + \gamma f^{i-1},\ \alpha f^{i-1}\}.$$

But $\alpha(f \circ (f \star e)) = 2$ and $\beta(f \circ (f \star e)) = 1$, according to Axiom 3. Thus $\alpha((f \circ (f \star e))^{n-i}) = n - i + 1,\ \beta((f \circ (f \star e))^{n-i}) = 1,\ \alpha f^{i-1} = i,\ \beta f^{i-1} = 1.$ Hence $\alpha e_i^n = \max\{n, i\} = n$.

Similarly we prove $\beta e_i^n = 1$. □

From the above proposition it follows that the equation

$$e_i^n \circ (e_1^n \star \cdots \star e_n^n) = e_i^n \tag{6.7.3}$$

is true for all $n \in \mathbb{N}$ and $i = 1, \ldots, n$.

Proposition 6.7.5. *For all $g_1, \ldots, g_n \in G$ such that $\alpha g_1 = \ldots = \alpha g_n$ and $\beta g_1 = \ldots = \beta g_n = 1$, we have*

$$e_i^n \circ (g_1 \star \cdots \star g_n) = g_i\,, \tag{6.7.4}$$

where $n \in \mathbb{N}$ and $i = 1, \ldots, n$ are arbitrary.

Proof. First let $n = 2$. If $i = 2$, then, according to Axiom 4, we have

$$e_2^2 \circ (g_1 \star g_2) = f \circ (g_1 \star g_2) = g_2\,.$$

If $i = 1$, then $e_1^2 \circ (g_1 \star g_2) = f \circ (f \star e) \circ (g_1 \star g_2)$. Hence, by Axioms 6(b) and 4, we obtain

$$e_1^2 \circ (g_1 \star g_2) = f \circ ((f \circ (g_1 \star g_2)) \star (e \circ g_1)) = f \circ (g_2 \star g_1) = g_1.$$

Now let $n > 2$, $i = 1, \ldots, n$. Then

$$\begin{aligned} e_i^n \circ (g_1 \star \cdots \star g_n) &= (e_1^2)^{n-i} \circ f^{i-1} \circ (g_1 \star \cdots \star g_n) \\ &= (e_1^2)^{n-i} \circ f^{i-2} \circ ((f \circ (g_1 \star g_2)) \star g_3 \star \cdots \star g_n) \\ &= (e_1^2)^{n-i} \circ f^{i-2} \circ (g_2 \star \cdots \star g_n). \end{aligned}$$

Repeating this procedure we obtain

$$\begin{aligned} e_i^n \circ (g_1 \star \cdots \star g_n) &= (e_1^2)^{n-i} \circ (g_i \star \cdots \star g_n) \\ &= (e_1^2)^{n-i-1} \circ ((e_1^2 \circ (g_i \star g_{i+1})) \star g_{i+2} \star \cdots \star g_n) \\ &= (e_1^2)^{n-i-1} \circ (g_i \star g_{i+2} \star \cdots \star g_n) = \ldots \\ &= e_1^2 \circ (g_i \star g_n) = g_i. \end{aligned}$$

This completes the proof. □

Proposition 6.7.6. *If $n = \beta x_1 + \cdots + \beta x_k$ and $m = \max\{\alpha x_1, \ldots, \alpha x_k\}$ for some $x_1, \ldots, x_k \in G$, then*

$$e_i^n \circ (x_1 \star \cdots \star x_k) = e_s^{\beta x_p} \circ x_p \circ (e_1^m \star \cdots \star e_{\alpha x_p}^m) \tag{6.7.5}$$

for all $i = 1, \ldots, n$, where $\sum\limits_{j=1}^{p-1} \beta x_j < i \leqslant \sum\limits_{j=1}^{p} \beta x_j$ and $s = i - \sum\limits_{j=1}^{p-1} \beta x_j$.

Proof. Let $n_i = \beta x_i$ for all $x_i \in G$, $i = 1,\dots,k$. By Axiom 3(b) we have $\alpha(x_1 \star \cdots \star x_k) = \max\{\alpha x_1,\dots,\alpha x_k\} = m$. Applying Axiom 2 we obtain

$$e_i^n \circ (x_1 \star \cdots \star x_k) =$$
$$e_i^n \circ (((e_1^{n_1} \star \cdots \star e_{n_1}^{n_1}) \circ x_1) \star \cdots \star ((e_1^{n_k} \star \cdots \star e_{n_k}^{n_k}) \circ x_k)) \circ (e_1^n \star \cdots \star e_m^m).$$

Further, by Axiom 5(b),

$$e_i^n \circ (x_1 \star \cdots \star x_k) = e_i^n \circ ((e_1^{n_1} \circ x_1) \star \cdots \star (e_{n_1}^{n_1} \circ x_1) \star \cdots \star (e_1^{n_k} \circ x_k) \star \cdots \star (e_{n_k}^{n_k} \circ x_k)) \circ (e_1^m \star \cdots \star e_m^m).$$

This, together with Axiom 6 and Proposition 6.7.5, implies

$$e_i^n \circ (x_1 \star \cdots \star x_k) = e_i^n \circ ((e_1^{n_1} \circ x_1 \circ (e_1^m \star \cdots \star e_{\alpha x_1}^m)) \star \cdots \star (e_{n_k}^{n_k} \circ x_1 \circ (e_1^m \star \cdots \star e_{\alpha x_k}^m))) = e_s^{\beta x_p} \circ x_p \circ (e_1^m \star \cdots \star e_{\alpha x_p}^m),$$

which completes the proof. □

Theorem 6.7.7. *An algebra $(G, \circ, \star, e, f)$ of type $(2,2,0,0)$ is isomorphic to some algebra of vector-valued functions if and only if it is a v-algebra.*

Proof. It is clear that any algebra $(G, \circ, \star, e, f)$ of type $(2,2,0,0)$, which is isomorphic to some algebra of vector-valued functions, satisfies the above six axioms. To prove the converse statement, assume that some abstract algebra $(G, \circ, \star, e, f)$ of type $(2,2,0,0)$ satisfies these six axioms. Let G_n be the set of all elements $g \in G$ such that $\alpha g = n$ and $\beta g = 1$. The set G_n is nonempty because $e_i^n \in G_n$ for every $i = 1,\dots,n$. Clearly $G_n \cap G_m = \varnothing$ for $n \neq m$. Let $\overline{G} = \bigtimes_{n\in\mathbb{N}} G_n$ be the Cartesian product of the family $(G_n)_{n\in\mathbb{N}}$.

For every $g \in G$, we define the vector-valued function $P_g : \overline{G}^{\,n} \to \overline{G}^{\,m}$, where $n = \alpha g$, $m = \beta g$, putting $P_g(\bar{x}_1,\dots,\bar{x}_n) = \bar{y}_1 \dots \bar{y}_m$ if and only if

$$\bar{y}_i(k) = e_i^m \circ g \circ (\bar{x}_1(k) \star \cdots \star \bar{x}_n(k)) \tag{6.7.6}$$

for all $i = 1,\dots,m$ and $k = 1,2,\dots$

Let us show that the mapping $P\colon g \mapsto P_g$ is an isomorphism between $(G, \circ, \star, e, f)$ and $(\Phi, \circ, \star, \Delta, F)$, where $\Phi = \{P_g \mid g \in G\}$.

First, observe that $P(e) = \Delta$ and $P(f) = F$. Indeed, if $P_e(\bar{x}) = \bar{y}$ for some $\bar{x}, \bar{y} \in \overline{G}$, then $\bar{y}(k) = e_1^1 \circ e \circ \bar{x}(k) = \bar{x}(k)$ for all $k = 1,2,\dots$, because $e_1^1 = e$ is the unit of $(G, \circ)$. Thus $\bar{y}(k) = \bar{x}(k)$, $k = 1,2,\dots$ So, $P_e(\bar{x}) = \bar{x}$. Hence $P_e = \Delta$. Similarly, from Axiom 4, we deduce $P_f = F$.

To prove $P(g_1 \circ g_2) = P(g_1) \circ P(g_2)$, i.e.,

$$P_{g_1 \circ g_2} = P_{g_1} \circ P_{g_2}, \tag{6.7.7}$$

let $n_i = \alpha g_i$, $m_i = \beta g_i$, $i = 1, 2$, $n = \max\{n_1 + \gamma g_2,\ n_2\}$ and $m = \max\{m_1,\ m_2 - \gamma g_1\}$. Since $n = \alpha(g_1 \circ g_2)$, $m = \beta(g_1 \circ g_2)$, by Axiom 3(c), the degree and the rank of the function $P_{g_1 \circ g_2}$ are equal n and m, respectively. Let

$$\bar{y}_1 \dots \bar{y}_m = P_{g_1 \circ g_2}(\bar{x}_1, \dots, \bar{x}_n)$$

for some $\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_m \in \overline{G}$. If $n_1 > m_2$, then $m = m_1$. Therefore, by (6.7.6), we have

$$\bar{y}_i(k) = e_i^m \circ g_1 \circ g_2 \circ (\bar{x}_1(k) \star \dots \star \bar{x}_n(k))$$

for all $i = 1, \dots, m$ and $k = 1, 2, \dots$ In view of the fact that

$$n_2 = \beta(\bar{x}_1(k) \star \dots \star \bar{x}_{n_2}(k))$$

and the Axiom 5(a) any $\bar{y}_i(k)$ can be written in the form

$$\bar{y}_i(k) = e_i^{m_1} \circ g_1 \circ ((g_2 \circ (\bar{x}_1(k) \star \dots \star \bar{x}_{n_2}(k))) \star \bar{x}_{n_2+1}(k) \star \dots \star \bar{x}_n(k)).$$

From this, by applying Axioms 2 and 6, we obtain

$$\begin{aligned} \bar{y}_i(k) &= e_i^{m_1} \circ g_1 \circ (((e_1^{m_2} \star \dots \star e_{m_2}^{m_2}) \circ g_2 \circ (\bar{x}_1(k) \star \dots \\ &\qquad\qquad \star \bar{x}_{n_2}(k))) \star \bar{x}_{n_2+1}(k) \star \dots \star \bar{x}_n(k)) \\ &= e_i^{m_1} \circ g_1 \circ ((e_1^{m_2} \circ g_2 \circ (\bar{x}_1(k) \star \dots \star \bar{x}_{n_2}(k))) \star \dots \\ &\quad \star (e_{m_2}^{m_2} \circ g_2 \circ (\bar{x}_1(k) \star \dots \star \bar{x}_{n_2}(k))) \star \bar{x}_{n_2+1}(k) \star \dots \star \bar{x}_n(k)). \end{aligned}$$

Let $\bar{z}_1 \dots \bar{z}_{m_2} = P_{g_2}(\bar{x}_1, \dots, \bar{x}_{n_2})$, i.e.,

$$\bar{z}_i(k) = e_i^{m_2} \circ g_2 \circ (\bar{x}_1(k) \star \dots \star \bar{x}_{n_2}(k))$$

for all $i = 1, \dots, m_2$ and $k = 1, 2, \dots$ Then

$$\bar{y}_i(k) = e_i^{m_1} \circ g_1 \circ (\bar{z}_1(k) \star \dots \star \bar{z}_{m_2}(k) \star \bar{x}_{n_2+1}(k) \star \dots \star \bar{x}_n(k))$$

for all $i = 1, \dots, m_1$ and $k = 1, 2, \dots$ Thus

$$\bar{y}_1 \dots \bar{y}_{m_1} = P_{g_1}(\bar{z}_1, \dots, \bar{z}_{m_2}, \bar{x}_{n_2+1}, \dots, \bar{x}_n).$$

Therefore

$$\bar{y}_1 \dots \bar{y}_{m_1} = P_{g_1}(P_{g_2}(\bar{x}_1, \dots, \bar{x}_{n_2}), \bar{x}_{n_2+1}, \dots, \bar{x}_n),$$

i.e., $\bar{y}_1 \dots \bar{y}_m = (P_{g_1} \circ P_{g_2})(\bar{x}_1, \dots, \bar{x}_n)$. This means that for $n_1 > m_2$, $m = m_1$, condition (6.7.7) is true.

In the case $n_1 \leqslant m_2$, we have $n = n_2$ and $m = m_2 - \gamma g_1$. Hence

$$\begin{aligned}\bar{y}_i(k) &= e_i^m \circ g_1 \circ g_2 \circ (\bar{x}_1(k) \star \cdots \star \bar{x}_n(k)) \\ &= e_i^m \circ g_1 \circ (e_1^{m_2} \star \cdots \star e_{m_2}^{m_2}) \circ g_2 \circ (\bar{x}_1(k) \star \cdots \star \bar{x}_{n_2}(k)) \\ &= e_i^m \circ g_1 \circ ((e_1^{m_2} \circ g_2 \circ (\bar{x}_1(k) \star \cdots \star \bar{x}_{n_2}(k))) \star \cdots \\ &\qquad \star (e_{m_2}^{m_2} \circ g_2 \circ (\bar{x}_1(k) \star \cdots \star \bar{x}_{n_2}(k)))) \\ &= e_i^m \circ g_1 \circ (\bar{z}_1(k) \star \cdots \star \bar{z}_{m_2}(k))\end{aligned}$$

for all $i = 1, \ldots, m,\ k = 1, 2, \ldots$ Whence, applying Axiom 5(a), we obtain

$$\bar{y}_i(k) = e_i^m \circ ((g_1 \circ (\bar{z}_1(k) \star \cdots \star \bar{z}_{n_1}(k))) \star \bar{z}_{n_1+1}(k) \star \cdots \star \bar{z}_{m_2}(k)).$$

This, according to Proposition 6.7.6, for $1 \leqslant i \leqslant m_1$ and $k \geqslant 1$, implies

$$\bar{y}_i(k) = e_i^{m_1} \circ g_1 \circ (\bar{z}_1(k) \star \cdots \star \bar{z}_{n_1}(k)).$$

Therefore

$$\bar{y}_1 \ldots \bar{y}_{m_1} = P_{g_1}(\bar{z}_1, \ldots, \bar{z}_{n_1}).$$

For $m_1 < i \leqslant m$ we have $\bar{y}_i(k) = \bar{z}_{i+\gamma g_1}(k)$, where $k = 1, 2, \ldots$ Whence $\bar{y}_{m_1+1} \ldots \bar{y}_m = \bar{z}_{n_1+1} \ldots \bar{z}_{m_2}$. So,

$$\bar{y}_1 \ldots \bar{y}_m = P_{g_1}(\bar{z}_1, \ldots, \bar{z}_{n_1})\bar{z}_{n_1+1} \ldots \bar{z}_{m_2}\,,$$

which, by Definition 6.7.1, gives $\bar{y}_1 \ldots \bar{y}_m = (P_{g_1} \circ P_{g_2})(\bar{x}_1, \ldots, \bar{x}_n)$. Thus

$$P_{g_1 \circ g_2}(\bar{x}_1, \ldots, \bar{x}_n) = (P_{g_1} \circ P_{g_2})(\bar{x}_1, \ldots, \bar{x}_n)$$

for all $\bar{x}_1, \ldots, \bar{x}_n \in \overline{G}$. This completes the proof of (6.7.7).

To prove $P(g_1 \star g_2) = P(g_1) \star P(g_2)$, i.e.,

$$P_{g_1 \star g_2} = P_{g_1} \star P_{g_2}, \tag{6.7.8}$$

assume that $n_i = \alpha g_i$, $m_i = \beta g_i$ for $i = 1, 2$, and $n = \max\{n_1, n_2\}$, $m = m_1 + m_2$. By Axiom 3(b), we have $n = \alpha(g_1 \star g_2) = \alpha(P_{g_1 \star g_2})$, $m = \beta(g_1 \star g_2) = \beta(P_{g_1 \star g_2})$. Let

$$\bar{y}_1 \ldots \bar{y}_m = P_{g_1 \star g_2}(\bar{x}_1, \ldots, \bar{x}_n)$$

for some $\bar{x}_1, \ldots, \bar{x}_n, \bar{y}_1, \ldots, \bar{y}_m \in \overline{G}$. Then, according to (6.7.6),

$$\bar{y}_i(k) = e_i^m \circ (g_1 \star g_2) \circ (\bar{x}_1(k) \star \cdots \star \bar{x}_n(k))$$

for all $i = 1, \ldots, m$ and $k = 1, 2, \ldots$

If $n_1 \leqslant n_2$, then $n = n_2$ and the last equation, by Axiom 6, can be written in the form

$$\bar{y}_i(k) = e_i^m \circ ((g_1 \circ (\bar{x}_1(k) \star \cdots \star \bar{x}_{n_1}(k))) \star (g_2 \circ (\bar{x}_1(k) \star \cdots \star \bar{x}_n(k)))),$$

whence, for $1 \leqslant i \leqslant m_1$, by Proposition 6.7.6, we get

$$\bar{y}_i(k) = e_i^{m_1} \circ g_1 \circ (\bar{x}_1(k), \ldots, \bar{x}_{n_1}(k)), \quad k = 1, 2, \ldots$$

Thus, $\bar{y}_1 \ldots \bar{y}_{m_1} = P_{g_1}(\bar{x}_1, \ldots, \bar{x}_{n_1})$.

In the same manner, for $m_1 < i \leqslant m$, we obtain

$$\bar{y}_i(k) = e_{i-m_1}^{m_2} \circ g_2 \circ (\bar{x}_1(k) \star \cdots \star \bar{x}_n(k)), \quad k = 1, 2, \ldots$$

and $\bar{y}_{m_1+1} \ldots \bar{y}_m = P_{g_2}(\bar{x}_1, \ldots, \bar{x}_n)$. So, $\bar{y}_1 \ldots \bar{y}_m = (P_{g_1} \star P_{g_2})(\bar{x}_1, \ldots, \bar{x}_n)$. Hence

$$P_{g_1 \star g_2}(\bar{x}_1, \ldots, \bar{x}_n) = (P_{g_1} \star P_{g_2})(\bar{x}_1, \ldots, \bar{x}_n)$$

for all $\bar{x}_1, \ldots, \bar{x}_n \in \overline{G}$. This means that in the case $n_1 \leqslant n_2$ the condition (6.7.8) is true. For $n_2 \leqslant n_1$ the proof is similar.

We have so proved that $P : g \mapsto P_g$ is a homomorphism of a v-algebra $(G, \circ, \star, e, f)$ onto $(\Phi, \circ, \star, \Delta, F)$, where $\Phi = \{P_g \,|\, g \in G\}$.

To prove that it is one-to-one let $P_{g_1} = P_{g_2}$ for some $g_1, g_2 \in G$. Then $\alpha g_1 = \alpha g_2$, $\beta g_1 = \beta g_2$. Thus

$$e_i^m \circ g_1 \circ (\bar{x}_1(k) \star \cdots \star \bar{x}_n(k)) = e_i^m \circ g_2 \circ (\bar{x}_1(k) \star \cdots \star \bar{x}_n(k))$$

for all $1 \leqslant i \leqslant m = \beta g_1$, $\bar{x}_1, \ldots, \bar{x}_n \in \overline{G}$, where $n = \alpha g_1$ and $k = 1, 2, \ldots$

This, for $\bar{x}_i = \bar{e}_i = (e_1^1, e_2^2, \ldots, e_i^i, e_i^{i+1}, e_i^{i+2}, \ldots) \in \overline{G}$, $i = 1, \ldots, n$ and $k = n$, gives us

$$e_i^m \circ g_1 \circ (e_1^n \star \cdots \star e_n^n) = e_i^m \circ g_2 \circ (e_1^n \star \cdots \star e_n^n)\,.$$

Thus $e_i^m \circ g_1 = e_i^m \circ g_2$ for all $i = 1, \ldots, m$. Consequently,

$$(e_1^m \circ g_1) \star \cdots \star (e_m^m \circ g_1) = (e_1^m \circ g_2) \star \cdots \star (e_m^m \circ g_2)\,,$$

i.e., $(e_1^m \star \cdots \star e_m^m) \circ g_1 = (e_1^m \star \cdots \star e_m^m) \circ g_2$. This implies $g_1 = g_2$. □

Definition 6.7.8. A homomorphism P of a v-algebra $(G, \circ, \star, e, f)$ into a v-algebra $(G', \circ', \star', e', f')$ is called a *v-homomorphism*, if for all $g \in G$ and $n \in \mathbb{N}$ from $g \neq g \circ (e_1^n \star \cdots \star e_n^n)$ it follows $P(g) \neq P(g \circ (e_1^n \star \cdots \star e_n^n))$.

It is easy to see that a v-homomorphism P keeps the degree and the rank, i.e., $\alpha g = \alpha P(g)$ and $\beta g = \beta P(g)$ for any $g \in G$. The converse is also true: each homomorphism of v-algebras which keeps the degree and the rank of all elements is a v-homomorphism.

Definition 6.7.9. A nonempty subset H of a v-algebra $(G, \circ, \star, e, f)$ is called a *v-ideal*, if $e_i^m \circ x \circ (h_1 \star \cdots \star h_n) \in H$ for all $x \in G$ and $h_i \in H$, where $i = 1, \ldots, n = \alpha x$, $m = \beta x$.

Generalizing the concept of dense v-subsystems (p. 336), we say that a v-ideal H of a v-algebra $(G, \circ, \star, e, f)$ is *dense* in this v-algebra, if

(a) any v-homomorphism from $(G, \circ, \star, e, f)$ onto other v-algebra which is not an isomorphism, induces on H a homomorphism which is not an isomorphism,

(b) if a v-algebra $(G', \circ', \star', e', f')$ contains a v-algebra $(G, \circ, \star, e, f)$ as a subalgebra, and H is a v-ideal of $(G', \circ', \star', e', f')$, then there exists a v-homomorphism from $(G', \circ', \star', e', f')$ onto some v-algebra which is not an isomorphism, but on H induces an isomorphism.

Theorem 6.7.10. *The set C_A of all unary operations $\varphi_a \in \mathcal{T}(A, A)$, $a \in A$, such that $\varphi_a(x) = a$ is a dense v-ideal of the algebra $(\mathcal{T}(A), \circ, \star, \Delta, F)$.*

Proof. Let $\psi \in \mathcal{T}(A)$, $\varphi_{a_1}, \ldots, \varphi_{a_n} \in C_A$, where $n = \alpha\psi$ and $a_1, \ldots, a_n \in A$. Suppose that

$$\psi(a_1, \ldots, a_n) = b_1 \ldots b_m$$

for some $b_1, \ldots, b_m \in A$, $m = \beta\psi$. Since $I_i^m(b_1, \ldots, b_m) = b_i$, we have also $(I_i^m \circ \psi)(a_1, \ldots, a_n) = b_i$. Thus

$$(I_i^m \circ \psi)(\varphi_{a_1}(c), \ldots, \varphi_{a_n}(c)) = \varphi_{b_i}(c),$$

for all $c \in A$, i.e., $(I_i^m \circ \psi \circ (\varphi_{a_1} \star \cdots \star \varphi_{a_n}))(c) = \varphi_{b_i}(c)$. So,

$$I_i^m \circ \psi \circ (\varphi_{a_1} \star \cdots \star \varphi_{a_n}) = \varphi_{b_i} \in C_A.$$

Therefore C_A is a v-ideal of $(\mathcal{T}(A), \circ, \star, \Delta, F)$.

Now let P be a v-homomorphism of $(\mathcal{T}(A), \circ, \star, \Delta, F)$ onto another v-algebra which is not an isomorphism. Hence, there are $\psi_1, \psi_2 \in \mathcal{T}(A)$ such that $\psi_1 \neq \psi_2$ and $P(\psi_1) = P(\psi_2)$. The last equation implies $\alpha P(\psi_1) = \alpha P(\psi_2)$ and $\beta P(\psi_1) = \beta P(\psi_2)$. Hence $\alpha\psi_1 = \alpha\psi_2 = n$, $\beta\psi_1 = \beta\psi_2 = m$. So, there are $a_1, \ldots, a_n \in A$ for which

$$\psi_1(a_1, \ldots, a_n) \neq \psi_2(a_1, \ldots, a_n).$$

Let $\psi_1(a_1, \ldots, a_n) = b_1 \ldots b_m$ and $\psi_2(a_1, \ldots, a_n) = c_1 \ldots c_m$. Then $b_i \neq c_i$, consequently $\varphi_{b_i} \neq \varphi_{c_i}$, for some $i = 1 \ldots, m$. But

$$\begin{aligned} P(\varphi_{b_i}) &= P(I_i^m \circ \psi_1 \circ (\varphi_{a_1} \star \cdots \star \varphi_{a_n})) \\ &= P(I_i^m) \circ P(\psi_1) \circ (P(\varphi_{a_1}) \star \cdots \star P(\varphi_{a_n})) \\ &= P(I_i^m) \circ P(\psi_2) \circ (P(\varphi_{a_1}) \star \cdots \star P(\varphi_{a_n})) \\ &= P(I_i^m \circ \psi_2 \circ (\varphi_{a_1} \star \cdots \star \varphi_{a_n})) = P(\varphi_{c_i}). \end{aligned}$$

This proves that P restricted to C_A is a v-homomorphism which is not an isomorphism.

Assume now that a v-algebra $(G, \circ, \star, \Delta, F)$ contains a proper subalgebra $(\mathcal{T}(A), \circ, \star, \Delta, F)$ and a v-ideal C_A. For each element $g \in G$ we define $\lambda_g \in \mathcal{T}(A)$ putting:

$$b_1 \ldots b_m = \lambda_g(a_1, \ldots, a_n) \longleftrightarrow \bigwedge_{i=1}^{m} I_i^m \circ g \circ (\varphi_{a_1} \star \cdots \star \varphi_{a_n}) = \varphi_{b_i}\,,$$

where $n = \alpha g,\ m = \beta g,\ a_1, \ldots, a_n, b_1, \ldots, b_m \in A$. It is not difficult to see that the mapping $P : g \mapsto \lambda_g$ is a v-homomorphism of $(G, \circ, \star, \Delta, F)$ onto $(\mathcal{T}(A), \circ, \star, \Delta, F)$. Since $\mathcal{T}(A) \subset G$ and $\mathcal{T}(A) \neq G$, for $g \in G \setminus \mathcal{T}(A)$ we have $g \neq P(g) = \lambda_g$. But directly from the definition of λ_g we obtain $P(\lambda_g) = \lambda_g$. So, $P(g) = P(\lambda_g)$. This means that P is not an isomorphism but P restricted to C_A is an isomorphism. □

Theorem 6.7.11. *A v-algebra $(G, \circ, \star, e, f)$ is isomorphic to some symmetrical algebra of vector-valued functions if and only if it contains a dense v-ideal H such that $(H, \circ)$ is a left zero semigroup and $\alpha h = \beta h = 1$ for every $h \in H$.*

Proof. The necessity of these conditions follows from Theorem 6.7.10. To prove their sufficiency we consider the mapping $P : G \to \mathcal{T}(H)$ defined by the formula

$$y_1 \ldots y_m = P(g)(x_1, \ldots, x_n) \longleftrightarrow \bigwedge_{i=1}^{m} y_i = e_i^m \circ g \circ (x_1 \star \cdots \star x_n),$$

where $g \in G,\ x_1, \ldots, x_n, y_1, \ldots, y_m \in H,\ n = \alpha g,\ m = \beta g$. It is not difficult to verify that $P(e) = \Delta$, $P(f) = F$ and P is a v-homomorphism which induces on H an isomorphism. But H is a dense v-ideal of $(G, \circ, \star, e, f)$, therefore, according to the condition (a) of the definition of a dense v-ideal, P must be an isomorphism. Hence, a v-ideal C_H is a dense v-ideal of a homomorphic image of $(G, \circ, \star, e, f)$, i.e., $(P(G), \circ, \star, \Delta, F)$, because $(C_H, \circ)$ is isomorphic to $(H, \circ)$. But, by Theorem 6.7.10, a v-ideal C_H is dense in $(\mathcal{T}(H), \circ, \star, \Delta, F)$, therefore $P(G) \subset \mathcal{T}(H)$ implies $P(G) = \mathcal{T}(H)$. This proves that $(G, \circ, \star, e, f)$ and $(\mathcal{T}(H), \circ, \star, \Delta, F)$ are isomorphic. □

6.8 Notes on Chapter 6

Menger systems were introduced by H. Whitlock [258] in order to investigate compositions of multiplace functions of various arities. Afterwards, in [74] Ja. V. Hion observed that the set of all multiplace endomorphisms of a universal algebra forms a Menger system. Based on this observation, E. Redi [147] proved that a Menger algebra is isomorphic to the system of all multiplace endomorphisms of a universal algebra if and only if it has a full system of selectors.

By defining on Menger systems some additional operations Ja. V. Hion [74] constructed a new type of algebras which are called m-ary $\mathbf{\Omega}$-ringoids or $\mathbf{\Omega}$-systems. Later on, these algebras were used to analyze Menger systems. For example, using these algebras J. Henno described free $\mathbf{\Omega}$-ringoids [63, 64], free Γ-commutative Menger systems [65], Green relations [57] and densely embedded ideals [59–62]. Group-like Menger systems are described in [58]. Results obtained by J. Henno are applied by E. Redi [148–151] to the study of polycategory and polyringoids of multiplace functions.

Menger T-systems were introduced in [202] but the first abstract characterization of ordered Menger T-systems of multiplace functions was given in [257].

Positional algebras were introduced by V. D. Belousov [6] for the study of functional equations. For this reason, positional algebras isomorphic to algebras of prequasigroup operations [201] are called Belousov algebras. Properties of such algebras are described in [201]; representations of positional algebras by multiplace functions in [242].

The study of iterative and pre-iterative Mal'cev-Post algebras was initiated in [106] (see also the books [107] and [108]). The system of axioms characterizing abstract pre-iterative Mal'cev-Post algebras was presented (without proof) by I. G. Rosenberg [153]. Elementary abstract characterizations of central iterative and pre-iterative Mal'cev-Post algebras were found by V. S. Trokhimenko in his papers [234] and [236]. In [234] these algebras are characterized by dense subsystems (see § 6.4).

Central semigroups of multiplace functions were characterized in [231]. Other results of § 6.5 have not been published in any journal yet.

Algebras of vector-valued functions were introduced by B. Schweizer and A. Sklar [191] following some problems in logic (see also [192] and [193]). The paper [191] contains results on properties of serial and parallel compositions and indications of possible applications of these results. Some applications in logic can be found in [198]. A connection between vector-valued semigroups, (m, n)-formal languages and semigroup automatons are described in [22] and [23]. Binary relations defined on the set of vector-valued operations were characterized in [158]; (i, j)-invertible vector-valued operations are described in [159]. Some other results can be found in [241].

Chapter 7

Open problems

7.1 Closure operations

An n-place function $f\colon \overset{n}{\times} \mathfrak{P}(A) \to \mathfrak{P}(A)$ is an *n-place $\cap$-transformation* of $\mathfrak{P}(A)$, if

$$f(H_1^{i-1}, X \cap Y, H_{i+1}^n) = f(H_1^{i-1}, X, H_{i+1}^n) \cap f(H_1^{i-1}, Y, H_{i+1}^n)$$

for all $i = 1, \ldots, n$, $X, Y, H_1, \ldots, H_n \in \mathfrak{P}(A)$, where H_i^j denotes the sequence $H_i, \ldots, H_j$ for $i \leqslant j$ and the empty symbol for $i > j$. Analogously is defined an n-place $\cup$-transformation.

7.1.1. Find an abstract characterization of n-place closure operations of subsets of A that are $\cap$-transformations ($\cup$-transformations).

7.1.2. Find a representation of Menger algebras by n-place closure operations of subsets of A that are $\cap$-transformations ($\cup$-transformations).

7.2 Menger algebras of functions

Let $(A, \prec)$ be an ordered set. Following V. V. Vagner [246], an n-place function $f\colon A^n \to A$ is called an *n-ary semiopening operation*, if

(a) $f(a_1, \ldots, a_n) \prec a_i$,

(b) $x \prec y \longrightarrow f(\overline{a}|_i x) \prec f(\overline{a}|_i y)$

holds for all $i = 1, \ldots, n$ and $a_1, \ldots, a_n, x, y \in A$. A semiopening operation f with the property $f[f \ldots f] = f$ is called an *opening operation*.

7.2.1. Describe a Menger algebra of n-place semiopening operations and a Menger algebra of n-place opening operations.

7.2.2. Describe Menger algebras of extensive n-place functions.

7.2.3. Find a characterization of Menger algebras of extensive n-place functions of linearly ordered sets.

7.2.4. Describe Menger algebras of n-place closure (opening) operations.

7.2.5. Find the necessary and sufficient conditions under which a semigroup will be isomorphic to the diagonal semigroup of some Menger algebra of rank $n \geqslant 2$.

7.2.6. Characterize Menger algebras of n-place functions generated by idempotent functions.

7.2.7. Describe Menger algebras of rank $n \geqslant 2$ generated by one element.

7.2.8. Describe i-solvable Dicker algebras (see p. 82).

7.3 Menger algebras of relations

7.3.1. Find an abstract characterization of Menger algebras of $(n+1)$-ary relations of the form

$$\rho = \mathop{\times}_{i=1}^{n} A_i \times B,$$

where $A_1, \ldots, A_n, B$ are subsets of the same set.

7.3.2. Find a characterization of a Menger algebra $(\Phi, O, \overset{n}{\chi}_\Phi)$ of reversive n-place functions.

7.3.3. Find an abstract characterization of Menger algebras $(\Phi, O, \overset{n}{\omega}_\Phi)$ and $(\Phi, O, \overset{n}{\omega}_\Phi, \overset{n}{\chi}_\Phi)$ of n-place (reversive) functions, where

$$(f_1, \ldots, f_n, g) \in \overset{n}{\omega}_\Phi \longleftrightarrow f_1 \cap f_2 \cap \cdots \cap f_n \subset g,$$

for $f_1, \ldots, f_n, g \in \Phi$. The same for Menger algebras of $(n+1)$-ary relations.

7.3.4. Characterize the relation $\overset{n}{\delta}_\Phi$ on a $\cap$-Menger algebra of n-place partial functions.

7.3.5. Find an abstract characterization of Menger algebras $(\Phi, O, \cup, \chi_\Phi)$, $(\Phi, O, \cup, \rhd)$, $(\Phi, O, \cap, \cup, \chi_\Phi)$, $(\Phi, O, \cap, \cup, \rhd)$ of (reversive) n-place functions.

7.3.6. Describe the subsets of transitive and reflexive $(n+1)$-ary relations in the class of Menger algebras of $(n+1)$-ary relations.

7.3.7. Find an abstract characterization of subtraction Menger algebras $(\Phi, O, \setminus, \rho)$ and $(\Phi, O, \setminus, \varnothing, \rho)$ of (reversive) n-place functions, where $\rho \in \{\rhd, \chi_\Phi, \xi_\Phi, \pi_\Phi, \gamma_\Phi, \varkappa_\Phi\}$.

7.4 $(2, n)$-semigroups

7.4.1. Find the necessary and sufficient conditions under which the relation γ will be faithful projection representable for a representable $(2, n)$-semigroup. The same for Menger $(2, n)$-semigroups.

7.4.2. Find the necessary and sufficient conditions under which the pairs (χ, γ) and (γ, π) of binary relations will be faithful projection representable for a representable $(2, n)$-semigroup. The same for Menger $(2, n)$-semigroups.

7.4.3. Find the necessary and sufficient conditions under which the triplet (χ, γ, π) of binary relations will be faithful projection representable for a representable $(2, n)$-semigroup. The same for Menger $(2, n)$-semigroups.

7.4.4. Find an abstract characterization of $(2, n)$-semigroups of all partial (full) n-place functions defined on a fixed set. The same for Menger $(2, n)$-semigroups.

7.4.5. Find an abstract characterization of $(2, n)$-semigroups of partial functions closed with respect to one or more of the relations: $\cap$, $\cup$, χ_Φ, $\triangleright$. The same for Menger $(2, n)$-semigroups.

7.4.6. Describe automorphisms and congruences of (Menger) $(2, n)$-semigroups.

7.5 Systems of multiplace functions

7.5.1. Characterize positional algebras $(\Phi, \overset{1}{+}, \overset{2}{+}, \cdots, \overset{n}{+}, \cdots, \zeta_\Phi, \chi_\Phi)$ of multiplace relations with ζ_Φ, χ_Φ defined for $f \subset A^n \times A$ and $g \subset A^m \times A$ in the following way:

$$(f, g) \in \zeta_\Phi \longleftrightarrow \Big(n \geqslant m\Big) \wedge (\forall a_1^n)\Big(f\langle a_1^n\rangle \neq \varnothing \longrightarrow f(a_1^n) = g(a_1^m)\Big),$$

$$(f, g) \in \chi_\Phi \longleftrightarrow \Big(n \geqslant m\Big) \wedge (\forall a_1^n)\Big(f\langle a_1^n\rangle \neq \varnothing \longrightarrow g\langle a_1^m\rangle \neq \varnothing\Big),$$

where $f\langle a_1^n\rangle \neq \varnothing$ means that $(a_1, \ldots, a_n, b) \in f$ for some $b \in A$.

7.5.2. Find an axioms system for Mal'cev–Post iterative algebras of partial multiplace functions.

7.5.3. Find an abstract characterization of Mal'cev–Post iterative algebras of multiplace relations introduced in [145].

Bibliography

Names written in cyrillic have been transcrypted to English in various ways over the years. For example, *Гарвацкий* has been written as *Garvackii* or *Garvatskij*, *Салий* as *Salii* or *Salij*, *Шайн* as *Schein*, *Shajn* or *Shajn*, *Трохименко* as *Trohimenko* or *Trokhimenko*. To prevent any misunderstandings we give their original spelling in brackets.

Similar case have been with Russian publications refered in Math. Review and Zentralblatt fúr Mathematik. *Математические исследования* have been transcrypted as *Mat. Issled.* or *Math. Research.* In non-regular publications only the title of a specific issue was often given, without the name of the journal.

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